6 Compact topological spaces

6.1 Basic properties

Definition 6.1 (Compact topological space).
A topological space E is called compact if each of its open covers has a finite sub-cover. A subset $A \subset E$ of a topological space is said compact if it is a compact topological space with respect to the subspace topology.

Remark 6.2 (Compactness formulated with closed sets).
A topological space is compact if and only if for every collection C of closed subsets of E such that $\cap_{\lambda \in \Lambda} C_\lambda = \emptyset$ there is a finite sub-collection $\lambda_1, \dots \lambda_n \in \Lambda$, such that $\cap_{i=1}^{n} C_{\lambda_i} = \emptyset$.

Proposition 6.3.
Compact subsets of a metric space are bounded.

Proof. Let (E, d) be a metric space and A a compact subset in E. One obtain an open covering of A if we put an open unite ball on each point of A. From the compactness of A there exists finitely many point $x_i \in A$, $(i \in \underline{n})$ such that $A \subset \cup_{i=1}^{n} B(x_i, 1)$. Then for any $x, y, \in A$, we have

$$d(x, y) \quad \leq \quad \max_{1 \leq i,j \leq n} d(x_i, x_j) + 2 \quad < \quad +\infty$$

that is A is bounded. □

Remark 6.4 (Compact sets of metric spaces).
As we will see later, the compact sets of Hausdorff separable topological space are closed. One obtain therefore that compact sets of metric spaces are closed and bounded.
The converse however is not true: in a metric space a closed and bounded set is not necessarily compact. Indeed, let us consider a non-finite set in a discrete metric space. It is closed and bounded, but not compact.

Proposition 6.5 (Closed subsets of a compact set).
The closed subsets of a compact set are compact.

Proof. Let E be a compact set and $A \subset E$ a closed set. If $\cup_{i \in I} U_i$ is an open covering of A, then

$$\cup_{i \in I} U_i \cup (E \setminus A)$$

is an open covering of E. Because of the compactness of E there exists a finite subset $J \subset I$ such that $\cup_{i \in J} U_i \cup E - A$ is an open covering of E, therefore $\cup_{i \in J} U_i$ is an open covering of A. □

Proposition 6.6.
Let A and B be two compact disjoint sets in a Hausdorff separable topological space $(E, \mathcal{O})$. Then there exist disjoint open sets U and V, such that $A \subset U$ and $B \subset V$.

Proof. Let $a \in A$ be fixed. Since E is Hausdorff separable, for any $b \in B$ there exist disjoint U_b and V_b open sets, such that $a \in U_b$, $b \in V_b$. Since B is compact, there exist finitely many $b_1, ..., b_n$ such that

$$B \subset V_{b_1} \cup \cdots \cup V_{b_n}.$$

Let $V_a = V_{b_1} \cup ... \cup V_{b_n}$ and $U_a = U_{b_1} \cap \cdots \cap U_{b_n}$. Both U_a and B_a are open, $a \in U_a$ and $B \subset V_a$ and $U_a \cap V_a = \emptyset$. We have $A \subset \cup_{a \in A} U_a$, and using the compactness of A, there exist finitely many $a_1, \ldots a_k$ such that $A \subset U_{a_1} \cap \cdots \cap V_{a_k}$. Then if one consider $U = U_{a_1} \cup ... \cup U_{a_k}$ and $V = V_{a_1} \cap ... \cap V_{a_k}$, we get U and V disjoint open set such that $A \subset U$, $B \subset V$. □

Proposition 6.7 (Compact set of Hausdorff spaces).
The compact subsets of a Hausdorff separable topological space are closed sets.

Proof. Let $(E, \mathcal{O})$ be Hausdorff separable topological space and $A \subset E$ a compact set and let $x \in E \setminus A$ The both A and $\{x\}$ are compact sets, and using Proposition 6.6 we get that there are disjoint open sets U_x and V such that $A \subset U_x$, $x \in V_x$. It follows that $A \cap V_x = \emptyset$, and

$$E \setminus A = \cup_{x \notin A} V_x$$

that is $E \setminus A$, therefore A is a closed set. □

Remark 6.8 (Compact sets are not necessarily closed sets).
Compact sets are not necessarily closed sets. To show this, one can consider the following examples:

- any one point set is compact, but it is not necessarily closed (the one point sets are closed if and only if the topological space satisfies the T_1 property.
- in a topology containing finitely many closed sets any set is compact, but naturally, they are not necessarily closed;

Definition 6.9 (Accumulation point of a sequence).
Let x_n be a sequence in the topological space. The point $x \in E$ is said an accumulation point of the sequence x_n if for any $p \in \mathbb{N}$ and open neighbourhood U of x there exists $n \in \mathbb{N}$ such that $n > p$ and $x_n \in U$.

