Definition (Locally Euclidean Spaces)

A topological space M is called *locally Euclidean of dimension n* when, for every point p, there is a neighborhood V of p that is homeomorphic to an open subset of $\mathbb{R}^n$.

Remark

- It can be shown that if an open set of $\mathbb{R}^n$ is homeomorphic to an open of $\mathbb{R}^m$ then $m = n$.
- This implies that the dimension of a manifold is well defined.

Definition (Topological Manifolds)

A *topological manifold of dimension n* is a locally Euclidean of dimension n that is Hausdorff and second countable.

Remark

See Problem 5.1 for an example of non-Hausdorff locally Euclidean space.

Definition (Local Charts)

Let M be locally Euclidean of dimension n.

1. A *(local) chart near a point* $p \in M$ is pair (U, ϕ) where U is a neighborhood of p and $\phi : U \to \mathbb{R}^n$ is a homeomorphism (from U onto its image).
2. The open U is called a *coordinate neighborhood* or *coordinate open set*.
3. The map ϕ is called a *coordinate map* or *coordinate system*.
4. We say that the chart (U, ϕ) is *centered at* p when $\phi(p) = 0$.

Remark

If $U \to \mathbb{R}^n$ is homeomorphism onto its image, then $\phi(U)$ must be an open subset of $\mathbb{R}^n$.

Example

- The Euclidean space $\mathbb{R}^n$ is covered by the single $(\mathbb{R}^n, \mathrm{id}_{\mathbb{R}^n})$, where $\mathrm{id}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. Thus, $\mathbb{R}^n$ is a topological manifold of dimension n.
- Every open subset $U \subset \mathbb{R}^n$ is a topological manifold as well, with the single chart (U, id_U).

Remark

Second countability and Hausdorff condition are "hereditary conditions", i.e., they are satisfied by subsets.

Example

Any open subset U a topological manifold M is automatically a topological manifold: if (V, ϕ) is a chart for M, then $(V \cap U, \phi_{|V \cap U})$ is a chart for U.

Topological Manifolds

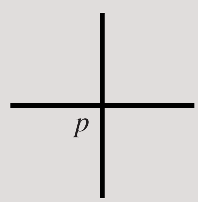
Example (A Cusp)

The graph of $y = x^{2/3}$ in $\mathbb{R}^2$ is a topological manifold (see below). It is homeomorphic to $\mathbb{R}$ via $(x, x^{2/3}) \to x$.

Topological Manifolds

Example (Example 5.4, A Cross)

The cross in $\mathbb{R}^2$ below is not locally Euclidean at the intersection p, and so it cannot be a topological manifold.



Compatible Charts

Facts

Let (U, ϕ) and (V, ψ) be two charts of a topological manifold.

1. $\phi(U \cap V)$ and $\psi(U \cap V)$ are open subsets of $\mathbb{R}^n$.
2. ϕ and ψ restricts to homeomorphisms,
$$\phi_{|U\cap V} : U \cap V \to \phi(U \cap V), \qquad \psi_{|U\cap V} : U \cap V \to \psi(U \cap V).$$
3. The compositions $(\psi_{|U\cap V}) \circ (\phi_{|U\cap V})^{-1}$ and $(\phi_{|U\cap V}) \circ (\psi_{|U\cap V})^{-1}$ and are denoted by $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$.
4. The maps $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$are inverses of each other.

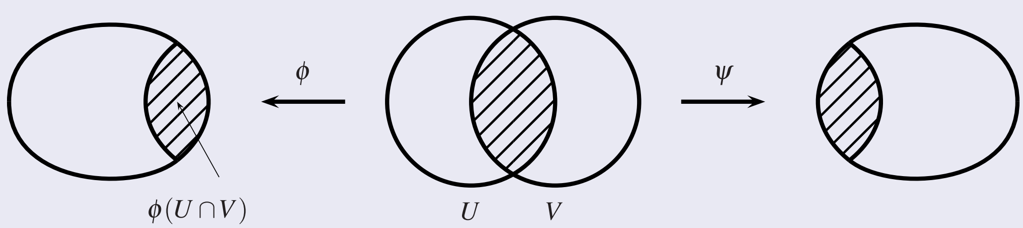
Compatible Charts

Definition (Transition Maps)

The maps

$$\psi\circ\phi^{-1} : \phi(U\cap V) \to \psi(U\cap V) \quad \text{and} \quad \phi\circ\psi^{-1} : \psi(U\cap V) \to \phi(U\cap V)$$

are called the *transition maps* of the charts (U, ϕ) and (V, ψ).



