1 Topological spaces

1.1 Topology, open sets

Definition 1.1 (Topology).
Let E be a nonempty set. We call a family $\mathcal{O}$ of subsets of E a topology on E, if the following properties are satisfied:

T.1 $\emptyset \in \mathcal{O}$ and $E \in \mathcal{O}$;

T.2 if $U_i \in \mathcal{O}$ for any $i \in I$, then $\cup_{i \in I} U_i \in \mathcal{O}$;

(the union of arbitrarily many elements of $\mathcal{O}$ is also an element of $\mathcal{O}$),

T.3 if $U_1, \dots U_n \in \mathcal{O}$, then $U_1 \cap \dots \cap U_n \in \mathcal{O}$.
(the intersection of finitely many elements of $\mathcal{O}$ is also an element of $\mathcal{O}$),

Terminology 1.2 (Topological space, open sets, open neighbourhoods).

1. The pair $(E, \mathcal{O})$ is called a topological space.
2. The elements of $\mathcal{O}$ are the open sets with respect to the topology $\mathcal{O}$,
3. The elements of $\mathcal{O}$ containing the point $x \in E$ are the open neighbourhoods of x. The set of open neighbourhoods of x is denoted by $\mathcal{O}(x)$. A set containing an open neighbourhood of x is said a neighbourhood of x.

Exercise 1.3 (Indiscrete topology).
Show that $\mathcal{O}_{\text{in}} = \{\emptyset, E\}$ is a topology on E. This topology is called the *indiscrete topology* of E.

Exercise 1.4 (Discrete topology).
Show that $\mathcal{O}_{\text{di}} = 2^E$, the set of all subsets of E is a topology on E. This topology is called the *discrete topology* of E.

Exercise 1.5 (Co-finite topology).
Show that $\mathcal{O}_{\text{co}} = \{U \subset E \mid E \setminus U \text{ is finite}\} \cup \{\emptyset\}$ is a topology on E. This topology is called the co-finite topology of E.

Definition 1.6 (Comparaison of topologies).
Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be two topologies on E. We call the topology $\mathcal{O}_2$ finer then the topology $\mathcal{O}_1$, or $\mathcal{O}_1$ coarser then the topology $\mathcal{O}_2$ if

$$\mathcal{O}_1 \subset \mathcal{O}_2.$$

Remark 1.7.
Topologies are not comparable in general.

1.2 Closed sets in a topological space

Definition 1.8 (Closed sets).
Let $(E, \mathcal{O})$ be a topological space. A set $A \subset E$ is called closed set (with respect to the topology $\mathcal{O}$), if its complementary set $E \setminus A$ is open, that is $E \setminus A \in \mathcal{O}$.

Exercise 1.9 (Properties of closed sets).

Z1. $\emptyset$ and E are closed;

Z2. The intersection of arbitrarily many closed set is closed,

Z3. The union of finitely many closed set is closed.

Theorem and definition 1.10 (The closure of a set).
Let $(E, \mathcal{O})$ be a topological space. If q $H \subset E$, then from (Z.2) we get that the intersection of all closed set containing H is a closed set. We call this set the *closure* of H with respect to the topology $\mathcal{O}$ and it is denoted by $\overline{H}^{\mathcal{O}}$ or

simply by $\overline{O}$:

$$\overline{H} = \cap \{ Z \mid H \subseteq Z \text{ and } Z \text{ is closed}\}$$

Remark 1.11.

Two trivial but useful remarks:

1. $\overline{H}$ is the smallest closed set containing H,
2. H is closed if and only if $H = \overline{H}$.

Exercise 1.12.

1. Let $\mathcal{O}_{di} = 2^E$ be the discrete topology on E and $H \subset E$. Determine $\overline{H}$.
2. Let $\mathcal{O}_{in}$ be the indiscrete topology on E and $H \subset E$. What is $\overline{H}$?

Exercise 1.13.

Let us consider the set $E = \{a, b, c\}$ and $\mathcal{O} := \big\{\emptyset, \{a\}, \{b, c\}, E\big\}$.

