1.2 Switching components

Exercise 1.2.1 *Construct the elementary ghost f_D for the direction set $D = (\underline{v}_1, \underline{v}_2, \underline{v}_3)$, where $\underline{v}_1 = (1,0)$, $\underline{v}_2 = (1,-1)$, $\underline{v}_3 = (3,1)$. Find the switching components f_{D+} and f_{D-} corresponding to f_D.*

Solution. We have the lattice directions

$$\underline{v}_1 = (s_1, t_1) = (1,0), \quad \underline{v}_2 = (s_2, t_2) = (1,-1), \quad \underline{v}_3 = (s_3, t_3) = (3,1).$$

The picture region of the elementary ghost for the directions D is $[0,m]\times[0,n]$, where

$$m = \sum_{k=1}^{3} s_k = 1+1+3 = 5, \quad \text{and} \quad n = \sum_{k=1}^{3} |t_k| = 0+1+1 = 2.$$

The polynomial corresponding to an arbitrary lattice direction $\underline{v} = (s,t)$ is defined as

$$p_{\underline{v}}(x,y) = \begin{cases} x^s y^t - 1, & \text{if } s > 0,\ t > 0, \\ x^s - y^{-t}, & \text{if } s > 0,\ t < 0, \\ x - 1, & \text{if } s = 1,\ t = 0, \\ y - 1, & \text{if } s = 0,\ t = 1, \end{cases}$$

Now the corresponding polynomials are

$$p_{\underline{v}_1}(x,y) = x - 1, \quad p_{\underline{v}_2}(x,y) = x - y, \quad p_{\underline{v}_3}(x,y) = x^3 y - 1,$$

The product of the above polynomials is

$$P_D(x,y) = p_{\underline{v}_1}(x,y) \cdot p_{\underline{v}_2}(x,y) \cdot p_{\underline{v}_3}(x,y)$$

$$= (x-1)(x-y)\left(x^3 y - 1\right) = x^5 y - x^4 y^2 - x^4 y + x^2 y^2 - x^2 + xy + x - y$$

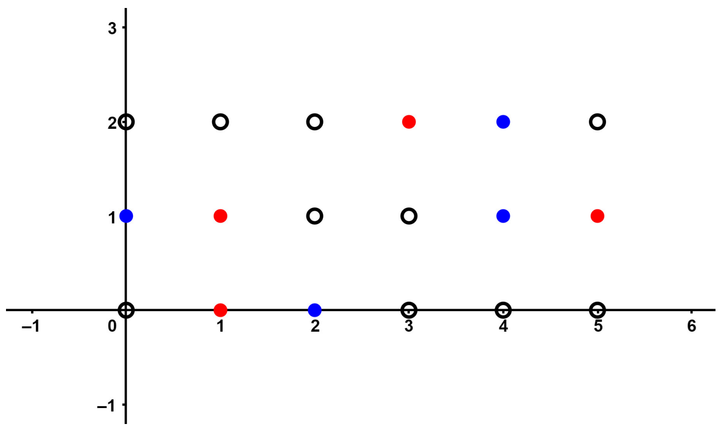
The ghost f_D is the picture function, which takes the value at the point $(i,j) \in \mathbb{Z}^2$ equal to the coefficient of the term $x^i y^j$ in the polynomial $P_D(x,y)$. Note that if the polynomial doesn't contain the term $x^i y^j$, then it means only that its coefficient is zero.