Definition (C^∞-Compatible Charts)

We say that two charts (U, ϕ) and (V, ψ) are C^∞-*compatible* when the transition maps $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are C^∞-maps.

Remark

As $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are inverses of each other, the above condition means that $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are C^∞-diffeomorphisms.

Definition (Atlas)

A C^∞-*atlas*, or simply an *atlas*, on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M, i.e., $M = \cup_\alpha U_\alpha$.

Remarks

1. The pairwise C^∞-compatibility means that, for all α, β, the transition maps $\phi_\beta \circ \phi_\alpha^{-1}$ are C^∞-maps.
2. This implies that every transition map $\phi_\beta \circ \phi_\alpha^{-1}$ is a C^∞-diffeomorphism, since its inverse is the transition map $\phi_\alpha \circ \phi_\beta^{-1}$, and hence is C^∞.

Example (Example 5.7, A C^∞-atlas on the circle)

We realize the circle $\mathbb{S}^1$ a subset of the complex plane,

$$\mathbb{S}^1 = \{z \in \mathbb{C};\ |z| = 1\} = \{e^{it};\ t \in [0, 2\pi]\}.$$

Let U_1 and U_2 be the open subsets,

$$U_1 = \{e^{it};\ t \in (-\pi, \pi)\} = \mathbb{S}^1 \setminus \{-1\},$$
$$U_2 = \{e^{it};\ t \in (0, 2\pi)\} = \mathbb{S}^1 \setminus \{1\}.$$

Define $\phi_1 : U_1 \to (-\pi, \pi)$ and $\phi_2 : U_2 \to (0, 2\pi)$ as the inverses of the maps $\psi_1 : (-\pi, \pi) \to U_1$ and $\psi_2 : (0, 2\pi) \to U_2$ given by

$$\psi_j(t) = e^{it}.$$

Then $\{(U_1, \phi_1), (U_2, \phi_2)\}$ is a C^∞ atlas for $\mathbb{S}^1$.

Definition

We say that a chart (V, ψ) is compatible with an atlas $\{(U_\alpha, \phi_\alpha)\}$ when it is compatible with every chart (U_α, ϕ_α) of the atlas.

Lemma (Lemma 5.8)

Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

Smooth Manifolds

Definition (Smooth manifolds; first definition)

A *smooth manifold*, or C^∞ *manifold* (of dimension n) is a topological manifold (of dimension n) that is equipped with a C^∞ atlas.

Remark

- A 1-dimensional manifold is called a *curve*.
- A 2-dimensional manifolds is called a *surface*.

Smooth Manifolds

Remarks

1. Two C^∞-atlases on a given topological manifold may define the same ring of C^∞-functions (see Section 6).
2. We would like to say that we have the same C^∞-manifold structure when this happens.
3. To deal with this issue it is convenient to use the notion of *maximal atlas*.

Definition (Maximal Atlas)

An atlas $\mathscr{M}$ of a locally Euclidean space is said to be *maximal* when it is not contained in another atlas, i.e., if $\mathscr{A}$ is an atlas containing $\mathscr{M}$, then it must agree with $\mathscr{M}$.

Proposition (Proposition 5.8)

Let $\mathscr{A} = \{(U_\alpha, \phi_\alpha)\}$ be a C^∞-atlas on a locally Euclidean space.

