3 Separation axioms

3.1 T_0, T_1, T_2 topological spaces

Definition 3.1 (T_0, T_1, T_2 separation axioms).
A topological space is called

1. T_0, if any two distinct points at least one of them has a neighbourhood that is not a neighbourhood of the other, or equivalently there is an open set that one point belongs to but the other point does not.
2. T_1 if any two distinct points are separated, that is if each of them has a neighbourhood that is not a neighbourhood of the other;
3. T_2 if any two distinct points are separated by neighbourhoods, that is if they have disjoint neighbourhoods.

Remark 3.2.
The T_2 topological space is also called *Hausdorff* topological space.

Remark 3.3.
The T_1 property is stronger than the T_0 property, that is a T_1 topological space is necessarily satisfying T_0. Similarly, the T_2 property is stronger than the T_1 property, that is a T_2 topological space is necessarily T_1.

Exercise 3.4 (Examples)**.**

1. Show that the indiscrete topology on a set with cardinality greater then 1 is not a T_0 topology.
2. Let E be a set with cardinality at least two and let $x_0 \in E$ be a fixed point. Than we consider: $\mathcal{O} = \{H \mid x_0 \notin H\} \cup \{E\}$. Show that $\mathcal{O}$ is a T_0 but not T_1 topology on E.
3. T_1 *but not* T_2 *topology*:
   Let E be an infinite set and $\mathcal{O}$ the co-finite topology. Show that $\mathcal{O}$ is a T_1 but not T_2 topology on E.
4. Show that any metrizable topology is T_2 .

Exercise 3.5 (Characterization of T_0 topologies).
A topology is T_0 if and only is for any distinct points their closures are different.

Indication.

- Let $(E, \mathcal{O})$ be a T_0 topological space and $x \neq y$. Then there exists an open set U which contains one but not the another. For the sake of concreteness, let say that $x \in U$ and $y \notin U$. Therefore $y \in E \setminus U$, which is a closed set, therefore $\overline{\{y\}} \subset E \setminus U$ and $x \in E \setminus \overline{\{y\}}$, therefore $\overline{\{x\}} \neq \overline{\{y\}}$.
- For the converse, let us suppose that we have the property that the closure of different points are different, that is

$$x \neq y \quad \Longrightarrow \quad \overline{\{x\}} \neq \overline{\{y\}}.$$

If $x \in \overline{\{y\}}$ and $y \in \overline{\{x\}}$ are both true, then

$$\left.\begin{aligned} x \in \overline{\{y\}} \Longrightarrow \overline{\{x\}} \subset \overline{\{y\}} \\ y \in \overline{\{x\}} \Longrightarrow \overline{\{y\}} \subset \overline{\{x\}} \end{aligned}\right\} \quad \Longrightarrow \quad \overline{\{x\}} = \overline{\{y\}}$$

It follows that either $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$. Let say we have $x \notin \overline{\{y\}}$. Then $E \setminus \overline{\{y\}}$ is an open neighbourhood of x which does not contain y.

□

Exercise 3.6 (Characterization of T_1 topologies).
A topology is T_1 if and only if every single-point set is a closed set.

Indication.

- Let $(E, \mathcal{O})$ be a T_1 topological space and $x \in E$. Because of the T_1 property, for any $y \neq x$ there exists an open neighbourhood U_y of y such that $x \notin U_y$. Then $F_y := E \setminus U_y$ is closed and contains x. Moreover

$$\cap\{\, F_y \mid y \in E \setminus \{x\}\} \;=\; \{x\},$$

and

$$\{x\} \;\subset\; \overline{\{x\}} \;\subset\; \cap\{\, F_y \mid y \neq x\} \;=\; \{x\}$$

showing that $\overline{\{x\}} = \{x\}$, that is a one point set $\{x\}$ is closed.

- For the converse, let us suppose that every single-point set is a closed set. If $x \neq y$, then $E \setminus \{x\}$ is an open neighbourhood of y and $E \setminus \{y\}$ is an open neighbourhood of x.

