4.2 Switching components

The problem of uniqueness for binary matrices can be answered in a different way, then in the previous section, which involves the so called switching components. If A is a binary matrix of size $m \times n$, then a **switching component** is a submatrix of size 2×2,

$$\begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} \\ a_{i_2,j_1} & a_{i_2,j_2} \end{pmatrix}$$

for some $i_1, i_2 \in \{1, 2, \dots m\}$, $i_1 < i_2$ and $j_1, j_2 \in \{1, 2, \dots m\}$, $j_1 < j_2$, where either

$$\begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} \\ a_{i_2,j_1} & a_{i_2,j_2} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} a_{i_1,j_1} & a_{i_1,j_2} \\ a_{i_2,j_1} & a_{i_2,j_2} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It's clear that, if we interchange the zeros and ones in a switching component, then none of the row sum or the column sums change. This operation is called an **elementary switch**. Thus, if a binary matrix contains a switching component, then it's not unique, since we can get another binary matrix with the same row sum and columns sum vectors by an elementary switch. What's really interesting, that the converse of this statement is also true. Furthermore if two binary matrices has the same row sum and column sum vectors, than they can be transformed into each other by finitely many elementary switches.

Theorem 9 *A binary matrix is uniquely determined by its row sums and columns sums if and only if it contains no switching component.*

Theorem 10 *If two binary matrices A and B has the same row sum and column sum vectors, then there's a finite sequence of elementary switches, which applied to A results in the matrix B.*

The next example shows a binary matrix on the left which contains no switching component, while the matrix on the right has many switching components, and on of these is highlighted.

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & \mathbf{0} & 1 & \mathbf{1} & 0 \\ 0 & \mathbf{1} & 1 & \mathbf{0} & 0 \\ 1 & 0 & 1 & 0 & 1 \end{pmatrix}$$

Now we give a method to find similar patterns, also called switching components, for a larger number of directions. Let $D = (\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_q)$ a sequence of distinct lattice directions. Recall that these are pairwise different integer vectors, whose Cartesian coordinates are coprime numbers, and the first component is non-negative. Let $\mathcal{L}_k$ denote the set of all lattice lines parallel to $\underline{v}_k$ and intersecting at least one point of $\mathbb{Z}^2$ within the picture region $R = [a, b] \times [c, d]$, for all $k = 1, 2, \ldots, q$.

Definition 12 *We say that a picture function $f \colon R \cap \mathbb{Z}^2 \to \mathbb{Z}$ is a **ghost** for the directions D, if*

$$P_{k,f}(l) = \sum_{(x,y) \in l \cap R \cap \mathbb{Z}^2} f(x, y) = 0$$

for all $k = 1, 2, \ldots, q$ and for all $l \in \mathcal{L}_k$. In other words f is a ghost, if its X-ray parallel to any direction in D is constant zero.

Note that the picture function above has integer values, including negative numbers. Thus its X-ray can be zero despite having nonzero values in the picture region. Now it's time to introduce some elementary ghosts. Suppose, that $\underline{v}_k = (s_k, t_k)$ for all $k = 1, 2, \ldots, q$, and let the picture region of the elementary ghost be $R = [0, m] \times [0, n]$, where

$$m = \sum_{k=1}^{q} s_k \quad \text{and} \quad n = \sum_{k=1}^{q} |t_k|.$$

If $\underline{v} = (s, t)$ is a vector with integer coordinates $s, t \in \mathbb{Z}$, then the corresponding polynomial is defined in the following way:

$$p_{\underline{v}}(x, y) = \begin{cases} x^s y^t - 1, & \text{if } s > 0,\ t > 0, \\ x^s - y^{-t}, & \text{if } s > 0,\ t < 0, \\ x - 1, & \text{if } s = 1,\ t = 0, \\ y - 1, & \text{if } s = 0,\ t = 1, \end{cases}$$

