4 Construction of topologies

4.1 Subspace

Exercise 4.1 (Induced topology on a subset of a tolpological space).
Let $(E, \mathcal{O})$ be a topological space and $M \subset E$. Show that the set $\mathcal{O}_M = \{U \cap M \mid U \in \mathcal{O}\}$ is a topology on M.

Hint.

1. $\emptyset = M \cap \emptyset, \quad M = M \cap E$;
2. $\cup_{\lambda \in \Lambda}(M \cap U_\lambda) = M \cap (\cup_{\lambda \in \Lambda} U_\lambda)$;
3. $\cap_{i=1}^{n}(M \cap U_i) = M \cap (\cap_{i=1}^{n} U_i)$;

□

Definition 4.2 (Subspace of a topological space).
Let $(E, \mathcal{O})$ be a topological space and $M \subset E$. The topological space $(M, \mathcal{O}_M)$

is called the topological subspace of $(E, \mathcal{O})$.

Exercise 4.3.
Show that

1. $C_M \subset M$ is closed in M if and only if it can be expressed as $C_M = C \cap M$, where C is closed in E.
2. if $A \subseteq M$, then the closure $\overline{A}^M$ in M is $\overline{A}^M = \overline{A} \cap M$.

Exercise 4.4 (Characterization of open and closed sets with the induced topology).
Show that

1. M is open in E if and only if any open set in M is open in E.
2. M is closed in E if and only if any closed set in M is closed in E.

Exercise 4.5 (Transitivity property).
Let $(E, \mathcal{O})$ be a topological space and $M \subset N \subset E$. Let $\mathcal{O}_M$ be the topology

induced on M by $\mathcal{O}$ and $(\mathcal{O}_N)_M$ the topology induced by $\mathcal{O}_N$ on M. Show that

$$(\mathcal{O}_N)_M = \mathcal{O}_M$$

Exercise 4.6 (Inherited topological properties).
Show that

1. a subspace of a T_0 (resp. T_1, T_2, T_3) space satisfies the T_0 (resp. T_1, T_2, T_3) property;
2. a closed subspace of a T_4 space is also a T_4 space;
3. a closed subspace of a normal space is normal;
4. a subspace of a metrizable space is also metrizable.

Exercise 4.7.
Show that the topological subspace of a separable space is not necessarily separable.

Hint. Consider a non-countable set E and chose a particular point $x_0 \in E$. Then $\mathcal{O} = \{ U \mid x_0 \notin U \} \cup \{ E \}$. Then the topology $\mathcal{O}$ is separable, because

$\overline{x_0} = E$ however, $M := E \setminus x_0$ considered as a subspace of E is not separable. (The induced topology on M is the discrete topology.) □

Notation 4.8 (The restriction of a function).
If $M \subset E$ and $f : E \to F$, then $f|_M : M \longrightarrow F$ denotes the restriction of f on M.

Proposition 4.9 (The restriction of a continuous function).
If $x \in M$ and $f : E \longrightarrow F$ is continuous at $x \in E$, then the restriction $f|_M$ is also continuous at x.

Proof. For any open neighbourhood V of $f(x)$ there exists an open neighbourhood $U \in \mathcal{O}(x)$, such that $f(U) \subset V$. It follows that $U_M := U \cap M$ is an open neighbourhood of x in M such that $f|_M(U_M) \subset V$. □

Corollary 4.10.
The restriction of a continuous function is a continuous function (with respect to the subspace topology).

Corollary 4.11.

- The subspace of a T_{3a} space is also T_{3a}.
- The subspace of a completely regular space is also regular.

4.2 Quotient space

Let $(X, \mathcal{O})$ be a topological space and $\sim$ an equivalence relation on X. The quotient set $Y = X/\sim$ is the set of equivalence classes of the elements of X. The equivalence class of x is denoted by $[x]$ and the canonical projection is denoted as $q : X \to Y$, $q(x) = [x]$. The quotient topology on the quotient space is introduced to be $q : X \to Y$ a continuous map: $U \subset Y$ is open with respect to the quotient topology if $q^{-1}(U) \in O_X$:

$$\mathcal{O}_\sim := \left\{ q^{-1}(U) \mid U \subset Y \right\}.$$

Exercise 4.12.
Show that $\mathcal{O}_\sim$ is a topology on $Y = X/\sim$.

4.3 Product space

Let $(E_i, \mathcal{O}_i)$ be a topological space for any $i \in I$ and consider $E_\pi = \Pi_{i\in I} E_i$. For any $x \in E$ we have $x = (x_i)_{i\in I}$, where $x_i \in E_i$, $i \in I$. We not by p_i the ith projection: $p_i : E_\pi \longrightarrow E_i$ where $p_i(x) = x_i$.

Theorem and definition 4.13.
On the product space $E_\pi = \Pi_{i\in I} E_i$ the family of sets

$$\mathcal{B} := \left\{ \bigcap_{j\in J} p_j^{-1}(U_j) \;\middle|\; J \text{ is finite, } U_j \in \mathcal{O}_j \right\}$$

forms a basis of a topology. This topology, denoted as $\mathcal{O}_\pi$, is the most coarser topology on the product space satisfying the property that each projection $p_i : E \longrightarrow E_i$ $i \in I$ is a continuous map. This topology is called the product topology. The elements of $\mathcal{B}$ will be called elementary open sets.