1. Show that $\mathcal{O}$ is a topology on E.
2. Find the closure of the set $\{ a \}$, $\{ b \}$, $\{ c \}$.

Proposition 1.14 (The characterization of the points in $\overline{H}$).

$$x \in \bar{H} \iff \forall \quad U \in \mathcal{O}(x): \; U \cap H \neq \emptyset.$$

or equivalently,

$$x \notin \bar{H} \iff \exists \quad U \in \mathcal{O}(x): \; U \cap H = \emptyset.$$

Proof. By definition, if x is not in $\bar{H}$ (which is the intersection of closed sets containing H), there exists at least one closed set F such that $H \subset F$ and $x \notin F$. Consequently, $x \in U := E \setminus F$ which is an open set and $H \cap U = \emptyset$.

□

Proposition 1.15.
Let $(E, \mathcal{O})$ be a topological space and H_1, H_2 two arbitrary subset in E. Then we have the following properties

1. $H_1 \subset H_2 \implies \overline{H_1} \subset \overline{H_2}$;
2. $\overline{H_1 \cap H_2} \subset \overline{H_1} \cap \overline{H_2}$;
3. $\overline{H_1 \cup H_2} = \overline{H_1} \cup \overline{H_2}$.

Proof. The proof is based on the Proposition 1.14.

1. If $x \in \overline{H_1}$, then for every $U \in \mathcal{O}(x)$ we have $U \cup H_1 \neq \emptyset$, that is, there exists $y \in U \cap H_1$. From the condition 1.) we have also $y \in H_2$, therefore $y \in U \cap H_2$, and $U \cap H_2 \neq \emptyset$. It follows that $x \in \overline{H_2}$. Since x was an arbitrary element of $\overline{H_1}$, we get that
$$\overline{H_1} \subset \overline{H_2}.$$
2. If $x \in \overline{H_1 \cap H_2}$, then for any $U \in \mathcal{O}(x)$ there exists $y \in U \cap (H_1 \cap H_2)$, that is $U \cap H_1 \neq \emptyset$ and $U \cap H_2 \neq \emptyset$. Consequently, $x \in \overline{H_1}$ and $x \in \overline{H_2}$, therefore
$$x \in \overline{H_1} \cap \overline{H_2}.$$
3. Firstly, $H_1 \subset \overline{H_1}$ and $H_2 \subset \overline{H_2}$, therefore $H_1 \cup H_2 \subset \overline{H_1} \cup \overline{H_2}$ and from 1. we get that
$$\overline{H_1 \cup H_2} \subset \overline{\overline{H_1} \cup \overline{H_2}} = \overline{H_1} \cup \overline{H_2}, \tag{1}$$
since the intersection of two closed set is also a closed set.

Secondly, $H_1 \subset H_1 \cup H_2$ and $H_2 \subset H_1 \cup H_2$ and from 1. again we obtain

$$\overline{H_1} \subset \overline{H_1 \cup H_2} \quad \text{and} \quad \overline{H_2} \subset \overline{H_1 \cup H_2}$$

and

$$\overline{H_1} \cup \overline{H_2} \subseteq \overline{H_1 \cup H_2} \subseteq \overline{H_1} \cup \overline{H_2} \tag{2}$$

From (1) and (2) we get 3).

□

Exercise 1.16.

1. Give an example, where $\overline{H_1} \cap \overline{H_2} = \overline{H_1 \cap H_2}$,
2. Give an example, where $\overline{H_1} \cap \overline{H_2} \subsetneq \overline{H_1 \cap H_2}$.

Definition 1.17 (Isolated point of a set).
Let $(E, \mathcal{O})$ be a topological space and $H \subset E$. $x \in H$ is an isolated point of H if there exists $U \in \mathcal{O}(x)$, such that $U \cap H = \{x\}$

Definition 1.18 (Accumulation point of a set)**.**
Let $(E, \mathcal{O})$ be a topological space and $H \subset E$, $x \in E$ is an accumulation point of H if for any $U \in \mathcal{O}(x)$, one has $(U \setminus \{x\}) \cap H \neq \emptyset$.

Proposition 1.19 (Characterization of closed sets with accumulation points)**.**
A set H is closed if and only if it contains all its accumulation points.

Proof. Let $(E, \mathcal{O})$ be a topological space and $H \subset E$.