Now the coefficient of $x^0 y^0 = 1$ in $P_D(x,y)$ is 0, which is just the (missing) constant term of $P_D(x,y)$. The coefficients of $x^0 y^1 = y$ and $x^0 y^2 = y^2$ are -1 and 0, respectively. The coefficients of $x^1 y^0 = x$, $x^1 y^1 = xy$ and $x^1 y^2 = xy^2$

are 1, 1 and 0, respectively. The coefficients of $x^2y^0 = x^2$, $x^2y^1 = x^2y$ and x^2y^2 are -1, 0 and 0, respectively. The coefficients of $x^3y^0 = x^3$, $x^3y^1 = x^3y$ and x^3y^2 are 0, 0 and 1, respectively. The coefficients of $x^4y^0 = x^4$, $x^4y^1 = x^4y$ and x^4y^2 are 0, -1 and -1, respectively. The coefficients of $x^5y^0 = x^5$, $x^5y^1 = x^5y$ and x^5y^2 are 0, 1 and 0, respectively. Hence the picture function f_D takes the following values in the picture region $[0, m] \times [0, n] = [0, 5] \times [0, 2]$:

$$\begin{array}{lll} f_D(0,0) = 0 & f_D(0,1) = -1 & f_D(0,2) = 0 \\ f_D(1,0) = 1 & f_D(1,1) = 1 & f_D(1,2) = 0 \\ f_D(2,0) = -1 & f_D(2,1) = 0 & f_D(2,2) = 0 \\ f_D(3,0) = 0 & f_D(3,1) = 0 & f_D(3,2) = 1 \\ f_D(4,0) = 0 & f_D(4,1) = -1 & f_D(4,2) = -1 \\ f_D(5,0) = 0 & f_D(5,1) = 1 & f_D(5,2) = 0 \end{array}$$

This is illustrated in the figure below, where solid red dots denote those points, where the values of f_D equal to 1, solid blue dots denote those points, where the values equal to -1, and empty dots denote those points, where the values equal to 0.



The elementary ghost f_D can also be presented by the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & 1 & 0 & 0 & -1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 0 \end{pmatrix}$$

The two switching components f_{D+} and f_{D-} are defined in the following way. f_{D+} equals to 1 in exactly those points, where f_D is 1, but otherwise f_{D+}

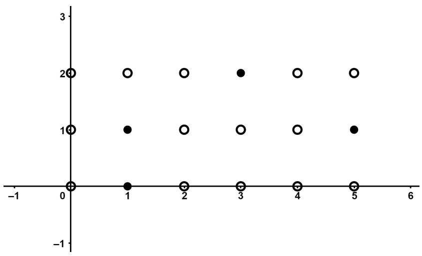
equals to 0, while f_{D-} equals to 1 in exactly those points, where f_D is -1, but otherwise f_{D-} is 0. Now f_{D+} and f_{D-} take the following values in the picture region $[0,m] \times [0,n] = [0,5] \times [0,2]$:

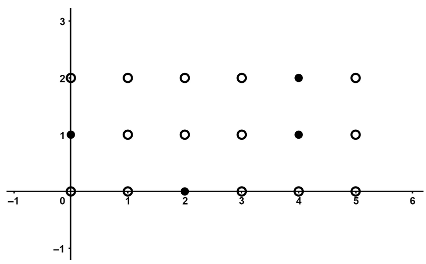
$$\begin{array}{lll} f_{D+}(0,0)=0 & f_{D+}(0,1)=0 & f_{D+}(0,2)=0 \\ f_{D+}(1,0)=1 & f_{D+}(1,1)=1 & f_{D+}(1,2)=0 \\ f_{D+}(2,0)=0 & f_{D+}(2,1)=0 & f_{D+}(2,2)=0 \\ f_{D+}(3,0)=0 & f_{D+}(3,1)=0 & f_{D+}(3,2)=1 \\ f_{D+}(4,0)=0 & f_{D+}(4,1)=0 & f_{D+}(4,2)=0 \\ f_{D+}(5,0)=0 & f_{D+}(5,1)=1 & f_{D+}(5,2)=0 \end{array}$$

and

$$\begin{array}{lll} f_{D-}(0,0)=0 & f_{D-}(0,1)=1 & f_{D-}(0,2)=0 \\ f_{D-}(1,0)=0 & f_{D-}(1,1)=0 & f_{D-}(1,2)=0 \\ f_{D-}(2,0)=1 & f_{D-}(2,1)=0 & f_{D-}(2,2)=0 \\ f_{D-}(3,0)=0 & f_{D-}(3,1)=0 & f_{D-}(3,2)=0 \\ f_{D-}(4,0)=0 & f_{D-}(4,1)=1 & f_{D-}(4,2)=1 \\ f_{D-}(5,0)=0 & f_{D-}(5,1)=0 & f_{D-}(5,2)=0 \end{array}$$