Remark 6.10 (The set of accumulation points of a sequence).
From the above definition it follows that x is an accumulation point on the sequence x_n if for any $p \in \mathbb{N}$ if it is an accumulation point of the set $S_p \doteq$

$\{x_n \mid n \geq p\}$. It follows that the set of accumulation points of the sequence can be characterized as

$$\{\text{accumulation points of } x_n\} = \cap_{p\geq 0}\overline{S_p}.$$

Proposition 6.11 (Sequences in compact sets).
In a compact space any sequence has an accumulation point.

Proof.

It is sufficient to show that the set $\cap_{p\geq 0}\overline{S_p} = A$ is a nonempty set. Let us prove by contradiction: suppose that A is empty. The form the compactness of E (using Remark 6.2) there exist finitely many $p_1 < p_2 < \cdots < p_k$ numbers such that

$$\overline{S_{p_1}} \cap \ \ldots \ \cap \overline{S_{p_k}} = \emptyset$$

which is a contradiction, since $\overline{S_{p_1}} \cap \ \ldots \ \cap \overline{S_{p_k}} = \overline{S_{n_k}}$ is a nonempty set. □

Definition 6.12 (Convergent sequence).
The sequence x_n in a topological space E is convergent if there is $x_0 \in E$ such

that for any open neighbourhood U of x_0 there exists $n_0 \in \mathbb{N}$ such that if $n > n_0$ then $x_n \in U$. The point x_0 is called the limit of the sequence x_n. (Notation: $\lim x_n = x_0$.)

Exercise 6.13 (Convergent sequence in Hausdorff space)**.**
Show that if a sequence is convergent, then its limit is unique and it is the only accumulation point of the sequence.

Proposition 6.14 (Convergent sequences in compact Hausdorff spaces)**.**
In a Hausdorff separable compact topological space a sequence is convergent if and only if it has exactly one accumulation point.

Proof. If a sequence x_n is convergent in a Hausdorff separable space, then its limit is the unique accumulation point (see Exercise 6.13).
On the other hand, let us suppose that a sequence x_n has exactly one accumulation point in a compact Hausdorff space. We denote this point by $a \in E$ and we will show that the sequence is convergent and $\lim x_n = a$. We argue by contradiction: let us suppose that x_n is not convergent. Then there is an

open neighbourhood U of $a \in E$ such that for any $k \in \mathbb{N}$ there exists $n_k > k$, such that $x_{n_k} \notin U$. We can obtain that way a sub-sequence $(x_{n_k})_{k=1}^{\infty}$ such that the sub-sequence has no point in U, therefore a cannot be an accumulation point of this sequence. From Theorem 6.11 however we get that this sub-sequence has an accumulation point. This accumulation, let it be denoted by $b \in E$, is different from a. One obtains therefore that the original sequence has two different accumulation points (a and b) which is a contradiction with the hypothesis. □

Theorem 6.15 (Tychonoff's theorem: product of compact spaces)**.**
The product of compact topological spaces is compact.

Proof. The proof of the theorem is using the notion of filter: $\mathcal{F}$ if a filter if the following two properties are satisfied: 1) for any $A,\ B \in \mathcal{F}$ one has $A \cap B \in \mathcal{F}$; 2) for any $A \in \mathcal{F}$ and $A \subset C$ one has $C \in \mathcal{F}$. A filter $\mathcal{M}$ is said maximal if from $\mathcal{M} \subset \mathcal{F}$ one has $\mathcal{M} = \mathcal{F}$.
In order to prove the theorem, we will use Remark 6.2 where the compactness was formulated in terms of closed sets. Let us consider the product space

$E = \Pi_{i \in I} E_i$ of the compact topological spaces E_i and let us suppose that we have a family of closed sets

$$(F_\lambda)_{\lambda \in \Lambda} \tag{4}$$

such that the intersection of any finite subfamily is nonempty. We will show, that the intersection of the family is nonempty which proves the compactness of E.