- Let H be a close set and $a \in E$ an accumulation point of H. Then for any $U \in \mathcal{O}(a)$ we have $U \cap H \neq \emptyset$, therefore $a \in \overline{H}$. Since H is closed, we have $H = \overline{H}$, so $a \in H$.
- Let us suppose that H contains its accumulation points. We will shot that the complementary set of H is open. Indeed, if $x \notin H$, then it is not an accumulation point of H, therefore there exists a neighbourhood $U_x \in \mathcal{O}(x)$ of x such that $U_x \cap H = \emptyset$, i.e. $U_x \subset E \setminus H$. This is true for all $x \notin H$, therefore we get
$$E \setminus H = \cup_{x \notin H} U_x.$$

which show that the complementary set of H is open (see T.2 property of Definition 1.1 and therefore H is closed.

□

Exercise 1.20.
If $a \in \overline{H}$, that is a is an element of the closure of H, then there are two possibilities:

- $a \in H$,
- $a \notin H$ but a is an accumulation point of H:

It follows that

$$\overline{H} = H \cup \{\text{ accumulation points of } H\}$$

Definition 1.21 (Dense subset).
Let $(E, \mathcal{O})$ be a topological space. The set $H \subset E$ is *dense* with respect to the topology $\mathcal{O}$ if

$$\overline{H} = E.$$

Property 1.22.

The set H is dense $(E, \mathcal{O})$ be a topological space if and only if for any nonempty set $U \in \mathcal{O}$ one has $U \cap H \neq \emptyset$.

Exercise 1.23.

1. Find the dense subsets with respect to the indiscrete topology.
2. Find the dense subsets with respect to the discrete topology.
3. Show that the set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect to the natural topology of $\mathbb{R}$.

Exercise 1.24.
Show that

1. Any nonempty set in the indiscrete topology is dense.
2. $\mathbb{Q}$ is dense in $\mathbb{R}$ with respect the natural topology, but it is not dense with respect to the discrete topology.

1.3 Induced topology on a metric space

Definition 1.25 (Metric and metric space).
The pair (E, d) is called a metric space if E is a nonempty set and is a metric, that is a function $d : E \times E \to \mathbb{R}$ satisfying the following properties: for any $x, y, z \in E$

1. $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$;
3. $d(x, y) \leq d(x, z) + d(z, y)$.

Example 1.26 (Canonical metric on $\mathbb{R}$).
On the set of real numbers $\mathbb{R}$ the usual metric is defined $d(x, y) = |x - y|$;

Example 1.27 (Metrics on $\mathbb{R}^n$).
On $\mathbb{R}^n$ let as consider the following :

1. $d(x, y) = \sqrt{\sum_{i=1}^{n}(x^i - y^i)^2}$

2. $d_1(x,y) = \sum_{i=1}^{n} |x^i - y^i|$;
3. $d_2(x,y) = max_i|x^i - y^i|$

Example 1.28 (Discrete metric).
Let E be a nonempty set and let us consider

$$d(x,y) = \begin{cases} 0, & \text{ha} \quad x = y \\ 1, & \text{ha} \quad x \neq y \end{cases}$$

Then d is a metric. It is called the *discrete metric* on E

Example 1.29 (Uniform metric).
Let X be a nonempty set and (E,d) a metric space. We denote by $\mathcal{B}(X,E)$ the set of bounded function defined on X with value in E (The function $f : X \longrightarrow E$ is bounded if

$$\delta(f) := \sup\{d(f(x), f(y)) : x, y \in X\}$$

exists.) Let us fix $a \in E$. Then for every $x \in E$ and $f, g \in \mathcal{B}(X,E)$ one has

$$d(f(x), g(x)) \leq d(f(x), f(a)) + d(f(a), g(a)) + d(g(a), g(x)) \leq \delta(f) + C + \delta(g),$$

therefore $d(f, g) := \sup d(f(x), g(x))$ exists. One can show that d is a metric on $\mathcal{B}(X, E)$.

Definition 1.30 (Metric induced on a subset).
Let (E, d) be a metric space and $A \subset E$ be a nonempty subset. Then on A one can consider the metric d_A as the restriction of the metric d on $A \times A$. The metric space (A, d_A) is called the subspace of (E, d).

