5 Connectedness and path connectedness

5.1 Connectedness

Definition 5.1.
The topological space $(E, \mathcal{O})$ is said to be connected if the only subsets of E that are both open and closed are the empty set $\emptyset$ and E itself. A subset of E is said to be connected if it is connected with respect of the induced subspace topology.

Exercise 5.2 (Equivalent formulations of the connectedness)**.**

1. $(E, \mathcal{O})$ is connected if and only if one writes $E = U \cup V$ as a union of two disjoint open sets, then U or V is necessarily the empty set.
2. $(E, \mathcal{O})$ is connected if and only if one writes $E = A \cup B$ as a union of two disjoint closed sets, then A or B is necessarily the empty set.

Exercise 5.3 (The image of conneted sets under continuous maps).
The image of a connected set under a continuous function is also connected.

Indication. Use the fact that the preimage of an open (resp. closed) set is also an open (resp. closed) set. □

Remark 5.4 (Separation).
One says that the space can be "separated" if it can be broken up into nonempty disjoint open sets. A topological space is connected if and only if it cannot be separated.

Exercise 5.5 (Connected subset).
Let A be a connected set in E, then

- the closure $\overline{\mathrm{A}}$ of A is connected;
- if $\mathrm{A} \subset \mathrm{B} \subset \overline{\mathrm{A}}$, then the set B is connected;
- if A is dense in E, then E is also connected.

Proof. It is enough to show that if $A \subset \mathrm{E}$ is a connected dense subset of E, then E is also connected. Indeed, let us suppose that $E = U \cup V$ where U and V are disjoint open sets. Then $A = (A \cap U) \cup (A \cap V)$ where $A \cap U$ and $A \cap V$ are open in the subspace topology of A. It follows that one of them must be the empty set. If $A \cap U = \emptyset$ (resp. $A \cap V = \emptyset$), then from the density of A it follows that U (resp. V) is empty.

□

Proposition 5.6.
The union of connected sets having pairwisely nonempty intersections is connected.

Proof. Let us suppose that $A_\alpha \subset E$ is connected and let us suppose that any pairwise intersection is nonempty. Let $A = \cup_\alpha A_\alpha$ denote the union of those sets and let us consider the separation $U \cup V$ of A with disjoint open sets. For any α we have $A_\alpha \subset U$ or $A_\alpha \subset V$ is open, because of the connectedness of A_α. Since any two sets has a common points, it follows that all A_α should be a subset of the same set: either U or V. Then necessarily the other set is the empty set. □

Exercise 5.7 (Connected set of the real line).
On the set of real numbers (with the usual topology) the connected sets are the intervals.

Indication. If A is not an interval, then there exists a number c such that c is not in A but A has greater and smaller elements then c. Then $(A \cap (-\infty, c)) \cup (A \cap (c, +\infty))$ is a separation of A, and A is not connected.
Moreover,

- for an closed interval $[a, b]$, let us suppose that $U \cup V$ is a separation. Let for example $a \in U$ and $c = \inf(V)$. Since U is open, therefore $c > a$. If $c \in U$, then U from the openness of U $c < \inf(V)$ and c wouldn't be the infinum of V. On the other hand, if $c \in V$, since an open set could not contain its infinum.
- for an open interval $]a, b[$, let us suppose that there is a separation. Then there is a closed interval in $]a, b[$ such that one part is in U, the other part is in V. That would give a separation of the closed interval which is impossible.

- Similar argument works for non-bounded intervals: for any separation one could chose a closed interval such that the two endpoints of the interval are in different sets of the separation ...

$\square$

Exercise 5.8.

1. $\mathbb{R}^n$ is a connected set.
2. If $K \subset \mathbb{R}^n$ is a star-shaped set, then it is connected.

Indication.

1. From Exercise 5.7 we know that $\mathbb{R}$ is connected. Moreover, $\mathbb{R}^n$ is just the union of straight line passing through $O \in \mathbb{R}^n$. From Proposition 5.6 we get the statement.
2. From Proposition 5.6 we get the statement.

$\square$

Proposition 5.9.