This is a polynomial of two variables x and y in each case. We can take the product of all the polynomials corresponding to the directions in D, which is denoted by $P_D(x, y)$. This can be expressed by the following formula:

$$P_D(x, y) = \prod_{\underline{v}_k \in D} p_{\underline{v}_k}(x, y)$$

Finally let f_D be the picture function, which takes the value at the point $(i,j) \in \mathbb{Z}^2$ equal to the coefficient of the term $x^i y^j$ in the polynomial $P_D(x,y)$. Note that if the polynomial doesn't contain the term $x^i y^j$, then it means only that its coefficient is zero. It's clear from the definition that all the values of f_D are either 0, 1, or -1 in the picture region $[0,m] \times [0,n]$. The picture function f_D can be easily extended to any picture region, which contains $[0,m] \times [0,n]$, by defining the values equal to zero outside of the region $[0,m] \times [0,n]$.

Theorem 11 *The picture function f_D defined above is a ghost for the directions D over any picture region, which contains $[0,m] \times [0,n]$.*

The picture function f_D is called the **elementary ghost** defined by the directions D. If (u,v) is an integer vector, then we can define another picture function $f_{u,v,D}$, by the formula $f_{u,v,D}(i,j) = f_D(i-u, j-v)$. We say $f_{u,v,D}$ is the ghost f_D shifted by the integer vector (u,v). Certainly, the shifted ghost $f_{u,v,D}$ is a ghost for the same directions D. Furthermore it's clear that any linear combination of shifted elementary ghosts is also a ghost for the same directions. The following theorem tells that any ghost for the directions D can be written as a linear combination of shifted elementary ghosts.

Theorem 12 *Let $R = [a,b] \times [c,d]$ be a picture region, and let $D = \big(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_q\big)$ a sequence of distinct lattice directions, where $\underline{v}_k = (s_k, t_k)$ for all $k = 1, 2, \ldots, q$. Put*

$$m = \sum_{k=1}^{q} s_k \quad \text{and} \quad n = \sum_{k=1}^{q} |t_k|,$$

and suppose that $m \le b-a$ and $n \le d-c$. Then any ghost $f\colon R \cap \mathbb{Z}^2 \to \mathbb{Z}$ can be uniquely written in the form

$$f = \sum_{u=a}^{b-m} \sum_{v=c}^{d-n} r_{u,v} \cdot f_{u,v,D}$$

where $r_{u,v}$ are real numbers, and $f_{u,v,D}$ are shifted elementary ghosts for the directions D. Moreover every such picture function f is a ghost for the directions D.

Finally we reveal how elementary ghosts can help us to find switching components for the directions D. We only need to recognize, that if f_D is the elementary ghost for the direction D, then any line l contains the same number of values equal to 1 and values equal to -1, since this is the only way how the sum of the elements equals to zero. We can define two binary picture functions f_{D+} and f_{D-}, where f_{D+} equals to 1 in exactly those points, where f_D is 1, but otherwise f_{D+} equals to 0, while f_{D-} equals to 1 in exactly those points, where f_D is -1, but otherwise f_{D-} is 0. Then f_{D+} and f_{D-} contains the same number of ones along any line l, which is parallel to one of the directions in D. Thus the values of the X-rays of f_{D+} and f_{D-} are the same for all the lines parallel to directions in D, which means they are tomographically equivalent. Furthermore, if we interchange the zeros and ones in f_{D+} in those positions, where f_D is nonzero, then we get exactly f_{D-}. Similarly, if we interchange the zeros and ones in f_{D-} in those positions, where f_D is nonzero, then we get exactly f_{D+}. That's why we may call f_{D+} and f_{D-} switching components for the directions D. Since the sequence of directions D is arbitrary, this shows, that a switching component exist for any finite set of directions, if the picture region is large enough. Consequently, arbitrary lattice sets are not determined by any finite set of directions.

