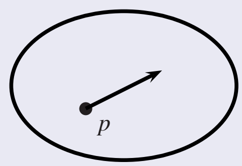
Tangent Vectors in $\mathbb{R}^n$

Reminder

In Section 2 we saw two equivalent ways to describe a tangent vector at a given point $p \in \mathbb{R}^n$:

(i) As an arrow emanating from p and represented by a column vector,



$$\begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}$$

(ii) As a derivation at p of C_p^∞, the algebra of germs of C^∞ functions at p.

Tangent Vectors in $\mathbb{R}^n$

Reminder

The correspondence between the two approaches is given by

$$\text{Vector } v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \longleftrightarrow \text{Derivation } D_v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p .$$

Remark

- The derivation approach is easier to generalize to manifolds.
- We are going to use this approach to define tangent vectors and the tangent space for manifolds.

The Tangent Space at a Point

Facts

- Let $\mathscr{F}_p(M)$ consists of pairs (U, f), where U is an open neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function.
- On $\mathscr{F}_p(M)$ we define an equivalence relation by

$$(U, f) \sim (U, g) \Longleftrightarrow f = g \text{ near } p.$$

Thus, $f \sim g$ means there is an open $W \subset U \cap V$ such that $p \in W$ and $f = g$ on W.

Definition

- The equivalence class of (U, f) is called the *germ of f at p*.
- The quotient $\mathscr{F}_p(M)/\sim$ is denoted $C_p^\infty(M)$; this is the set of germs of C^∞ functions at p.

The Tangent Space at a Point

Facts

- $C_p^\infty(M)$ is a vector space with respect to the scalar multiplication and the addition given by

$$\lambda \cdot (\text{germ at } p \text{ of } f) = \text{germ at } p \text{ of } \lambda f, \quad \lambda \in \mathbb{R},$$
$$(\text{germ at } p \text{ of } f) + (\text{germ at } p \text{ of } g) = \text{germ at } p \text{ of } f + g.$$

- $C_p^\infty(M)$ is also an algebra with respect to the multiplication given by

$$(\text{germ at } p \text{ of } f) \cdot (\text{germ at } p \text{ of } g) = \text{germ at } p \text{ of } fg.$$

The Tangent Space at a Point

Remark

Let U be an open set in M containing p.

- As $\mathscr{F}_p(U) \subset \mathscr{F}_p(M)$ we get an inclusion,
$$C_p^\infty(U) \subset C_p^\infty(M).$$
- As $(V, f) \in \mathscr{F}_p(M)$ and $(V \cap U, f_{|V\cap U})$ are equivalent, we actually have an equality. That is,
$$C_p^\infty(U) = C_p^\infty(M).$$

The Tangent Space at a Point

Definition

A *derivation at* p is any linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that
$$D(fg) = (Df)g(p) + f(p)Dg.$$

Remarks

1. By abuse of notation, we use the same letter f or g to denote a function and its germ at p.
2. The set of all derivations at p is a subspace of the space of linear maps $C_p^\infty \to \mathbb{R}$.

Definition

- The *tangent space of* M *at* p, denoted $T_p(M)$ or T_pM, is the vector space of all derivations at p.
- An element of $T_p(M)$ is now called a *tangent vector at* p.

Example

For $M=\mathbb{R}^n$ we recover the description of the tangent space $T_p(\mathbb{R}^n)$ in terms of derivations.

Example (see Remark 8.2)

- Let U be an open in M containing p. As $C_p^\infty(U)=C_p^\infty(U)$, we see that
$$T_p(U)=T_p(M).$$
- In particular, if $M=\mathbb{R}^n$, then
$$T_p(U)=T_p(\mathbb{R}^n)\simeq\mathbb{R}^n.$$

Example

Let $(U,\phi)=(U,x^1,\ldots,x^n)$ be a chart about p in M. Denote by $(r^1,\ldots,r^n)$ the standard coordinates in $\mathbb{R}^n$ (so that $x^i=r^i\circ\phi$).