1. Let $\mathcal{F}$ be the subset of E containing a finite intersection of (4). Then $\mathcal{F}$ is a filter. Moreover, there exists a maximal filter $\mathcal{M}$ containing $\mathcal{F}$.
2. Let $i \in I$ be an arbitrary index. For any finite set $A_1, \ldots, A_n$ of elements in $\mathcal{M}$ we have $A_1 \cap \cdots \cap A_n \neq \emptyset$, therefore

   $$\emptyset \neq p_i(A_1 \cap \cdots \cap A_n) \subset p_i(A_1) \cap \cdots \cap p_i(A_n) \subset \overline{p_i(A_1)} \cap \cdots \cap \overline{p_i(A_n)}.$$

   It follows that $\{\overline{p_i(A)} \mid A \in \mathcal{M}\}$ is a family of closed sets such that any finite subfamily is nonempty. From the compactness of E_i we obtain that their intersection cannot be empty, that is there exists $a_i \in E_i$ such that

   $$a_i \in \overline{p_i(A)} \quad \text{for any} \quad A \in \mathcal{M}.$$

3. If $V_i \subset E_i$ is an open neighbourhood of a_i, then $p^{-1}(V_i) \in \mathcal{M}$, since for every $A \in \mathcal{M}$ one has $a_i \in \overline{p_i(A_i)}$, that is $V_i \cap p_i(A) \neq \emptyset$ i.e. $p_i^{-1}(V_i) \cap A \neq \emptyset$. Since $\mathcal{M}$ is maximal,
$$p_i^{-1}(V_i) \in \mathcal{M}.$$

4. Let us consider in the product space $a = (a_i)_{i \in I}$. We will prove that
$$a \in \cap_{\lambda \in \Lambda} F_\lambda.$$
We remark that it is enough tho show that any open neighbourhood of a intersects F_λ (since that would show that the closure of F_λ contains a, but the closure if F_λ is itself since F_λ is closed).
Let V be an elementary open (elementary open sets form a basis of the product topology) set containing a: $V = \cap_{i \in J} p_i^{-1}(V_i)$ where J is finite, and V_i are open neighbourhood of a_i in E_i. From item 3.) we know that $p_i^{-1}(V_i) \in \mathcal{M}$, therefore $V \in \mathcal{M}$. Since $F_\lambda \in \mathcal{F} \subset \mathcal{M}$ we have also $V \cap F_\lambda \in \mathcal{M}$, and $V \cap F_\lambda \neq \emptyset$.

□

6.2 Continuous functions on compact sets

Theorem 6.16 (Continuous image of a compact set)**.**
The image of a compact set under a continuous map is compact.

Proof. Let $(E, \mathcal{O}_E)$ be a compact topological space and $f : E \longrightarrow F$ a continuous map into the topological space $(F, \mathcal{O}_F)$. Let $\cup_{i\in I} V_i$ an open covering of the image set $f(E)$. Then

$$\cup_{i\in I} f^{-1}(V_i)$$

is an open covering of E. Using the compactness of E, there exists a finite subset J of I such that $E \subset \cup_{j\in J} f^{-1}(V_j)$ Then $\cup_{j\in J} V_j$ give a finite open covering of $f(E)$. □

Proposition 6.17.
Let $(E, \mathcal{O}_E)$ compact, $(F, \mathcal{O}_F)$ Hausdorff separable, and $f : E \longrightarrow F$ a continuous bijection. Then f is a homeomorphism.

Proof. For any A closed set in E we have $f(A)$ compact in $f(E) = F$, therefore (using the Hausdorff property of F) the set $f(A)$ is closed. It follows that f^{-1} is a continuous function. $\square$

Exercise 6.18 (Extreme value theorem).
Let $f : E \longrightarrow \mathbb{R}$ be a continuous function defined on a compact set. Then $f(E)$ is bounded and there exists $a \in E$ and $b \in E$ such that

$$f(a) = \inf\ f(E) \quad \text{and} \quad f(b) = \sup\ f(E),$$

Proof. Since $f(E)$ is compact, it is bounded, and (using the Hausdorff property of $\mathbb{R}$ it is also closed. Because of the boundedness $f(E)$ has an infinum and a supremum, and because of the closedness $f(E)$ contains them ... $\square$