Notation 1.31 (Open ball).
Let $x \in E$ and $r < 0$. $B(x, r)$ denotes the open ball centered at x with radius r:

$$B(x, r) := \{y \in E \mid d(x, y) < r\};$$

Definition 1.32 (Open sets of a metric space).
Let (E, d) be a metric space. The set $U \subset E$ is called open set with respect to the metric d if it is the empty set of if for any $x \in U$ there exist $r > 0$, such that $B(x, r) \subset U$. The set of open sets with respect to the metric d will be denoted as $\mathcal{O}_d$.

Exercise 1.33.
Show that in a metric space (E, d) open balls are open sets with respect to the metric, that is for any $x \in E$ and $r > 0$ one has $B(x, r) \in \mathcal{O}_d$.

Theorem and definition 1.34 (Topology induced by a metric).
In a metric space (E, d), the sets $\mathcal{O}_d$ of open sets with respect to the metric d satisfies the condition (T.1) – (T.3) of Definition 1.1, therefore $\mathcal{O}_d$ is a topology on E. $\mathcal{O}_d$ is called the topology induced by the metric d.

Definition 1.35 (Metrizable topological space).
A topology $\mathcal{O}$ on E is called metrizable if there exists a metric d on E such that $\mathcal{O}$ coincides with the topology $\mathcal{O}_d$ induced by the metric d, that is $\mathcal{O} = \mathcal{O}_d$.

Definition 1.36 (Equivalent metrics).
Two metrics d_1 and d_2 on the set E are called equivalent if the induced metrics coincide, that is $\mathcal{O}_{d_1} = \mathcal{O}_{d_2}$.

Exercise 1.37.
Show that on $\mathbb{R}$ the canonical and the discrete metrics (see Example 1.26 and 1.28) are not equivalent.

Exercise 1.38 (Natural topology of $\mathbb{R}^n$).
Show that on $\mathbb{R}^n$ the metrics introduced in Example 1.27 are equivalent. This topology is denoted by $\mathcal{O}_T$ and called the natural topology of $\mathbb{R}^n$.

Exercise 1.39.
Show that any finite set in a metric space is closed.

Exercise 1.40.

1. Show that the closed ball $B(x, r] := \{ y \in E \mid d(x, y) \leq r \}$ is a closed set.
2. Show that the closure of the open ball $B(x, r]$ and the closed ball $B(x, r]$ are not necessarily equal.
3. Show that in a normed vector space the open ball $B(x, r]$ and the closed ball $B(x, r]$ are equal.

1.4 Basis and local basis of a topology

Definition 1.41 (Basis of a topology)**.**
Let $(E, \mathcal{O})$ be a topological space. The subset $\mathcal{B}$ of $\mathcal{O}$ is a basis of the topology $\mathcal{O}$ if any open sets can be written as a union of elements of $\mathcal{B}$.

Exercise 1.42.
Show that in a metric space the open balls form a basis.

Exercise 1.43.
Let us consider in $\mathbb{R}$ the natural topology $\mathcal{O}$ and let $I_m(q) :=]q - \frac{1}{m}; \; q + \frac{1}{m}[$ be the open interval, where $m \in \mathbb{N}$ and $q \in \mathbb{Q}$. Show that the set

$$\mathcal{B} := \{ I_m(q) \mid m \in N, q \in \mathbb{Q} \}$$

is a countable basis of $\mathcal{O}$.

Exercise 1.44.
Show that the collection of a subset $\mathcal{B}$ in a a topological space $(E, \mathcal{O})$ is a basis of the topology $\mathcal{O}$ if and only if the following two properties are satisfied:

1. $\mathcal{B} \subset \mathcal{O}$,
2. for any $x \in E$ and $U \in \mathcal{O}_x$ there exists $B \in \mathcal{B}$ such that $x \in V \subset U$.

Indication.

$\Rightarrow$ When $\mathcal{B}$ is a basis, then 1. is satisfied by the definition. Moreover if $U \in \mathcal{O}_x$ then U can be write as a union $\cup_{\lambda \in \Lambda} B_\lambda$ where B_λ are elements of $\mathcal{B}$. Pick for V one, containing x.