- By definition, if f is C^∞ at p, then
$$\left.\frac{\partial}{\partial x^i}\right|_p f=\left.\frac{\partial}{\partial r^i}\right|_{\phi(p)}\left(f\circ\phi^{-1}\right)\in\mathbb{R}.$$
- We have $\left.\frac{\partial}{\partial x^i}\right|_p(fg)=(\left.\frac{\partial}{\partial x^i}\right|_p f)g(p)+f(p)\left.\frac{\partial}{\partial x^i}\right|_p g$.
- If $f=g$ near p, then $\left.\frac{\partial}{\partial x^i}\right|_p f=\left.\frac{\partial}{\partial x^i}\right|_p g$.
- Thus, $\left.\frac{\partial}{\partial x^i}\right|_p$ induces a map,
$$\left.\frac{\partial}{\partial x^i}\right|_p:C_p^\infty\longrightarrow\mathbb{R}.$$

We obtain a derivation at p, i.e., a tangent vector at p.

The Tangent Space at a Point

Remarks

- We sometimes write $\frac{\partial}{\partial x^i}$ instead $\frac{\partial}{\partial x^i}\Big|_p$ when it is understood that derivatives are evaluated at the point p.
- When M has dimension 1 and t is a local coordinate, we write $\frac{d}{dt}$ instead of $\frac{\partial}{\partial t}$.
- We will see later that $\left\{\frac{\partial}{\partial x^1}\Big|_p, \ldots, \frac{\partial}{\partial x^n}\Big|_p\right\}$ is a basis of $T_p(M)$.

The Differential of a Map

Facts

Let $F : M \rightarrow N$ be a C^∞ map, where M and N are manifolds.

- Given $X \in T_p(M)$ define $F_*(X) : C^\infty_{F(p)}(N) \rightarrow \mathbb{R}$ by
$$F_*(X)f = X(F \circ f), \qquad f \in C^\infty_{F(p)}(N).$$
- $F_*(X)$ is a linear map.
- As X is a derivation at p, we have
$$\begin{aligned} F_*(X)(fg) &= X\left[(f \circ F)(g \circ F)\right] \\ &= X\left[(f \circ F)\right](g \circ F)(p) + (f \circ F)(p) X\left[(g \circ F)\right] \\ &= \left[F_*(X)f\right] g\left(F(p)\right) + f\left(F(p)\right)\left[F_*(X)g\right]. \end{aligned}$$
That is, $F_*(X)$ is a derivation at $F(p)$, i.e., $F_*(X) \in T_{F(p)}(N)$.
- We thus get a map $F_* : T_p(M) \rightarrow T_{F(p)}(N)$, $X \rightarrow F_*(X)$.

The Differential of a Map

Fact

The map $F_* : T_p(M) \to T_{F(p)}(N)$ is linear, since we have

$$F_*(\lambda X)f = \lambda X(f \circ F) = \lambda F_*(X)f,$$
$$F_*(X+Y)f = X(f \circ F) + Y(f \circ F) = F_*(X)f + F_*(Y)f.$$

Definition

The linear map $F_* : T_p(M) \to T_{F(p)}(N)$ is called the *differential of F at p*.

Remarks

1. To emphasize the dependence on the point p we sometimes $F_{*,p}$ for F_*.
2. There are various notations for the differential. For instance, it also denoted d_pF, $dF(p)$, D_pF or even $F'(p)$.

The Differential of a Map

Example

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a C^∞ map. Denote by $(x^1, \ldots, x^n)$ the coordinates on $\mathbb{R}^n$ and by $(y^1, \ldots, y^m)$ the coordinates on $\mathbb{R}^m$. Set $F = (F^1, \ldots, F^m)$.

- Let $p \in \mathbb{R}^n$. Then $\left\{\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p\right\}$ is a basis of $T_p(\mathbb{R}^n)$.
- Likewise, $\left\{\frac{\partial}{\partial y^1}\big|_{F(p)}, \ldots, \frac{\partial}{\partial y^m}\big|_{F(p)}\right\}$ is a basis of $T_{F(p)}(\mathbb{R}^m)$.
- Given any $f \in C^\infty_{F(p)}(\mathbb{R}^m)$, we have

$$F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right)f = \frac{\partial}{\partial x^j}\bigg|_p (f \circ F) = \sum_{i=1}^m \frac{\partial F^i}{\partial x^j}(p) \frac{\partial}{\partial y^i}\bigg|_{F(p)} f.$$

- This means that $F_*\left(\frac{\partial}{\partial x^j}\big|_p\right) = \sum_{i=1}^m \frac{\partial F^i}{\partial x^j}(p) \frac{\partial}{\partial y^i}\big|_{F(p)}$.
- In other words, the matrix of F_* relative to the bases $\left\{\frac{\partial}{\partial x^j}\big|_p\right\}$ and $\left\{\frac{\partial}{\partial y^i}\big|_{F(p)}\right\}$ is precisely the Jacobian matrix $\left[\frac{\partial F^i}{\partial x^j}(p)\right]$.