For any $a \in E$ there exists the largest connected set containing a.

Proof. Indeed, consider the set

$$\cup\{A \subset E \mid a \in A \text{ and } A \text{ is connected}\}.$$

It is connected and contains a. □

Definition 5.10 (Components of a topological space).
The maximal connected subsets of the topological space E are called components of E.

Proposition 5.11.
Any component of a topological space is a closed set.

Proof. Indeed if C is a component of a topological space E, the its closure $\overline{C}$ is connected, and because of the maximal property we have $C = \overline{C}$. □

Remark 5.12.
If a topological space has finitely many components, then those components are

not only closed, but open sets as well. However if the number of components is not finite, then it may happen that the components are not open sets. Indeed, let us consider the set $\mathbb{Q} \subset \mathbb{R}$ with the subspace topology. The maximal connected components are the one-point sets. On the other hand, those sets are not open sets.

5.2 Path-connectedness

Definition 5.13 (Paths of a topological space).
A path in a topological space is a continuous map $f : [0, 1] \longrightarrow E$. $f(0)$ is the starting point, $f(1)$ is the ending point of the path f. The path is called loop if $f(0) = f(1)$.

Definition 5.14.
If f and g are path of E and $f(1) = g(0)$, then let $f \star g$ be the product of the two path defined as follows:

$$f \star g(t) = \begin{cases} = f(2t), & 0 \leq t \leq \dfrac{1}{2} \\ = g(2t-1), & \dfrac{1}{2} \leq t \leq 1 \end{cases}$$

Lemma 5.15 (Gluing/pasting rule).
If $E = C_1 \cup \cdots \cup C_n$ where the sets C_i are closed, and the restriction of $f : E \longrightarrow Y$ on each C_i is continuous, then f is continuous.

Proof. We show that the preimage of any closed set is closed. Let H be a closed set. Then

$$f^{-1}(H) = \cup_{i=1}^{n} (f|_{C_i})^{-1}(H),$$

where the ith preimage is closed in C_i, therefore in E as well. The intersection of finitely many closed sets is also closed. $\square$

Remark 5.16.
Using Lemma 5.15 it is clear that $f \star g$ introduced in Definition 5.14 is a path in E.

Definition 5.17.
A topological space E is said to be path connected if every pair of points of E can be joined by a path in E.

Proposition 5.18.
If E is path connected, then it is also connected.

Proof. Let us fix a point x_0 in E. Any $x \in E$ can be connected with x_0 with a path f_x. The union of the images of the paths is E, and according to the proposition 5.3, it is connected. $\square$

Remark 5.19.
There connected but not path connected topological spaces. One example on such topological space, let consider

$$E := (0,0) \cup \left\{ \left(x, \sin \frac{1}{x} \right) \right\}_{x \in]0,1[}.$$

On $E \subset \mathbb{R}^2$ one considers the subspace topology. Then E is connected but not path connected. Indeed if one suppose that $(0,0)$ and $(1, \sin \frac{1}{1})$ can be connected with a continuous path, then one could consider a sequence x_n converging to 0 such that $\sin(x_n)$ converges to 1. From the continuity of the sin function this is impossible.

Exercise 5.20 (The continuous image of a path connected topological space).
The continuous image of a path connected topological space is also path connected.

Indication. Let $y_0 = f(x_0)$ and $y_1 = f(x_1)$ be two points in the image and consider the image of the path connecting x_0 and x_1... $\square$

Definition 5.21 (Path connected components)**.**
The path connected components of a topological space are the maximal path connected subsets.

Remark 5.22 (Path connected and connected partitioning)**.**
The path connected components give partitioning of a topological space. This partitioning is usually finer then the partitioning given by connected components (see the example in Remark 5.19).

Proposition 5.23.
If E is connected and locally path connected (i.e. any point has a path connected open neighbourhood), then it is path connected.

Proof. From the locally path connected property we get that the path connected component is open, and from the partitioning it is also closed (since its complementary set is the union of open sets). It follows that such a component is open and close at the same time. From the connectedness of E it follows that the pathconnected component is E. $\square$