(i) *There is a unique maximal C^∞-atlas $\mathscr{M}$ that contains $\mathscr{A}$.*

(ii) *$\mathscr{M}$ consists of all local charts (V, ψ) that are C^∞-compatible with all the charts (U_α, ϕ_α).*

Definition (Smooth Structure, C^∞-Manifold; 2nd definition)

- A *smooth structure*, or *C^∞-structure*, on a topological manifold is given by the datum of a maximal C^∞-atlas.
- A C^∞-manifold is a topological manifold equipped with a C^∞-structure (i.e., a maximal C^∞-atlas).

Remark

The two definitions of C^∞-manifolds are equivalent.

- A C^∞-atlas $\mathscr{A}$ on a topological manifold M is contained in a unique maximal C^∞-atlas $\mathscr{M}$.
- It thus defines a unique C^∞-structure on M (given by the maximal atlas $\mathscr{M}$).

Smooth Manifolds

Remark

Two C^∞-manifolds agree if and only if they agree as sets and have the same topology and C^∞-structure (i.e., maximal C^∞-atlas).

Fact

Let $\mathscr{A} = \{(U_\alpha, \phi_\alpha)\}$ and $\mathscr{B} = \{(V_\beta, \psi_\beta)\}$ be C^∞-atlases on a topological manifold M. TFAE:

(i) *$\mathscr{A}$ and $\mathscr{B}$ define the same C^∞-structure on M.*

(ii) *$\mathscr{A}$ and $\mathscr{B}$ are contained in the same maximal C^∞-atlas.*

(iii) *The charts of $\mathscr{A}$ and $\mathscr{B}$ are pairwise C^∞-compatible, i.e., for all α, β the charts (U_α, ϕ_α) and (V_β, ψ_β) are C^∞-compatible.*

Smooth Manifolds

Remark

- In practice we may forget about maximal atlases.
- In order to verify that a topological space M is a C^∞-manifold we only need to check that
  (a) M is Hausdorff and second countable.
  (b) M has a C^∞-atlas.

Remarks

1. In what follows, by a "manifold" it will be always meant a "smooth manifold'.
2. By a *chart* (U, ϕ) *about* p in a (smooth) manifold M, we shall mean a chart in the maximal C^∞ atlas of M such that $p \in U$.

Smooth Manifolds

Notation

We denote by $(r^1, \ldots, r^n)$ the standard coordinates in $\mathbb{R}^n$,

Definition (Local Coordinates)

- If (U, ϕ) is a chart of a (smooth) manifold, we let $x^i = r^i \circ \phi$ be the i-th coordinate of ϕ.
- The functions $x^1, \ldots, x^n$ are called *local coordinates on U*.

Remarks

1. If $p \in U$, then $(x^1(p), \ldots, x^n(p))$ is a point in $\mathbb{R}^n$.
2. We often omit p from the notation, so that, depending on context, $(x^1, \ldots, x^n)$ may denote local coordinates (functions) or a point in $\mathbb{R}^n$.

Examples of Manifolds

Example (Example 5.11; Euclidean Spaces)

The Euclidean space $\mathbb{R}^n$ is a smooth manifold with single chart $(\mathbb{R}^n, r^1, \ldots, r^n)$, where $r^1, \ldots, r^n$ are the standard coordinates in $\mathbb{R}^n$.

Examples of Manifolds

Example (Vector Spaces)

Let E be a (real) vector space of dimension n. Any basis $(e_1, \ldots, e_n)$ of E defines a chart (E, ϕ), where $\phi : E \to \mathbb{R}^n$ is defined by

$$\phi(r^1 e_1 + \cdots + r^n e_n) = (r^1, \ldots, r^n), \qquad r^i \in \mathbb{R}.$$

This is a linear isomorphism with inverse,

$$\phi^{-1}(r^1, \ldots, r^n) = r^1 e_1 + \cdots + r^n e_n.$$

Therefore, E is a smooth manifold with single chart (E, ϕ).

Remarks

1. The topology of E is such that the open subsets are of the form $\phi^{-1}(U)$, where U ranges over open subsets of $\mathbb{R}^n$.
2. The topology and smooth structure of E do not depend on the choice of the basis $e_1, \ldots, e_n$.