Example 1

Let $D = (\underline{v}_1, \underline{v}_2, \underline{v}_3)$, where

$$\underline{v}_1 = (s_1, t_1) = (1, 0), \quad \underline{v}_2 = (s_2, t_2) = (0, 1), \quad \underline{v}_3 = (s_3, t_3) = (2, 1).$$

Then the corresponding polynomials are

$$p_{\underline{v}_1}(x, y) = x - 1, \quad p_{\underline{v}_2}(x, y) = y - 1, \quad p_{\underline{v}_3}(x, y) = x^2 y - 1,$$

and

$$m = \sum_{k=1}^{q} s_k = 1 + 0 + 2 = 3 \quad n = \sum_{k=1}^{q} |t_k| = 0 + 1 + 1 = 2.$$

The product of the above polynomials is

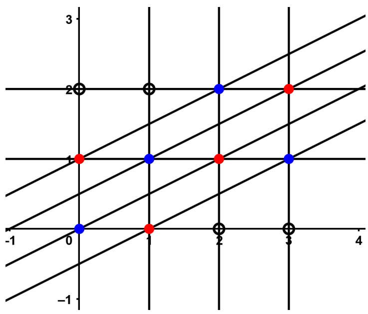
$$P_D(x, y) = p_{\underline{v}_1}(x, y) \cdot p_{\underline{v}_2}(x, y) \cdot p_{\underline{v}_3}(x, y)$$

$$= (x - 1)(y - 1)\left(x^2 y - 1\right) = x^3 y^2 - x^3 y - x^2 y^2 + x^2 y - xy + x + y - 1$$

Here the coefficient of $x^0 y^0 = 1$ in $P_D(x, y)$ is -1, which is just the constant term of $P_D(x, y)$. The coefficients of $x^0 y^1 = y$ and $x^0 y^2 = y^2$ are 1 and 0, respectively, as the term y^2 is missing in the polynomial $P_D(x, y)$. The coefficients of $x^1y^0 = x$, $x^1y^1 = xy$ and $x^1y^2 = xy^2$ are 1, −1 and 0, respectively. The coefficients of $x^2y^0 = x^2$, $x^2y^1 = x^2y$ and x^2y^2 are 0, 1 and −1, respectively. The coefficients of $x^3y^0 = x^3$, $x^3y^1 = x^3y$ and x^3y^2 are 0, −1 and 1, respectively. Hence the picture function f_D takes the following values in the picture region $[0, m] \times [0, n] = [0, 3] \times [0, 2]$:

$$\begin{array}{lll} f_D(0,0) = -1 & f_D(0,1) = 1 & f_D(0,2) = 0 \\ f_D(1,0) = 1 & f_D(1,1) = -1 & f_D(1,2) = 0 \\ f_D(2,0) = 0 & f_D(2,1) = 1 & f_D(2,2) = -1 \\ f_D(3,0) = 0 & f_D(3,1) = -1 & f_D(3,2) = 1 \end{array}$$

This is illustrated in the figure below, where solid red dots denote those points, where the values of f_D equal to 1, solid blue dots denote those points, where the values equal to −1, and empty dots denote those points, where the values equal to 0.



We can clearly see that each line parallel to the directions v_1, v_2, v_3 contain equal number of red and blue dots. The lines, which contain at least one such point, are presented in the figure. The elementary ghost f_D can also be presented by the following matrix:

$$\begin{pmatrix} 0 & 0 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 0 & 0 \end{pmatrix}$$

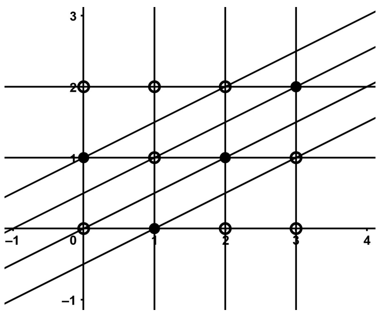
The picture functions f_{D+} and f_{D-} take the following values in the picture region $[0,m] \times [0,n] = [0,3] \times [0,2]$:

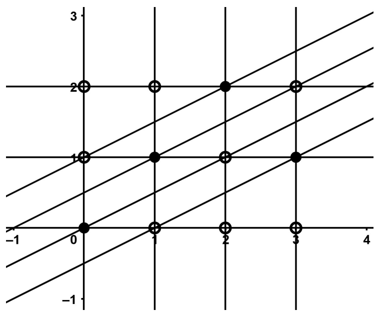
$$\begin{array}{lll} f_{D+}(0,0) = 0 & f_{D+}(0,1) = 1 & f_{D+}(0,2) = 0 \\ f_{D+}(1,0) = 1 & f_{D+}(1,1) = 0 & f_{D+}(1,2) = 0 \\ f_{D+}(2,0) = 0 & f_{D+}(2,1) = 1 & f_{D+}(2,2) = 0 \\ f_{D+}(3,0) = 0 & f_{D+}(3,1) = 0 & f_{D+}(3,2) = 1 \end{array}$$

and

$$\begin{array}{lll} f_{D-}(0,0) = 1 & f_{D-}(0,1) = 0 & f_{D-}(0,2) = 0 \\ f_{D-}(1,0) = 0 & f_{D-}(1,1) = 1 & f_{D-}(1,2) = 0 \\ f_{D-}(2,0) = 0 & f_{D-}(2,1) = 0 & f_{D-}(2,2) = 1 \\ f_{D-}(3,0) = 0 & f_{D-}(3,1) = 1 & f_{D-}(3,2) = 0 \end{array}$$

The binary picture function f_{D+} is presented in the figure below on the left, and te picture function f_{D-} is presented on the right. Solid dots denote those points, where the values of f_{D+} or f_{D-} equal to 1, and empty dots denote those points, where the values equal to 0.





The picture functions f_{D+} and f_{D-} can also be presented by the following binary matrices:

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Example 2

Let $D = (\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4)$, where

$$\begin{array}{ll} \underline{v}_1 = (s_1, t_1) = (1,1) & \underline{v}_2 = (s_2, t_2) = (1,-1) \\ \underline{v}_3 = (s_3, t_3) = (1,2) & \underline{v}_4 = (s_4, t_4) = (3,-1) \end{array}$$

Then the corresponding polynomials are

$$p_{\underline{v}_1}(x,y) = xy - 1 \qquad p_{\underline{v}_2}(x,y) = x - y$$
$$p_{\underline{v}_3}(x,y) = xy^2 - 1 \qquad p_{\underline{v}_4}(x,y) = x^3 - y$$

and

$$m = \sum_{k=1}^{q} s_k = 1+1+1+3 = 6 \quad n = \sum_{k=1}^{q} |t_k| = 1+1+2+1 = 5.$$

The product of the above polynomials is

$$P_D(x,y) = p_{\underline{v}_1}(x,y) \cdot p_{\underline{v}_2}(x,y) \cdot p_{\underline{v}_3}(x,y) \cdot p_{\underline{v}_4}(x,y)$$

$$= (xy-1)(x-y)\left(xy^2-1\right)\left(x^3-y\right)$$

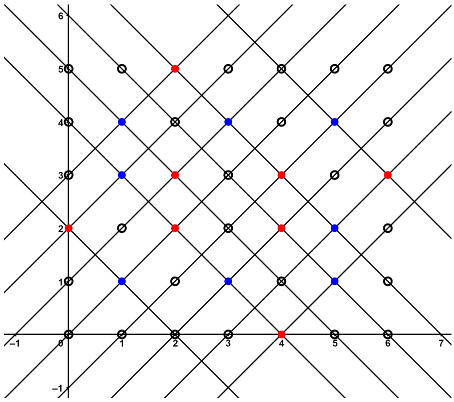
$$= x^6y^3 - x^5y^4 - x^5y^2 + x^4y^3 - x^3y^4 + x^2y^5 - x^5y$$