$\Leftarrow$ If $\mathcal{B}$ satisfies the conditions 1. and 2. then let us consider an arbitrary $U \in \mathcal{O}$. Then

$$U \quad \subset \quad \cup\{B \mid B \in \mathcal{B} \text{ and } B \subset U\} \quad \subset \quad U.$$

that is

$$U = \cup\{B \mid B \in \mathcal{B} \text{ and } B \subset U\}$$

□

Proposition 1.45 (Existence of a topology having given $\mathcal{B}$ as a basis)**.**
A family $\mathcal{B}$ is a basis for some topology if and only if the following two conditions are satisfied:

1. If $U, V \in \mathcal{B}$ and $x \in U \cap V$, then there exists $W \in \mathcal{B}$ such that $x \in W \subset U \cap V$.
2. $\mathcal{B}$ is covering E, that is $\cup\{B \mid \ \in \mathcal{B}\} = E$

Proof.
- The condition is necessary. Indeed if U V are element of $\mathcal{B}$ which is a basis of a topology, the both are open sets, therefore $U \cap V$ is an open neighbourhood of x. Since $\mathcal{B}$ is a basis, then from 2.) if Exercise 1.44 there exists W such that $x \in W \subset U \cap V$ which show 1.) Moreover, E is an open set, therefore is can be written as a union of elements of $\mathcal{B}$. We have therefore 2.) as well.
- The conditions are also sufficient. Let $\mathcal{O}$ be the family of subsets of E

which can be written as a union of of elements of $\mathcal{B}$:

$$H \in \mathcal{O} \Longleftrightarrow H = \cup B_\lambda.$$

Then $\mathcal{O}$ is a topology. Indeed, the first two properties of the topology (see Definition 1.1) are trivial:

T.1 poperty: $\emptyset$ and $E \in \mathcal{O}$,

T.2 poperty: if $H_\gamma \in \mathcal{O}$ $(\gamma \in \Gamma) \Longleftrightarrow \cup_{\gamma\in\Gamma} \quad \{H_\gamma \mid \gamma \in \Gamma\} \in \mathcal{O}$

T.3 poperty: this condition (the finite intersection of elements of $\mathcal{O}$ are also elements of $\mathcal{O}$) is a bit more interesting to show. Let us consider $U, V \in \mathcal{O}$. If the intersection is not empty, then both can be written as a union of elements in $\mathcal{B}$. Consequently, there exist $B_U \in \mathcal{B}$ and $B_V \in \mathcal{B}$ such that $x \in B_U \subset U$ and $x \in B_V \subset V$. From the second condition of the proposition, there exists also $W_x \in \mathcal{B}$, such that $x \in W_x \subset B_U \cap B_V$, and therefore $x \in W_x \subset U \cap V$. Considering this for all $x \in U \cap V$ we have

$$U \cap V \quad \subset \quad \cup_{x\in U\cap W} B_x \quad \subset \quad U \cap V.$$

that is $U \cap V = \cup_{x\in U\cap W} B_x$ and $\cap V \in \mathcal{O}$.

□

Exercise 1.46.
Show that if $\mathcal{B}$ is a family of subset of E verifying the property that it is covering of E and it contains all finite intersections of their elements, then there exists a topology, such that $\mathcal{B}$ a basis for it.

Exercise* 1.47 (The construction of a topology).
Let $\mathcal{A}$ be a family of subset of E. Find the most coarser topology such that the elements of $\mathcal{A}$ are open sets.

Indication. Let suppose that $E \in \mathcal{A}$ (if not, then add it) and consider the family of subset of E defined as

$$\mathcal{B}_{\mathcal{A}} = \{B \subset E | \; \exists \; A_1 \ldots A_k \in \mathcal{A} \; : \; A_1 \cap \cdots \cap A_k = B\}$$

Show that there is a topology $\mathcal{O}_{\mathcal{A}}$ such that $\mathcal{B}_{\mathcal{A}}$ is a basis. This topology is the most coarser topology such that the elements of $\mathcal{A}$ are open sets. □

Exercise 1.48.
What is the most coarser topology on $\mathbb{R}$ such that

1. the set $[0,1]$, $[2,3]$, and $[4,5]$ are open sets?
2. contains the closed intervals $\mathcal{A} = \{\, [a,b] \mid a,b \in \mathbb{R}\,\}$?
3. contains the open intervals $\mathcal{A} = \{\,]a,b[\mid a,b \in \mathbb{R}\,\}$?

Indication.

1. use the result of the previous exercise.
2. use the result of the previous exercise: since the intersections contain the one-point sets, the topology will be the discrete topology.
3. Show that the answer is the natural topology of $\mathbb{R}$.