Example (Example 5.12; Open subset of a manifold)

An open subset V of a smooth manifold M is a smooth manifold. If $\{(U_\alpha, \phi_\alpha)$ is a C^∞-atlas for M, then $\{(U_\alpha \cap V, \phi_{\alpha|V\cap U_\alpha})\}$ is a C^∞-atlas for V.

Example (Example 5.13; Manifolds of dimension 0)

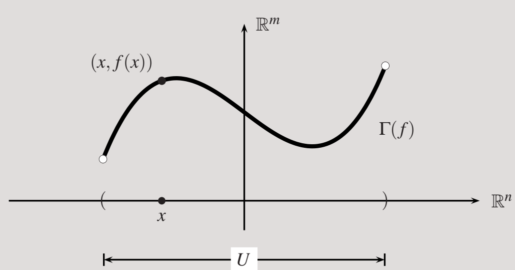
Let M be a 0-dimensional manifold. Then

- For every point $p \in M$, the singleton $\{p\}$ is homeomorphic to $\mathbb{R}^0 = \{0\}$, and hence is open. Therefore, M is discrete.
- Second countability then implies that M is countable.
- The charts $(\{p\}, p \to 0)$, $p \in M$, form a C^∞-atlas.

Example (Example 5.14; Graph of a smooth function)

Let $f : U \to \mathbb{R}^m$ a C^∞ function, where U is an open subset. The graph of f is

$$\begin{aligned}\Gamma(f) &= \{(x, f(x))\,;\; x \in U\} \\ &= \{(x, y) \in U \times \mathbb{R}^m;\; y = f(x)\}\,.\end{aligned}$$



This is a smooth manifold with single chart $(\Gamma(f), \phi)$, where $\phi : \Gamma(f) \to U$ is defined by

$$\phi(x, f(x)) = x, \qquad x \in U.$$

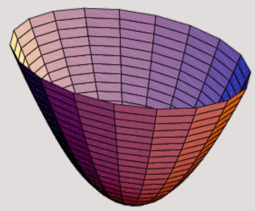
Here $\phi^{-1} : U \to \Gamma(f)$ is just $x \to (x, f(x))$.

Examples of Manifolds

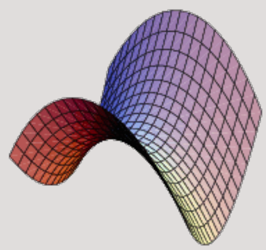
Examples

The following surfaces are graphs of smooth functions, and hence are C^∞-manifolds:

- Elliptic paraboloid: $z = x^2 + y^2$.



- Hyperbolic paraboloid: $z = y^2 - x^2$.



Example (Example 5.15; Real matrices)

- Let $\mathbb{R}^{m\times n}$ be the space of $m \times n$ matrices $A = (a_{ij})$ with real entries. This is smooth manifold, since this is a vector space. Its dimension is mn.
- The *real linear group* is
$$\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{R}^{n\times n};\ \det(A) \neq 0\} = \det{}^{-1}(\mathbb{R} \setminus 0).$$
This is an open subset of $\mathbb{R}^{n\times n}$, since the determinant map $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ is continuous. Therefore, $\mathrm{GL}(n, \mathbb{R})$ is a smooth manifold of dimension n^2.

Examples of Manifolds

Example (Example 5.15; Complex matrices)

- Let $\mathbb{C}^{m\times n}$ be the space of $m \times n$ matrices $A = (a_{ij})$ with real entries. This is smooth manifold, since this is a real vector space. It has complex dimension mn, and so its real dimension is $2mn$.
- The *complex linear group* is

$$\mathrm{GL}(n, \mathbb{R}) = \{A \in \mathbb{C}^{n\times n};\ \det(A) \neq 0\} = \det{}^{-1}(\mathbb{C} \setminus 0).$$

As in the real case, this is an open subset of $\mathbb{C}^{n\times n}$, and so $\mathrm{GL}(n, \mathbb{C})$ is a smooth manifold of dimension $2n^2$.