□

3.2 T_3 and regular topological spaces

Definition 3.7 (T_3 topology).
A topology is T_3 if, given any closed set F and any point x that does not belong to F, there exists an open neighbourhood U of x and an open neighbourhood V of F that are disjoint. Concisely put, it must be possible to separate x and F with disjoint open neighborhoods.

Proposition 3.8 (Characterization of T_3 topology).
The topological space $(E, \mathcal{O})$ is T_3 if and only if for any $x \in E$ and $x \in V \in \mathcal{O}$ there exists $U \in \mathcal{O}$, such that $x \in U \subset \overline{U} \subset V$.

Proof.

- Let E a T_3 topological space, $x \in E$ and $x \in V \in \mathcal{O}$. From T_3 there exist U_1 and U_2 disjoint open sets such that $x \in U_1$ and $E - V \subset U_2$. It follows

$U_1 \subset E - U_2 \subset V$. The set $E \setminus U_2$ is closed, we have

$$x \in U_1 \subset \overline{U}_1 \subset E - U_2 \subset V,$$

since from $F \subset U_2$ we have $E \setminus U_2 \subset E \setminus F = V$.

- Conversely, let us suppose now that the property of the proposition is satisfied. If $x \in E$ and F is a closed set such that $x \notin F$, then $x \in E - F \in \mathcal{O}$ and by the hypotheses there exists an open set U, such that $x \in U \subset \overline{U} \subset E - F$. Then U and $E \setminus \overline{U}$ are disjoint open sets, and $x \in U$, $F \subset E - \overline{U}$.

□

Remark 3.9 ($T_3 \not\Rightarrow T_0$).
From the property T_3 does not follow the property T_0 (and therefore neither the property T_1, nor the property T_2): see Example 3.10.

Example 3.10.
Let us consider the set $E = \{a, b, c\}$ and $\mathcal{O} = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. Then $\mathcal{O}$ is a topology on E, and we have the particular property that any open set is also closed. One can verify that the T_3 property is trivially satisfied. However, the

topology $\mathcal{O}$ does not satisfy the T_0 property, since b and c cannot be separated as required for the T_0 property.

Proposition 3.11.
If a topology satisfies both the T_0 and T_3 properties, then it also satisfies the T_2 property, that is

$$T_0 \text{ and } T_3 \quad \Longrightarrow \quad T_2.$$

Proof. Let $(E, \mathcal{O})$ be a topology satisfying both the T_0 and T_3 properties. From the T_0 if x and y are different points, then at least one of them has a neighbourhood that is not a neighbourhood of the other. For the seek of the argument let x be a point having a neighbourhood V not containing y. From the T_3 property there exists $U \in \mathcal{O}_x$, such that $\overline{U} \subset V$ is satisfied. Then $U_x = U$ and $U_y := V \setminus \overline{U}$, are disjoint open neighbourhood of $x \in E$ and $y \in E$. □

Remark 3.12 (T_1 and T_3) $\Rightarrow$ T_2).
As a corollary of Proposition 3.11 we get T_1 and T_3 together ensure the property T_2.

Remark 3.13 ($T_2 \not\Rightarrow T_3$).
The property T_2 does not imply the property T_3. In order to show this, we consider the following example: on $\mathbb{R}$ we consider the set

$$K = \{1, \tfrac{1}{2}, \tfrac{1}{3}, \dots\}$$

and

$$\mathcal{A} \doteq \{U \subset \mathbb{R} \mid U \in \mathcal{O}_0 \text{ or } U = U' \setminus K \text{ for some } U' \in \mathcal{O}_0\}$$

where $\mathcal{O}_0$ is the natural topology of $\mathbb{R}$ and we consider the coarser topology $\mathcal{O}$ such taht $\mathcal{A} \subset \mathcal{O}$, i.e. the coarser topology in which the elements of $\mathcal{A}$ are open sets. Then, by the construction, this topology contains the elements of the natural topology $\mathcal{O}_0$ of $\mathbb{R}$, therefore it is satisfying the T_2 property. Moreover, Z is closed, since $\mathcal{A}$ (and therefore also $\mathcal{O}$) contains $\mathbb{R} \setminus Z$, that is the complementary set of Z. One can remark that one can consider as a local basis for $\mathcal{O}$, one can choose

$$\mathcal{B}(x) = \begin{cases} \{U \mid \text{ where } U \in \mathcal{O}_T(x)\} & \text{if } x \neq 0; \\ \{U_K \mid U_K = U \setminus K \text{ where } U \in \mathcal{O}_T(0)\} & \text{if } x = 0. \end{cases}$$

One can show that it is impossible to separate the set Z and the point 0 with disjoint open sets of $\mathcal{O}$, therefore $\mathcal{O}$ does not satisfy the property T_3.