□

Definition 1.49 (Local basis of a topology).
Let $\mathcal{O}$ be a topology on E and let $x \in E$. $\mathcal{B}(x)$ is called a local basis of the topology $\mathcal{O}$ at x if $\mathcal{B}(x) \subset \mathcal{O}(x)$ and for every open neighbourhood U of x there exists $B \in \mathcal{B}(x)$, such that $V \subset U$.

Example 1.50.
For a topology $\mathcal{O}_d$ generated by a metric d, the x-centered balls

$$\mathcal{B}(x) := \{ B(x, r) \mid r > 0 \}$$

form a local basis at x.

Exercise 1.51 (Basis $\Rightarrow$ local basis)**.**
Let $(E, \mathcal{O})$ be a topological space. Show that if $\mathcal{B}$ is a basis of $\mathcal{O}$ and $\mathcal{B}(x)$ are the element of $\mathcal{B}$ containing a fixed $x \in E$, then $\mathcal{B}(x)$ is a local basis of $\mathcal{O}$ at x.

Exercise 1.52 (Local basis at any point $\Rightarrow$ basis)**.**
Let $(E, \mathcal{O})$ be a topological space. Show that if for any $x \in E$ the $\mathcal{B}(x)$ is a local basis of $\mathcal{O}$ at x, then the collection $\mathcal{B} = \cup_{x \in E} \mathcal{B}(x)$ is a basis of $\mathcal{O}$.

1.5 Countability axioms and their consequences

Definition 1.53 (Countability axioms).
One say that a topological space satisfies the

- first countability axiom (shortly: a first-countable space) if any points has a countable local basis,
- second countability axiom (shortly: a second-countable space) if it has a countable basis,

Remark 1.54.
The second countability axiom implies the first countability axiom, that is if a topology satisfies the second countability axiom, then it also satisfies the first countability axiom.

Exercise 1.55 (Non first-countable topology).
Show that on a non-countable set E the co-finite topology does not satisfy the

first countability axiom.

Exercise 1.56 (First-countable topology).
Show that metric spaces are first-countable spaces.

Exercise 1.57 (First-countable but not second-countable topology).
Show that on a non-countable set, the discrete topology satisfies the first countability axiom, but does not satisfies the second countability axiom.

Exercise 1.58 (Second-countable topology).
Show that on $\mathbb{R}$ the natural topology satisfies the second countability axiom.

Theorem 1.59 (Lindelöf).
Any open covering of a second-countable topological space contains a countable sub-covering.

Proof. Let $(E, \mathcal{O})$ be a second-countable topological space and $\mathcal{L} = \{L_\lambda \mid \lambda \in \Lambda\}$ an open covering of E. There exists a countable $\mathcal{B}_0 = \{U_1, U_2, ...\}$ basis for the topology $\mathcal{O}$. We denote

$$\mathcal{B}_{\mathcal{L}} = \{U \mid U \in \mathcal{B} \text{ and } \exists\ L \in \mathcal{L} : U \subset L\}$$

Then $\mathcal{B}_{\mathcal{L}}$ is countable. Let us consider for each element $B \in \mathcal{B}_{\mathcal{L}}$ of let us choose one element of $\mathcal{L}$ and the family of chosen elements is

$$\mathcal{L}_0 = \{L_{\lambda(U)} \mid U \in \mathcal{B}_{\mathcal{L}} \text{ and } U \subset L_{\lambda(U)}\}.$$

The set $\mathcal{L}_0$ is countable. Moreover, we shall show that it is also a covering of E. Let $p \in E$. Since L is a covering, there exists $L \in \mathcal{L}$ such that $p \in L$. Since $\mathcal{B}$ is a basis, there exists $U \in \mathcal{B}$ such that $x \in U \subset L$. U is then an element of $\mathcal{B}_{\mathcal{L}}$. Moreover, from the construction of $\mathcal{L}_0$ there is an element $L_{\lambda(U)}$ in $\mathcal{L}_0$ associated to U verifying $U \subset L_{\lambda(U)}$. Since $p \in U$, we have also $p \in L_{\lambda(U)}$. Since $p \in E$ is arbitrary, the theorem is proven. □

Theorem 1.60 (Alexandrov)**.**
Every basis of a second countable space contains a countable basis.