Examples of Manifolds

Example (Spheres; Example 5.16 and Problem 5.3)

The *unit sphere* of $\mathbb{R}^{n+1}$ is

$$\mathbb{S}^n = \left\{(x^1, \ldots, x^{n+1}) \in \mathbb{R}^{n+1};\ (x^1)^2 + \cdots + (x^{n+1})^2 = 1\right\}.$$

This is a smooth manifold of dimension n. An atlas is $\{(U_i^\pm, \phi_i^\pm)\}_{i=1}^{n+1}$, where

$$U_i^\pm = \left\{(x^1, \ldots, x^{n+1}) \in \mathbb{S}^n;\ \pm x^i > 0\right\},$$

and $\phi_i^\pm : U_i^\pm \rightarrow \mathbb{B}^n$ is defined by

$$\phi_i^\pm(x^1, \ldots, x^{n+1}) = \left(x^1, \ldots, x^i, x^{i+1}, \ldots, x^{n+1}\right).$$

Here $\mathbb{B}^n$ is the unit ball of $\mathbb{R}^n$. The inverse map of $\phi_i^\pm$ is

$$(\phi_i^\pm)^{-1}(u^1, \ldots, u^n) = \left(u^1, \ldots, u^{i-1}, \pm\sqrt{1 - (u^1)^2 - \cdots - (u^n)^2}, u^i, \ldots, u^n\right).$$

Examples of Manifolds

Remarks

1. The above smooth structure on $\mathbb{S}^n$ is called its *standard smooth structure*.
2. For $n = 1$ it agrees with the previous smooth structure.
3. It can be shown that $\mathbb{S}^7$ admits exactly 28 distinct smooth structures.

Remarks

1. It is known that any topological manifold of dimension ≤ 3 admits at most one smooth structure.
2. It can be also shown that (compact) topological manifold of dimension ≥ 5 admits at most a finitely many smooth structures.
3. In dimension 4 the situation remains unsettled.

Examples of Manifolds

Definition

Let M and N be locally Euclidean spaces of respective dimensions m and n. If (U, ϕ) is a chart for M and (V, ψ) is a chart for V, then the map $\phi \times \psi : U \times V \to \mathbb{R}^{m+n}$ is defined by

$$(\varphi \times \phi)(x, y) = (\phi(x), \psi(y)) \in \mathbb{R}^{m+n}, \qquad x \in U, \ y \in V.$$

Remark

$\phi \times \psi$ is a homeomorphism from $U \times V$ onto the open subset $\phi(U) \times \psi(V) \subset \mathbb{R}^{m+n}$.

Fact (Corollary A.21 and Proposition A.22)

If M and N are both Hausdorff second countable topological spaces, then the product $M \times N$ is again Hausdorff and second countable.

Proposition (Proposition 5.18, Example 5.17)

Suppose that M and N are smooth manifolds of respective dimensions m and n. Let $\{(U_\alpha, \phi_\alpha)\}$ be a C^∞-atlas for M and $\{(V_\beta, \psi_\beta)\}$ a C^∞-atlas for N. Then

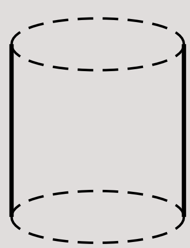
1. *The collection $\{(U_\alpha \times V_\beta, \phi_\alpha \times \psi_\beta)\}$ is a C^∞ atlas for $M \times N$.*
2. *The product $M \times N$ is a smooth manifold of dimension $m + n$.*

Remark

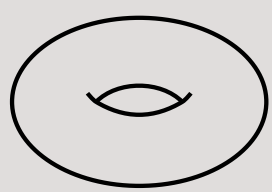
The smooth structure of $M \times N$ does not depend on the choices of the atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$.

Example

The infinite cylinder $\mathbb{S}^1 \times \mathbb{R}$ and the torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ are both smooth manifolds of dimension 2, since they are product of 1-dimensional smooth manifolds.



Infinite cylinder.



Torus.

Remark

More generally, if $M_1, \ldots, M_k$ are smooth manifolds, then their $M_1 \times \cdots \times M_k$ is a smooth manifold of dimension $\dim M_1 + \cdots + \dim M_k$.

Example

The n-torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n times) is a smooth manifold of dimension n.