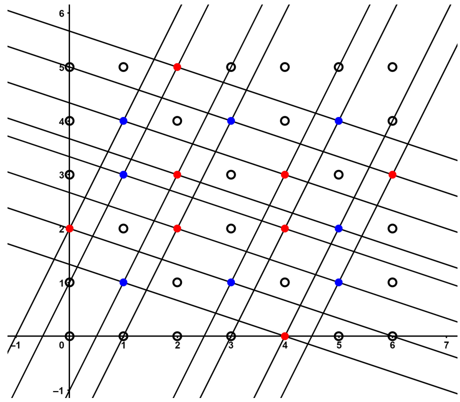
$$+x^4y^2 + x^2y^3 - xy^4 + x^4 - x^3y + x^2y^2 - xy^3 - xy + y^2$$

We can see, that only the coefficients of $x^0y^2 = y^2$, $x^1y^1 = xy$, $x^1y^3 = xy^3$, $x^1y^4 = xy^4$, x^2y^2, x^2y^3, x^2y^5, $x^3y^1 = x^3y$, x^3y^4, $x^4y^0 = x^4$, x^4y^2, x^4y^3, $x^5y^1 = x^5y$, x^5y^2, x^5y^4 and x^6y^3 are different from zero. Thus the picture function f_D has nonzero values only in the points $(0,2)$, $(1,1)$, $(1,3)$, $(1,4)$, $(2,2)$, $(2,3)$, $(2,5)$, $(3,1)$, $(3,4)$, $(4,0)$, $(4,2)$, $(4,3)$, $(5,1)$, $(5,2)$, $(5,4)$, $(6,3)$ of the picture region $[0,m] \times [0,n] = [0,6] \times [0,5]$. These values are 1, -1, -1, -1, 1, 1, 1, -1, -1, 1, 1, 1, -1, -1, -1, and 1, respectively. Hence f_D has the following values in all the points of the picture region $[0,6] \times [0,5]$:

$f_D(0,0)=0$	$f_D(0,1)=0$	$f_D(0,2)=1$	$f_D(0,3)=0$	$f_D(0,4)=0$	$f_D(0,5)=0$
$f_D(1,0)=0$	$f_D(1,1)=-1$	$f_D(1,2)=0$	$f_D(1,3)=-1$	$f_D(1,4)=-1$	$f_D(1,5)=0$
$f_D(2,0)=0$	$f_D(2,1)=0$	$f_D(2,2)=1$	$f_D(2,3)=1$	$f_D(2,4)=0$	$f_D(2,5)=1$
$f_D(3,0)=0$	$f_D(3,1)=-1$	$f_D(3,2)=0$	$f_D(3,3)=0$	$f_D(3,4)=-1$	$f_D(3,5)=0$
$f_D(4,0)=1$	$f_D(4,1)=0$	$f_D(4,2)=1$	$f_D(4,3)=1$	$f_D(4,4)=0$	$f_D(4,5)=0$
$f_D(5,0)=0$	$f_D(5,1)=-1$	$f_D(5,2)=-1$	$f_D(5,3)=0$	$f_D(5,4)=-1$	$f_D(5,5)=0$
$f_D(6,0)=0$	$f_D(6,1)=0$	$f_D(6,2)=0$	$f_D(6,3)=1$	$f_D(6,4)=0$	$f_D(6,5)=0$

This is illustrated in the figures below, where solid red dots denote those points, where the values of f_D equal to 1, solid blue dots denote those points, where the values equal to -1, and empty dots denote those points, where the values equal to 0.





We can clearly see that each line parallel to the directions v_1, v_2, v_3, v_4 contain equal number of red and blue dots. The lines, which contain at least one such point and parallel to v_1 or v_2, are presented in the figure on the left, while the lines, which are parallel to v_3 or v_4, are presented on the right. The elementary ghost f_D can also be presented by the following matrix:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & -1 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

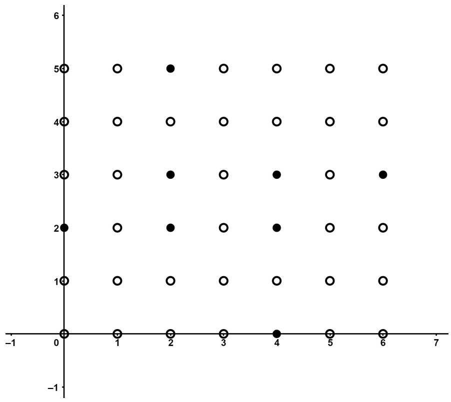
The picture functions f_{D+} and f_{D-} take the following values in the picture region $[0,6] \times [0,5]$:

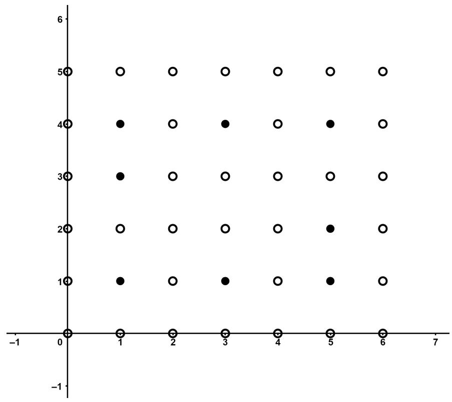
$f_D(0,0)=0$	$f_D(0,1)=0$	$f_D(0,2)=1$	$f_D(0,3)=0$	$f_D(0,4)=0$	$f_D(0,5)=0$
$f_D(1,0)=0$	$f_D(1,1)=0$	$f_D(1,2)=0$	$f_D(1,3)=0$	$f_D(1,4)=0$	$f_D(1,5)=0$
$f_D(2,0)=0$	$f_D(2,1)=0$	$f_D(2,2)=1$	$f_D(2,3)=1$	$f_D(2,4)=0$	$f_D(2,5)=1$
$f_D(3,0)=0$	$f_D(3,1)=0$	$f_D(3,2)=0$	$f_D(3,3)=0$	$f_D(3,4)=0$	$f_D(3,5)=0$
$f_D(4,0)=1$	$f_D(4,1)=0$	$f_D(4,2)=1$	$f_D(4,3)=1$	$f_D(4,4)=0$	$f_D(4,5)=0$
$f_D(5,0)=0$	$f_D(5,1)=0$	$f_D(5,2)=0$	$f_D(5,3)=0$	$f_D(5,4)=0$	$f_D(5,5)=0$
$f_D(6,0)=0$	$f_D(6,1)=0$	$f_D(6,2)=0$	$f_D(6,3)=1$	$f_D(6,4)=0$	$f_D(6,5)=0$

and

$f_D(0,0)=0$	$f_D(0,1)=0$	$f_D(0,2)=0$	$f_D(0,3)=0$	$f_D(0,4)=0$	$f_D(0,5)=0$
$f_D(1,0)=0$	$f_D(1,1)=1$	$f_D(1,2)=0$	$f_D(1,3)=1$	$f_D(1,4)=1$	$f_D(1,5)=0$
$f_D(2,0)=0$	$f_D(2,1)=0$	$f_D(2,2)=0$	$f_D(2,3)=0$	$f_D(2,4)=0$	$f_D(2,5)=0$
$f_D(3,0)=0$	$f_D(3,1)=1$	$f_D(3,2)=0$	$f_D(3,3)=0$	$f_D(3,4)=1$	$f_D(3,5)=0$
$f_D(4,0)=0$	$f_D(4,1)=0$	$f_D(4,2)=0$	$f_D(4,3)=0$	$f_D(4,4)=0$	$f_D(4,5)=0$
$f_D(5,0)=0$	$f_D(5,1)=1$	$f_D(5,2)=1$	$f_D(5,3)=0$	$f_D(5,4)=1$	$f_D(5,5)=0$
$f_D(6,0)=0$	$f_D(6,1)=0$	$f_D(6,2)=0$	$f_D(6,3)=0$	$f_D(6,4)=0$	$f_D(6,5)=0$

The binary picture function f_{D+} is presented in the figure below on the left, and the picture function f_{D-} is presented on the right. Solid dots denote those points, where the values of f_{D+} or f_{D-} equal to 1, and empty dots denote those points, where the values equal to 0.





The picture functions f_{D+} and f_{D-} can be also presented by the following binary matrices:

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$