Proof. Let on a second-countable topological space $(E, \mathcal{O})$ be $\mathcal{B}_0$ a countable basis such that $\emptyset, E \in \mathcal{B}_0$, and $\mathcal{B}$ an arbitrary basis. We consider the subset of $B_0 \times B_0$ composed by (U, V) pairs such that there exists $W \in \mathcal{B}$ with $U \subset W \subset V$. This subset is nonempty, since it contains at least the pair $(\emptyset, E)$, and countable, therefore one can write as a

$$(U_1, V_1), \ldots, (U_n, V_n), \ldots$$

For each $n \in \mathbb{N}$ let us consider $W_n \in \mathcal{B}$ with the property $U_n \subset W_n \subset V_n$, and consider

$$\mathcal{B}_1 \quad = \quad \{W_n \mid n \in \mathbb{N}\}.$$

It is easy to verify the conditions of Exercise 1.44 are satisfied. Indeed, the elements of $\mathcal{B}_1$ are open sets. Moreover if G is an open set and $p \in G$, then (because $\mathcal{B}_0$ is a basis) there exists $V \in \mathcal{B}_0$ such that $p \in V \subset G$. Then, (because $\mathcal{B}$ is a basis), there exists $W \in \mathcal{B}$ such that $p \in W \subset V$. then (because $\mathcal{B}_0$ is a basis) there exists $U \in \mathcal{B}_0$ such that $p \in U \subset W$. It follows that $(U, V) \in \mathcal{B}_0 \times \mathcal{B}_0$ is a pair with the property that there an element of $\mathcal{B}$ can be written between them, therefore there exists $n \in \mathbb{N}$, such that $(U, V) = (U_n, V_n)$. Moreover, for the chosen W_n we have

$$p \in U_n \subset V_n \subset W_n \subset G,$$

we have $p \in W_n \subset G$ showing the condition 2.) of Exercise 1.44. □

1.6 Separability of topological spaces

Definition 1.61.
The topological space $(E, \mathcal{O})$ is called *separable* if there exists in E a countable dense subset.

Exercise 1.62.
Second-countable topological spaces are separable.

Indication. Consider a countable basis $\mathcal{B}$ and for every nonempty $B \in \mathcal{B}$ choose $x_B \in B$. Then the set $\{ x_B \mid B \in \mathcal{B} \}$ is a countable and dense set. □

Exercise 1.63.
Show that the converse of the statement of Exercise 1.62 is in general not true, that is there exists a separable topologyical space which is not satisfying the second-countablilty axiom.

Indication. Consider the co-finite topology $\mathcal{O}_{co}$ on a set E. It is separable.

Indeed if E is finite, then this is trivial, when E is not finite, then any infinite sets (in particular the countable infinite sets) are separable.
On the other hand if E is a non-countable set, then $(E, \mathcal{O}_{\text{co}})$ is not a first-countable (therefore not a second-countable) topological space. Indeed if it would be a a first-countable topological space, the let fix $x \in E$ and a countable local basis $\mathcal{B}(x)$ in x. Then

$$\cap\{U \mid x \in U \in \mathcal{O}\} = \{x\},$$

since

$$x \neq y \Longleftrightarrow x \in E \setminus \{y\} \in \mathcal{O}.$$

It follows that

$$\cap\{B \mid x \in B \in \mathcal{B}\} = \{x\},$$

and

$$\{x\} \subset \cap\{B \mid x \in B \in \mathcal{B}\} \subset \cap\{U \mid x \in U \in \mathcal{O}\} = \{x\},$$

therefore

$$E \setminus \{x\} = \cup\{E \setminus B \mid x \in B \in \mathcal{B}\},$$

where on the right-hand side there is a countable union of finite sets. It follows that $E \setminus \{x\}$ is countable, consequently E is countable, which is a contradiction. □

Exercise 1.64.
Separable metrizable space are second countable.

Indication. Let $(E, \mathcal{O})$ be a separable metrizable space and let $A \subset E$ be a countable dense subset. Using open balls defined by the metric,

$$\mathcal{B} = \{B(x, q) | \; x \in A, \;\; q \in \mathbb{Q}\}$$

is a countable basis for the topology. □