Fact

Let $F : N \to M$ and $G : M \to P$ be C^∞ maps. Given any $p \in N$ the differentials $F_{*,p}$ and $G_{*,F(p)}$ are linear maps,

$$T_p(M) \xrightarrow{F_{*,p}} T_{F(p)}(N) \xrightarrow{G_{*,F(p)}} T_{G(F(p))}(P).$$

Theorem (Theorem 8.5; Chain Rule)

If $F : N \to M$ and $G : M \to P$ are C^∞ maps, then, for every $p \in N$, we have

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}.$$

Remark

Let $\mathbb{1}_M : M \to M$ be the identity, and let $p \in m$. Given $X \in T_p(M)$ and $f \in C_p^\infty$, we have

$$(\mathbb{1}_M)_*(X)f = X(f \circ \mathbb{1}_M) = Xf.$$

Thus, $(\mathbb{1}_M)_*(X) = X$, and so the differential $(\mathbb{1}_M)_*$ is the identity map $\mathbb{1}_{T_p(M)} : T_p(M) \longrightarrow T_p(M)$.

Corollary (Corollary 8.6)

If $F : N \to M$ is a diffeomorphism, then, for every $p \in N$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

Corollary (Corollary 8.7; Invariance of Dimension)

If an open $U \subset \mathbb{R}^n$ is diffeomorphic to an open $V \subset \mathbb{R}^m$, then we must have $n = m$.

Bases for the Tangent Space at a Point

Facts

Let $(U,\phi) = (U, x^1, \dots, x^n)$ be a chart about p in M. Denote by $(r^1, \dots, r^n)$ the coordinates in $\mathbb{R}^n$. Then:

- The map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.
- The differential $F_{*,p}$ is an isomorphism from $T_p(U) = T_p(M)$ to $T_{\phi(p)}(U) = T_{\phi(p)}\mathbb{R}^n$.
- $\{\frac{\partial}{\partial r^1}\big|_{\phi(p)}, \dots, \frac{\partial}{\partial r^n}\big|_{\phi(p)}\}$ is a basis of $T_{\phi(p)}(\mathbb{R}^n)$.
- By definition of ϕ_* and $\frac{\partial}{\partial x^i}\big|_p$, if $f \in C^\infty_{\phi(p)}(\mathbb{R}^n)$, then

$$\phi_*\left(\frac{\partial}{\partial x^i}\bigg|_p\right) f = \frac{\partial}{\partial x^i}\bigg|_p (f \circ \phi) = \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} \left[(f \circ \phi) \circ \phi^{-1}\right] = \frac{\partial}{\partial r^i}\bigg|_{\phi(p)} f.$$

Thus,
$$\phi_*\left(\frac{\partial}{\partial x^i}\bigg|_p\right) = \frac{\partial}{\partial r^i}\bigg|_{\phi(p)}.$$

Bases for the Tangent Space at a Point

Facts

To sum up:

- The differential $\phi_{*,p} : T_pM \to T_{\phi(p)}\mathbb{R}^n$ is an isomorphism.
- It maps the family $\{\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\}$ in T_pM to the basis $\{\frac{\partial}{\partial r^1}\big|_{\phi(p)}, \dots, \frac{\partial}{\partial r^n}\big|_{\phi(p)}\}$ of $T_{\phi(p)}\mathbb{R}^n$.

We deduce from this the following result:

Proposition (Proposition 8.9)

If $(U, x^1, \dots, x^n)$ is a chart about p in M, then $\{\frac{\partial}{\partial x^1}\big|_p, \dots, \frac{\partial}{\partial x^n}\big|_p\}$ is a basis of T_pM. In particular, T_pM has dimension n.