Exercise 4.14.
Let consider the case when $I = \{1, 2\}$, $E_1 = E_2 = \mathbb{R}$, $E_\pi = \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

1. Draw the pictures of the elementary open sets in $\mathbb{R}^2$.
2. Show that the product topology and the natural topology (determined by the standard metric) coincide.

Exercise 4.15.
Let consider the case when $I = \{1, \ldots, n\}$, $E_i = \mathbb{R}$, $E_\pi = \mathbb{R} \times \cdots \times \mathbb{R} = \mathbb{R}^n$. Show that the product topology and the natural topology (determined by the standard metric) of $\mathbb{R}^n$ coincide.

Proposition 4.16 (Functions with image in a product space)**.**
Let $(F, \mathcal{O}_F)$ be a topological space and consider the function

$$f : F \longrightarrow E_\pi = \prod_{i \in I} E_i$$

with value in the product space. We denote for any $i \in I$ the coordinate function $f_i = p_i \circ f$ of the function f. Then f is continuous if and only if the functions f_i are continuous $(i \in I)$.

Indication.

- If f is continuous, then $f_i = p_i \circ f$ is a composition of continuous functions, therefore it is also continuous.
- Let $V \in \mathcal{B}$ be an elementary open set in the product space. If $V = \bigcap_{j\in J} p_j^{-1}(U_j)$ then:

$$f^{-1}(V) = f^{-1}\left(\bigcap_{j\in J} p_j^{-1}(U_j)\right) = \bigcap_{j\in J} f^{-1}\left(p_j^{-1}(U_j)\right) = \bigcap_{j\in J}\left(f_j^{-1}(U_j)\right)$$

  which is an open set, since the functions f_j are continuous. Since the preimages of a basis are open sets, we get that f is continuous (see Corollary 2.4).

□

Exercise 4.17.
A topological space E is Hausdorff separable if and only if the diagonal

$$\Delta_E = \{(x, x) \mid x \in E\} \tag{3}$$

is closed in the product space $E \times E$.

Indication. The diagonal Δ_E is closed if and only if the complementary set $(E \times E) \setminus \Delta_E$ is open in $E \times E$.

$\Rightarrow$) Let us suppose that Δ_E is closed. Then for any $x \neq y$ the set $(E \times E) \setminus \Delta_E$ is an open neighbourhood of (x, y). Since on the elementary sets form a basis of the topology, there exists an elementary set $U \times V$ (U and V are open in E) such that

$$(x, y) \in U \times V \quad \text{and} \quad U \times V \subset (E \times E) \setminus \Delta_E$$

Then $U \cap V = \emptyset$, $x \in U$, $y \in V$, that is E satisfies the T_2 property that is Hausdorff separable.

$\Leftarrow$) Let us suppose that E satisfies the Hausdorff property. Then for any $x \neq y$, there exists U_x and V_y open set in E, such that $U_x \cap V_y = \emptyset$. Then $U_x \times V_y$ is an open neighbourhood of (x, y) in $E \times E$ and

$$(U_x \times V_y) \cap \Delta_E = \emptyset.$$

It follows that

$$(E \times E) \setminus \Delta_E \subset \bigcup_{x \neq y} (U_x \times V_y) \subset (E \times E) \setminus \Delta_E$$

that is

$$\bigcup_{x \neq y} (U_x \times V_y) = (E \times E) \setminus \Delta_E$$

which shows that

□

Proposition 4.18 (Closed graph theorem).
If $f : E \longrightarrow F$ is a continuous function into a Hausdorff separable topological space F, then its graph

$$G_f = \big\{ (x, f(x)) \mid x \in E \big\}$$

is a closed set in $E \times F$.

Proof. Let $g : E \times F \longrightarrow F \times F$ be the function defined as $g(x, y) = (f(x), y)$. Since the coordinate functions of g are continuous, then g is continuous (see Proposition 4.16). It follows that $G_f = g^{-1}(\Delta_F)$ is closed. □

Proposition 4.19 (Prolongation of equality with a continous function). Let h and g two continuous functions on E with values in F, where F is Hausdorff separable. If there exists a dense subset $A \subset E$ where $h(x) = g(x)$ for any $x \in A$, then $h \equiv g$, that is $h(x) = g(x)$ for any $x \in E$.

Proof. Let

$$f : E \longrightarrow F \times F \qquad f(x) = (h(x), g(x)).$$

Since the coordinate functions of f are continuous, therefore f is continuous (Proposition 4.16), and $f^{-1}(\Delta_F)$, that is the points, where g and h are equal, is closed containing A. It follows that $f^{-1}(\Delta_F)$ contains also $\overline{A}$, and we have

$$A \subset \overline{A} = E \subset f^{-1}(\Delta_F) \subset E.$$

□

Exercise 4.20 (Prolongation of an inequality with a continous function). Let h and g be two function defined on E with value in $\mathbb{R}$. If there exists dense subset $A \subset E$ where $h(x) \leq g(x)$ for any $x \in A$, then the inequality holds for any $x \in E$.

Proof. Let $B = \{x \in E \mid h(x) \leq g(x)\}$. Then $A \subset B$. Let us consider the following two functions:

$$\begin{array}{rcl} E & \longrightarrow & \mathbb{R} \times \mathbb{R} \\ x & \longrightarrow & (h(x), g(x)) \end{array} \qquad \begin{array}{rcl} \mathbb{R} \times \mathbb{R} & \longrightarrow & \mathbb{R} \\ \lambda \times \mu & \longrightarrow & \lambda - \mu \end{array}$$

The composition of the two functions is also continuous. Consider the preimage of the closed set $[0, +\infty)$. $\square$