Definition 3.14 (Regular topology)**.**
A topology called regular if it satisfies T_1 and T_3 properties.

Remark 3.15.
Using 3.11, any regular topological space satisfies T_2. One can show however, that there are non-regular topological spaces satisfying the T_2 property.

Property 3.16.
Metrizable topology is is regular.

Proof. Indeed, let $\mathcal{O}$ be a metrizable topology on E. If $x \in U$ where $U \in \mathcal{O}$, then there exists $\varepsilon > 0$ such that $x \in B(x, \varepsilon)$, and it follows that

$$x \in B(x, \varepsilon/2) \subset \overline{B(x, \varepsilon/2)} \subset B(x, \varepsilon) \subset U.$$

According to Proposition 3.8, the topology satisfies the T_3 property. □

3.3 T_{3a} and completely regular topological spaces

Definition 3.17 (T_{3a} property).
A topological space $(E, \mathcal{O})$ is called regular or T_{3a} if for every closed subset C of E and every point x in E such that $x \notin C$ there exists a continuous function $f : E \to [0, 1]$ such that $f(x) = 0$ and $f(C) = 1$.

Exercise 3.18.
Show that every T_{3a} topology is also T_3.

Hint: For $x \in E$ and closed C ($x \notin C$) consider the sets $U_1 = f^{-1}([0, \frac{1}{2}))$ and $U_2 = f^{-1}((\frac{1}{2}, 1])$ . . . □

Definition 3.19 (Completely regular topology).
A topology is called completely regular if it satisfies the T_1 and T_{3a} properties.

3.4 T_4 and normal topological spaces

Definition 3.20 (T_4 topology).
A topological space $(E, \mathcal{O})$ is called T_4 space if for any every two disjoint closed sets A and B have disjoint open neighborhoods, i.e. there exist $U, V \in \mathcal{O}$ disjoint open sets, such that $A \subset U$ and $B \subset V$.

Remark 3.21 ($T_2 \not\Rightarrow T_4$).
From the property T_4 does not follow the property T_0 (and therefore neither the property T_1, nor the property T_2): the topology presented in Example 3.10 satisfies T_4 but not the T_0 property.

Definition 3.22 (Normal topology).
A topological space $(E, \mathcal{O})$ called normal if its topology satisfies the T_1 and T_4 properties.

Example 3.23.

Hausdorff separable compact topological space are normal spaces (see Chapter 6: Theorem 6.6, Proposition 6.7).

Exercise 3.24.
Metrizable topological spaces are normal spaces.

Hint. Show that if A are B closed disjoint sets, then the function

$$f(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}$$

is continuous and separates the A and B sets. □

Proposition 3.25.
Show that

1. normal spaces are completely regular, that is $(T_1 + T_4) \Rightarrow (T_1 + T_{3a})$.
2. normal spaces are Hausdorff separable, that is $(T_1 + T_4) \Rightarrow T_2$.

Theorem 3.26 (Urysohn's lemma)**.**
If a topological space $(E, \mathcal{O})$ satisfies the T_4 property, then any two disjoint

closed sets A and B can be separated by a continuous function, that is there exists a continuous function $f : E \longrightarrow [0, 1]$ such that $f(a) = 0$ for $a \in A$ and $f(b) = 1$ if $b \in B$.

Remark 3.27.
Urysohn's Lemma shows the rather surprising fact that being able to separate closed sets from one another with a continuous function is not stronger than being able to separate them with open sets.

Exercise 3.28.
Show that T_0, T_1, T_2, T_3, T_4 are topological properties, that is invariant with respect to a homeomorphism.