The switching component f_{D+} is presented in the figure below on the left, and the switching component f_{D-} is presented on the right. Solid dots denote those points, where the values of f_{D+} or f_{D-} equal to 1, and empty dots denote those points, where the values equal to 0.





The switching components f_{D+} and f_{D-} can also be presented by the following binary matrices:

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

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Exercise 1.2.2 *Construct the elementary ghost* f_D *for the direction set* $D = (\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4)$, *where* $\underline{v}_1 = (0,1)$, $\underline{v}_2 = (1,2)$, $\underline{v}_3 = (1,-2)$, $\underline{v}_4 = (3,2)$. *Find the switching components* f_{D+} *and* f_{D-} *corresponding to* f_D.

Solution. We have the lattice directions

$$\underline{v}_1 = (s_1, t_1) = (0,1), \quad \underline{v}_2 = (s_2, t_2) = (1,2),$$
$$\underline{v}_3 = (s_3, t_3) = (1,-2), \quad \underline{v}_4 = (s_4, t_4) = (3,2).$$

The picture region of the elementary ghost for the directions D is $[0,m]\times[0,n]$, where

$$m = \sum_{k=1}^{4} s_k = 0+1+1+3 = 5, \quad \text{and} \quad n = \sum_{k=1}^{4} |t_k| = 1+2+2+2 = 7.$$

The corresponding polynomials are

$$p_{\underline{v}_1}(x,y) = y-1, \quad p_{\underline{v}_2}(x,y) = xy^2-1, \quad p_{\underline{v}_3}(x,y) = x-y^2, \quad p_{\underline{v}_3}(x,y) = x^3y^2-1$$

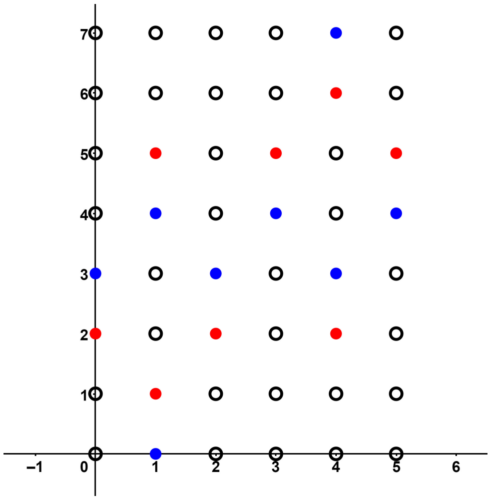
The product of the above polynomials is

$$\begin{aligned} P_D(x,y) &= p_{\underline{v}_1}(x,y) \cdot p_{\underline{v}_2}(x,y) \cdot p_{\underline{v}_3}(x,y) \cdot p_{\underline{v}_4}(x,y) \\ &= (y-1)\left(x^2y-1\right)\left(x-y^2\right)\left(x^3y^2-1\right) \\ &= -x^4y^7 + x^5y^5 + x^4y^6 - x^5y^4 + x^3y^5 - x^4y^3 - x^3y^4 + x^4y^2 \\ &\quad +xy^5 - x^2y^3 - xy^4 + x^2y^2 - y^3 + xy + y^2 - x \end{aligned}$$

The ghost f_D is the picture function, which takes the value at the point $(i,j) \in \mathbb{Z}^2$ equal to the coefficient of the term x^iy^j in the polynomial $P_D(x,y)$. Hence the picture function f_D takes the following values in the picture region $[0,m] \times [0,n] = [0,5] \times [0,7]$:

$f_D(0,0) = 0$	$f_D(1,0) = -1$	$f_D(2,0) = 0$	$f_D(3,0) = 0$	$f_D(4,0) = 0$	$f_D(5,0) = 0$
$f_D(0,1) = 0$	$f_D(1,1) = 1$	$f_D(2,1) = 0$	$f_D(3,1) = 0$	$f_D(4,1) = 0$	$f_D(5,1) = 0$
$f_D(0,2) = 1$	$f_D(1,2) = 0$	$f_D(2,2) = 1$	$f_D(3,2) = 0$	$f_D(4,2) = 1$	$f_D(5,2) = 0$
$f_D(0,3) = -1$	$f_D(1,3) = 0$	$f_D(2,3) = -1$	$f_D(3,3) = 0$	$f_D(4,3) = -1$	$f_D(5,3) = 0$
$f_D(0,4) = 0$	$f_D(1,4) = -1$	$f_D(2,4) = 0$	$f_D(3,4) = -1$	$f_D(4,4) = 0$	$f_D(5,4) = -1$
$f_D(0,5) = 0$	$f_D(1,5) = 1$	$f_D(2,5) = 0$	$f_D(3,5) = 1$	$f_D(4,5) = 0$	$f_D(5,5) = 1$
$f_D(0,6) = 0$	$f_D(1,6) = 0$	$f_D(2,6) = 0$	$f_D(3,6) = 0$	$f_D(4,6) = 1$	$f_D(5,6) = 0$
$f_D(0,7) = 0$	$f_D(1,7) = 0$	$f_D(2,7) = 0$	$f_D(3,7) = 0$	$f_D(4,7) = -1$	$f_D(5,7) = 0$

This is illustrated in the figure below, where solid red dots denote those points, where the values of f_D equal to 1, solid blue dots denote those points, where the values equal to -1, and empty dots denote those points, where the values equal to 0.



The elementary ghost f_D can also be presented by the following matrix:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & 0 & -1 \\ -1 & 0 & -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

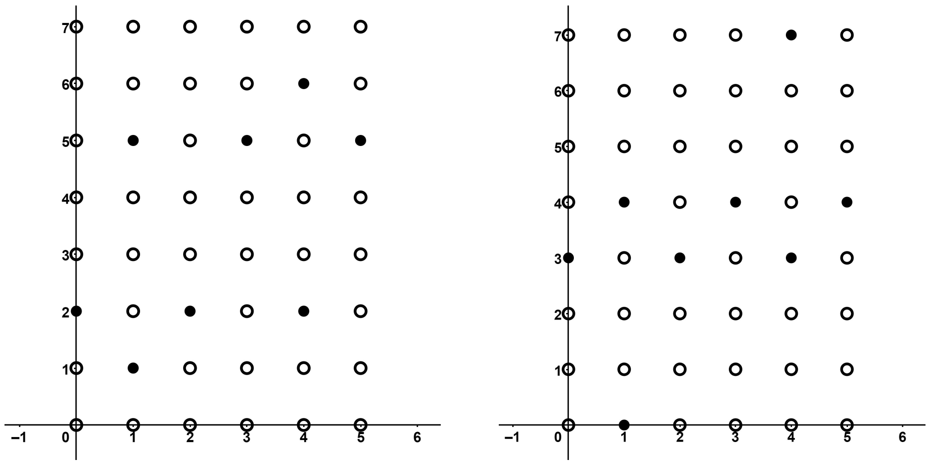
The switching components f_{D+} and f_{D-} take the following values in the picture region $[0,m] \times [0,n] = [0,5] \times [0,7]$:

$f_{D+}(0,0)=0$	$f_{D+}(1,0)=0$	$f_{D+}(2,0)=0$	$f_{D+}(3,0)=0$	$f_{D+}(4,0)=0$	$f_{D+}(5,0)=0$
$f_{D+}(0,1)=0$	$f_{D+}(1,1)=1$	$f_{D+}(2,1)=0$	$f_{D+}(3,1)=0$	$f_{D+}(4,1)=0$	$f_{D+}(5,1)=0$
$f_{D+}(0,2)=1$	$f_{D+}(1,2)=0$	$f_{D+}(2,2)=1$	$f_{D+}(3,2)=0$	$f_{D+}(4,2)=1$	$f_{D+}(5,2)=0$
$f_{D+}(0,3)=0$	$f_{D+}(1,3)=0$	$f_{D+}(2,3)=0$	$f_{D+}(3,3)=0$	$f_{D+}(4,3)=0$	$f_{D+}(5,3)=0$
$f_{D+}(0,4)=0$	$f_{D+}(1,4)=0$	$f_{D+}(2,4)=0$	$f_{D+}(3,4)=0$	$f_{D+}(4,4)=0$	$f_{D+}(5,4)=0$
$f_{D+}(0,5)=0$	$f_{D+}(1,5)=1$	$f_{D+}(2,5)=0$	$f_{D+}(3,5)=1$	$f_{D+}(4,5)=0$	$f_{D+}(5,5)=1$
$f_{D+}(0,6)=0$	$f_{D+}(1,6)=0$	$f_{D+}(2,6)=0$	$f_{D+}(3,6)=0$	$f_{D+}(4,6)=1$	$f_{D+}(5,6)=0$
$f_{D+}(0,7)=0$	$f_{D+}(1,7)=0$	$f_{D+}(2,7)=0$	$f_{D+}(3,7)=0$	$f_{D+}(4,7)=0$	$f_{D+}(5,7)=0$

and

$f_{D-}(0,0)=0$	$f_{D-}(1,0)=1$	$f_{D-}(2,0)=0$	$f_{D-}(3,0)=0$	$f_{D-}(4,0)=0$	$f_{D-}(5,0)=0$
$f_{D-}(0,1)=0$	$f_{D-}(1,1)=0$	$f_{D-}(2,1)=0$	$f_{D-}(3,1)=0$	$f_{D-}(4,1)=0$	$f_{D-}(5,1)=0$
$f_{D-}(0,2)=0$	$f_{D-}(1,2)=0$	$f_{D-}(2,2)=0$	$f_{D-}(3,2)=0$	$f_{D-}(4,2)=0$	$f_{D-}(5,2)=0$
$f_{D-}(0,3)=1$	$f_{D-}(1,3)=0$	$f_{D-}(2,3)=1$	$f_{D-}(3,3)=0$	$f_{D-}(4,3)=1$	$f_{D-}(5,3)=0$
$f_{D-}(0,4)=0$	$f_{D-}(1,4)=1$	$f_{D-}(2,4)=0$	$f_{D-}(3,4)=1$	$f_{D-}(4,4)=0$	$f_{D-}(5,4)=1$
$f_{D-}(0,5)=0$	$f_{D-}(1,5)=0$	$f_{D-}(2,5)=0$	$f_{D-}(3,5)=0$	$f_{D-}(4,5)=0$	$f_{D-}(5,5)=0$
$f_{D-}(0,6)=0$	$f_{D-}(1,6)=0$	$f_{D-}(2,6)=0$	$f_{D-}(3,6)=0$	$f_{D-}(4,6)=0$	$f_{D-}(5,6)=0$
$f_{D-}(0,7)=0$	$f_{D-}(1,7)=0$	$f_{D-}(2,7)=0$	$f_{D-}(3,7)=0$	$f_{D-}(4,7)=1$	$f_{D-}(5,7)=0$

The binary picture function f_{D+} is presented in the figure below on the left, and the picture function f_{D-} is presented on the right. Solid dots denote those points, where the values of f_{D+} or f_{D-} equal to 1, and empty dots denote those points, where the values equal to 0.



The picture functions f_{D+} and f_{D-} can also be presented by the following binary matrices:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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