2 Continuous functions

Definition 2.1 (Continuous function).

- Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be topological spaces. The function $f : E \longrightarrow F$ is continuous at $a \in E$ if for any open neighbourhood V of $f(a)$ there exists an open neighbourhood U of $a \in E$ such that

$$f(U) \subset V.$$

- The function $f : E \longrightarrow F$ is continuous if it is continuous at any point of E.

Proposition 2.2.
Let $f : E \longrightarrow F$ be continuous at $a \in E$. Then:

$$a \in \overline{A} \quad \Rightarrow \quad f(a) \in \overline{f(A)}.$$

Proof. Let V be an open neighbourhood of $f(a)$. Because of the continuity, there exists a neighbourhood U of a such that $f(U) \subset V$. On the other hand, since $a \in \overline{A}$ there exists $y \in A \cap U$, that is $f(y) \in V$ or

$$V \cap f(A) \neq \emptyset.$$

□

Proposition 2.3 (Equivalent formulation of the continuity property).
Let $f : E \longrightarrow F$ be a function. Then the following statement are equivalent:

1. f is continuous on E;
2. for any open set V of F, the preimage set $f^{-1}(V)$ is open in E;
3. for any closed set B of F, the preimage set $f^{-1}(B)$ is closed in E;
4. for any set A of E one has $f\left(\overline{A}\right) \subset \overline{f(A)}$

Proof.

$1 \Rightarrow 4$: One obtain immediately by using Proposition 2.2.

4. $\Rightarrow$ 3. Let B be a closed set of F and $B_E = f^{-1}(B)$. Then from 4. we get:

$$f(\overline{B_E}) \subset \overline{f(B_E)} \subset \overline{B} = B$$

since B is closed. Then $\overline{B_E} \subset B_E$. On the other hand, $\overline{B_E}$ is the smallest closed set containing B_E, that is $\overline{B_E} \supset B_E$ therefore we get that $B_E = \overline{B_E}$ and B_E is a closed set in E.

3. $\Rightarrow$ 2. For any open set $V \subset F$ we have $F \setminus V$ a close set in F. Then from 3. we get $E \setminus f^{-1}(V) = f^{-1}(F \setminus V)$ is a closed set in E. It follows that $f^{-1}(V)$ is open in E.

2. $\Rightarrow$ 1. At an open neighbourhood V of $f(x)$ one can chose $U = f^{-1}(V)$ which is an open neighbourhood of x, and clearly $f(f^{-1}(V)) \subset V$.

□

Corollary 2.4 (The continuity porperty with a basis).
A function $f : E \to F$ is continuous if and only if there exists a basis $\mathcal{B}_F$ of the topology of F such that the preimages of its elements are open sets in E.

Proof. Let $(E, \mathcal{O}_E)$, and $(F, \mathcal{O}_F)$ be topological spaces and $\mathcal{B}_F$ a basis of $\mathcal{O}_F$.

$\Rightarrow$) If $f : E \longrightarrow F$ is continuous, then from 2.) of Proposition 2.3 we get that the preimage of any open set is also open. In particular the elements of $\mathcal{B}_F$ are open set, therefore their preimages are open set in E.

$\Leftarrow$) Let us now suppose that there $\mathcal{B}_F$ is a basis such that the preimages of their elements are open and let us consider an arbitrary open set $V \in \mathcal{O}_F$ in F. Since $\mathcal{B}_F$ is a basis, there exists V_λ $(\lambda \in \Lambda)$ in $\mathcal{B}_F$ such that $V = \cup_{\lambda \in \Lambda} B_\lambda$. Since

$$f^{-1}(V) \; = \; f^{-1}\big(\cup_{\lambda\in\Lambda} B_\lambda\big) \; = \; \cup_{\lambda\in\Lambda} f^{-1}(B_\lambda)$$

we get that $f^{-1}(V)$ is open.

$\square$

Exercise 2.5 (Continuiuous map between metric spaces).
Let (E_1, d_1) and (E_2, d_2) be metric spaces. Show that the function $f : E_1 \longrightarrow E_2$ is continuous (with respect to the induced topologies) at $x \in E_1$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$, such that if $y \in E_1$ and $d(x, y) < \delta$ then

$d(f(x), f(y)) < \varepsilon$.

Exercise 2.6 (The continuity property using open balls).
$f : E_1 \longrightarrow E_2$ is continuous at $x \in E_1$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $B(x, \delta)$

$$f(B(x, \delta)) \subset B(f(x), \varepsilon).$$

Exercise 2.7 (Continuity with convergent sequences).
Let $(E, \mathcal{O}_E)$ be a metrizable topological space and $(F, \mathcal{O}_F)$ a Hausdorff separable (T_2) topological space. Then the function $f : E \longrightarrow F$ is continuous at $a \in E$ if and only if for any x_n sequence in E converging to a the sequence $f(x_n)$ converges to $f(a)$.

Exercise 2.8 (Composition of continuous functions).
Let $(E, \mathcal{O}_E)$, and $(F, \mathcal{O}_F)$ be $(G, \mathcal{O}_G)$ topological spaces and let $f : E \longrightarrow F$ and $g : F \longrightarrow G$ be functions. If f is continuous at $a \in E$ and g is continuous at $f(a) \in F$, then $g \circ f : E \longrightarrow G$ is continuous at $a \in E$.

Definition 2.9 (Homeomorphism).

1. Let $(E, \mathcal{O}_E)$ and $(F, \mathcal{O}_F)$ be topological spaces. The function $f : E \longrightarrow F$ is called *homeomorphism* if it is continuous, invertible, and its inverse is also continuous.
2. Two topological spaces are homeomorphic if there exists a homeomorphism between them.

A property P is called *topological* if it is invariant with respect to homeomorphism, that is a topological space satisfies the P if and only if any other topological space homeomorphic with it satisfy the property.