Corollary (Invariance of Dimension)

If M and N are diffeomorphic manifolds, then $\dim M = \dim N$.

Bases for the Tangent Space at a Point

Remarks

1. There are alternative definitions of the tangent space T_pM.
2. What is important to keep in mind is the following:
   - The tangent space T_pM is a vector space that has basis $\left\{\frac{\partial}{\partial x^1}\big|_p, \ldots, \frac{\partial}{\partial x^n}\big|_p\right\}$, where $(x^1, \ldots, x^n)$ are local coordinates.
   - We keep the same vector space upon changing local coordinates.

Local Expression for the Differential

Facts

Let $F : N \rightarrow M$ be a C^∞ map and $p \in N$. Let $(U, x^1, \ldots, x^n)$ be a chart around p in N and $(V, y^1, \ldots, y^m)$ a chart around $F(p)$ in M.

- As $\left\{\frac{\partial}{\partial x^j}\big|_p\right\}$ and $\left\{\frac{\partial}{\partial y^i}\big|_{F(p)}\right\}$ are bases of T_pN and $T_{F(p)}M$, there are constants a_j^k such that

$$F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \sum_{k=1}^m a_j^k \frac{\partial}{\partial y^k}\bigg|_{F(p)}, \quad j = 1, \ldots, n.$$

- Set $F^i = y^i \circ F$. We have

$$F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) y^i = \frac{\partial}{\partial x^j}\bigg|_p (y^i \circ F) = \frac{\partial F^i}{\partial x^j}(p).$$

Facts

- As $\frac{\partial y^i}{\partial y^k} = \delta^i_k$ (see Proposition 6.22), we get

$$\left(\sum_{k=1}^{m} a_j^k \frac{\partial}{\partial y^k}\bigg|_{F(p)}\right) y^i = \sum_{k=1}^{m} a_j^k \frac{\partial y^i}{\partial y^k}\bigg|_{F(p)} = \sum_{k=1}^{m} a_j^k \delta_k^i = a_j^i.$$

- This shows that $a_j^i = \frac{\partial F^i}{\partial x^j}(p)$. Thus,

$$F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \sum_{i=1}^{m} \frac{\partial F^i}{\partial x^j}(p) \frac{\partial}{\partial y^i}\bigg|_{F(p)}, \quad j = 1, \ldots, n.$$

Therefore, we have obtained the following result:

Proposition (Proposition 8.11)

Let $F : N \to M$ be a C^∞ map and $p \in N$. Let $(U, x^1, \ldots, x^n)$ be a chart around p in N and $(V, y^1, \ldots, y^m)$ a chart around $F(p)$ in M. Then, relative to the bases $\left\{\frac{\partial}{\partial x^j}\big|_p\right\}$ and $\left\{\frac{\partial}{\partial y^i}\big|_{F(p)}\right\}$ of T_pN and $T_{F(p)}M$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ has matrix

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right]_{\substack{1 \le i \le m \\ 1 \le j \le n}} \qquad \text{where } F^i = y^i \circ F.$$

Local Expression for the Differential

Remark (Remark 8.12)

- The inverse function theorem for manifolds (Theorem 6.26) asserts that F is locally invertible at p if and only if $\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] \neq 0$.
- Therefore, we obtain the following coordinate-free description of this result:

Theorem (Inverse Function Theorem)

Let $F : N \to M$ be a C^∞ map and $p \in N$. TFAE:

1. *F is locally invertible at p.*
2. *The differential $F_* : T_pN \to T_{F(p)}M$ is an isomorphism.*

Curves in a Manifold

Remark

- In classical differential geometry (i.e., differential geometry of surfaces) the tangent plane at a point p of surface $S \subset \mathbb{R}^3$ is defined in terms of tangent vectors of curves in S through p.
- We shall now see an analogous description of the tangent space for general manifolds.

Curves in a Manifold

Definition

- A *smooth curve in manifold M* is any smooth map $c: I \to M$, where I is some open interval in $\mathbb{R}$.
- We say that a smooth curve $c: I \to M$ starts at a given point $p \in M$ when $0 \in I$ and $c(0) = p$.

Definition (Velocity Vector)

Let $c: I \to M$ be a smooth curve and $t_0 \in M$. Its *velocity vector* (or *velocity*) at $t = t_0$ is

$$c'(t_0) := c_*\left(\frac{d}{dt}\bigg|_{t_0}\right) \in T_{c(t_0)}M.$$

Remarks

1. We also say that $c'(t_0)$ is the velocity vector at $c(t_0)$.
2. The velocity vector $c'(t_0)$ is also denoted $\frac{dc}{dt}(t_0)$ and $\frac{d}{dt}\big|_{t_0} c$.

Curves in a Manifold

Example

Suppose that $M = \mathbb{R}^n$ with coordinates $(x^1, \ldots, x^n)$. Let $c: I \to \mathbb{R}^n$ be a smooth curve, and set $c(t) = (c^1(t), \ldots, c^n(t))$.

- Given $t_0 \in I$, if $f \in f \in C^\infty_{c(t_0)}(\mathbb{R}^n)$, then

$$c'(t_0)f = c_*\left(\frac{d}{dt}\bigg|_{t_0}\right)f = \frac{d}{dt}\bigg|_{t_0} f\left(c(t)\right) = \sum_{i=1}^n \frac{dc^i}{dt}(t_0)\frac{\partial f}{\partial x^i}\left(c(t_0)\right).$$

- Thus,

$$c'(t_0) = \sum_{i=1}^n \frac{dc^i}{dt}(t_0)\frac{\partial}{\partial x^i}\bigg|_{c(t_0)}.$$

- That is, $c'(t_0)$ is the derivation at $c(t_0)$ defined by the vector,

$$\left(\frac{dc^1}{dt}(t_0), \ldots, \frac{dc^n}{dt}(t_0)\right) = \frac{dc}{dt}(t_0) \in \mathbb{R}^n.$$

  Therefore, we recover the usual notion of velocity vector from calculus and classical differential geometry.

Curves in a Manifold

Proposition (Proposition 8.16; see Tu's book)

For any point $p \in M$ and any tangent vector $X \in T_pM$, there is a curve $c : (-\epsilon, \epsilon) \to M$ starting at p with initial velocity $c'(0) = X$.

Corollary

For every $p \in M$, we have

$$T_pM = \left\{ c'(0);\ \ c : I \to M \text{ smooth curve starting at } p \right\}.$$

Remark

This interpretation of the tangent space is the analogue for smooth manifolds of the description of the tangent plane of surfaces in classical differential geometry in terms of tangent vectors of curves.

Curves in a Manifold

Remark (see Proposition 8.17)

Given $p \in M$ and $X \in T_pM$, let $c : I \to M$ is a smooth curve starting at p such that $c'(0) = X$. Then, for every $f \in C_p^\infty(M)$, we have

$$Xf = c'(0)f = c_*\left(\left.\frac{d}{dt}\right|_0 \right) f = \left.\frac{d}{dt}\right|_{t=0} (f \circ c)(t).$$

This provides us a more geometric description of tangent vectors as directional derivatives.

Computing the Differential Using Curves

Facts

Let $F : N \to M$ be a smooth map. Given $p \in N$ and $X \in T_pN$ let $c : I \to N$ be a smooth curve such that $c(0) = p$ and $c'(0) = X$.

- $F \circ c : I \to M$ is a smooth curve in M starting at $F(c(0)) = F(p)$.
- By the Chain Rule $(F \circ c)_{*,0} = F_{*,c(0)} \circ c_{*,0} = F_{*,p} \circ c_{*,0}$.
- Therefore, the velocity vector of $F \circ c$ at $t = 0$ is

$$(F \circ c)'(0) = (F \circ c)_* \left(\frac{d}{dt}\bigg|_0 \right) = F_{*,p}\left[c_{*,0}\left(\frac{d}{dt}\bigg|_0 \right) \right] = F_{*,p}\left[c'(0) \right].$$

- As $c'(0) = X$, we get

$$(F \circ c)'(0) = F_*(X).$$

Computing the Differential Using Curves

Proposition (Proposition 8.18)

Let $F : N \to M$ be a smooth map. Given $p \in N$ and $X \in T_pN$. Given $p \in N$ and $X \in T_pN$, for any smooth curve $c : I \to N$ starting at p and with velocity vector X at p, we have

$$F_*(X) = (F \circ c)'(0).$$

That is, $F_*(X)$ is the velocity vector at $F(p)$ of the curve $F \circ c : I \to M$.

Remark

This description of the differential of a smooth map is the analogue of the definition of the differential in terms of curves in Classical Differential Geometry.

Submersions and Immersions

Definition (Immersion)

Let $F : M \to N$ be a smooth map.

1. We say that F is an *immersion at a point* $p \in N$ when the differential $F_* : T_pN \to T_{F(p)}M$ is injective.
2. We say thar F is an *immersion* when it is an immersion at every point $p \in N$.

Remark

If F is an immersion at p, then $\dim N \leq \dim M$.

Example

If $n \leq m$, then the inclusion of $\mathbb{R}^n$ into $\mathbb{R}^m$

$$(x^1, \ldots, x^n) \longrightarrow (x^1, \ldots, x^n, 0, \ldots, 0)$$

is an immersion.

Submersions and Immersions

Definition (Submersion)

Let $F : M \to N$ be a smooth map.

1. We say that F is a *submersion at a point* $p \in N$ when the differential $F_* : T_pN \to T_{F(p)}M$ is surjective.
2. We say that F is a *submersion* when it is a submersion at every point $p \in N$.

Remark

If F is a submersion at p, then $\dim N \geq \dim M$.

Example

If $n \geq m$, then the projection of $\mathbb{R}^n$ into $\mathbb{R}^m$

$$(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n) \longrightarrow (x^1, \ldots, x^m)$$

is a submersion.

Example

- If U is an open in $\mathbb{R}^n$, the inclusion of U into $\mathbb{R}^n$ is both an immersion and submersion.
- This example shows that a submersion need not be onto.

Remark

A deeper in-depth analysis of immersions and submersions will be carried out in Section 11.

Definition (Rank of a Smooth Map)

If $F : N \to M$ is a smooth map and $p \in N$, the rank of the differential $F_* : T_pN \to T_{F(p)}M$ is called the *rank of F at p* and is denoted by $\operatorname{rk} F(p)$.

Remark

We have

$$\begin{aligned} F \text{ is an immersion at } p &\Longleftrightarrow \operatorname{rk} F(p) = \dim N, \\ F \text{ is a submersion at } p &\Longleftrightarrow \operatorname{rk} F(p) = \dim M. \end{aligned}$$

Definition (Critical and Regular Points)

Let $F : N \to M$ be a smooth map, and $p \in N$.

- We say that p is a *critical point of* F when the differential $F_* : T_pN \to T_{F(p)}M$ is not surjective.
- Otherwise we say that F is a *regular point of* F.

Remark

We have

$$\begin{aligned} p \text{ is a regular point} &\Longleftrightarrow F \text{ is a submersion at } p \\ &\Longleftrightarrow \operatorname{rk} F(p) = \dim M. \end{aligned}$$

Definition (Critical and Regular Points)

Let $F : N \to M$ be a smooth map, and $c \in M$.

- We say that c is a *critical value of* F when it is the image of a critical point, i.e., the preimage $F^{-1}(c)$ contains a critical point.
- Otherwise we say that c is a *regular value of* F.

Remarks

1. If qc is a critical value, then its preimage may contain regular points, but it contains at least one critical points.
2. Any point of $M \setminus F(N)$ is a regular value.
3. If $c \in F(M)$, then c is a regular value if and only if every point of $F^{-1}(c)$ is a regular point.

Rank, Critical and Regular Points

Fact

Let $f : M \to \mathbb{R}$ be a smooth function, and $p \in M$.

- *The differential $f_{*,p}$ is a linear map from T_pM to $T_{f(p)}\mathbb{R} \simeq \mathbb{R}$.*
- *Therefore, it is either onto or zero.*

Proposition (Proposition 8.23)

Let $f : M \to \mathbb{R}$ be a smooth function. Given any $p \in M$, TFAE:

1. *p is a critical point of f.*
2. *The differential $f_{*,p}$ is zero.*
3. *There is a chart $(U, x^1, \ldots, x^n)$ about p in M such that*

$$\frac{\partial f}{\partial x^i}(p) = 0, \qquad i = 1, \ldots, n.$$