4.1 Reconstruction of binary matrices with prescribed row and column sums

We investigate the problems of consistency, uniqueness and reconstruction in the case when the set of directions is $D = (\underline{v}_1, \underline{v}_2)$,, where $\underline{v}_1 = (1, 0)$ and $\underline{v}_1 = (0, 1)$. This means we use the X-rays of lattice sets parallel to the coordinate directions of a Cartesian coordinate system. A lattice set in the plane can naturally characterized by a binary matrix. A matrix A is called **binary matrix**, if all elements of A are either one or zero. It presents the lattice set, where the positions of the ones in the matrix determine the positions of the points of the lattice set. Then the values of the X-ray of the lattice set parallel to horizontal direction $\underline{v}_1 = (1, 0)$ equal to the row sums of the binary matrix. Similarly, the values of the X-ray of of the lattice set parallel to vertical direction $\underline{v}_1 = (0, 1)$ equal to the column sums of the binary matrix. The **row sum vector** of the binary matrix A of size $m \times n$ is the vector $R = (r_1, r_2, \ldots, r_m)$, where

$$\sum_{j=1}^{n} a_{i,j} = r_i, \qquad i \in \{1, 2, \ldots, m\}$$

In other words r_i equals to number of ones in the i-th row of A for each $i \in \{1, 2, \ldots, m\}$. The **column sum vector** of the binary matrix A of size $m \times n$ is the vector $S = (s_1, s_2, \ldots, s_n)$, where

$$\sum_{i=1}^{m} a_{i,j} = s_j, \qquad j \in \{1, 2, \ldots, n\}$$

In other words s_j equals to number of ones in the j-th column of A for each $j \in \{1, 2, \ldots, n\}$.

Now we can formulate the problems of consistency, uniqueness and reconstruction in terms of binary matrices.

- Consistency: Given two vectors R and S, is there any binary matrix whose row sum vector is R and columns sum vector is S?
- Uniqueness: Given a binary matrix A with row sum vector R and column sum vector S, is there another matrix B with the same row sum and column sum vectors?

- Reconstruction: Given two vectors R and S, find a binary matrix whose row sum vector is R and columns sum vector is S.

It's easy to see, that if A is any binary matrix of size $m \times n$, and its row and column sum vectors are R and S, then

1. all elements of R and S are non-negative integers,
2. R has no element which is larger than n, and S has no element which is larger than m,
3. the sum of the elements in R equals to the sum of the elements in S, that is
$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j.$$

The first two statements is quite trivial, while the third is just the consequence of the fact, that both $\sum_{i=1}^{m} r_i$ and $\sum_{j=1}^{n} s_j$ equals to the total number of ones in the matrix A.

Definition 11 *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be two vectors, whose elements are non-negative integers. The vectors R and S are called compatible if*

1. $r_i \leq n$ *for all* $i \in \{1, 2, \ldots, m\}$,

2. $s_j \leq m$ *for all* $j \in \{1, 2, \ldots, n\}$,

3.
$$\sum_{i=1}^{m} r_i = \sum_{j=1}^{n} s_j.$$

Note that if the vectors $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ are not compatible, then there's no binary matrix of size $m \times n$ whose row sum vector is R and column sum vector is S.

Given the row sum vector $R = (r_1, r_2, \ldots, r_m)$ and the number of columns n the maximal matrix corresponding to R is the binary matrix A of size $m \times n$ which satisfies

$$a_{ij} = \begin{cases} 1 & \text{if } j \leq r_i, \\ 0 & \text{if } r_i < j. \end{cases} \qquad \text{for all } i \in \{1, 2, \ldots, m\}$$

In other words the first r_i elements of the i-th row of the maximal matrix equal to one, while the rest of the elements in the i-th row equal to zero. The maximal matrix with n columns corresponding to the row sum vector R is denoted by $\overline{A}$, and the column sum vector of the maximal matrix is denoted by $\overline{S} = (\overline{s}_1, \overline{s}_2, \ldots, \overline{s}_n)$. Furthermore let's denote the nonincreasing permutation of any column sum vector S by $S' = (s'_1, s'_2, \ldots, s'_n)$.

Theorem 8 *Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be two compatible integer vectors. There exists a binary matrix with row sum vector R and column sum vector S if and only if*

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j \qquad \textit{for all } k \in \{1, 2, \ldots, n\}$$

Furthermore this binary matrix is unique if and only if all the above inequalities are satisfied with equalities.

The above theorem can be used to decide whether there's a binary matrix with given row sum and column sum vector or not, and also to decide whether the solution is unique or not. However it tells nothing about how to find such matrix. If we know that there's a solution of the problem then the following procedure can be used.

Let $R = (r_1, r_2, \ldots, r_m)$ and $S = (s_1, s_2, \ldots, s_n)$ be two compatible integer vectors that satisfy the condition

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j \qquad \text{for all } k \in \{1, 2, \ldots, n\}$$

Let the matrix A be equal to the maximal matrix $\overline{A}$ at the beginning of the procedure. Then

1. If there's no column index k such that s'_k is larger than the sum of the elements of A in the k-th column, then go to step 6.

2. Choose the largest column index k such that s'_k is larger than the sum of the elements of A in the k-th column.

3. Choose the largest column index l, such that it's less than k and the l-th column of A contains at least one nonzero element.

4. Let i denote the largest row index such that $a_{i,l} = 1$. Then change the value of $a_{i,l}$ to zero and change the value of $a_{i,k}$ to one (which must be zero before the change if the above conditions are satisfied).

5. Repeat steps (2)-(4) until there's no column index k such that s'_k is larger than the sum of the elements of A in the k-th column.

6. Now the matrix A must have row sum vector R and column sum vector S'. Find a permutation that transforms S' into S and apply the same permutation for the columns of A.

By the end of the above procedure A has row sum vector R and column sum vector S.

Example 1

Let $R = (2, 6, 4, 3, 3)$ and $S = (3, 4, 6, 2, 2, 1)$. Then there's no binary matrix of size 5×6 which has row sum vector R and column sum vector S, because S has an element larger than 5, and hence the vectors R and S are not compatible.

Example 2

Let $R = (5, 4, 2, 3, 3)$ and $S = (2, 3, 1, 4, 4, 2)$. Then there's no binary matrix of size 5×6 which has row sum vector R and column sum vector S, because

$$\sum_{i=1}^{5} r_i = 5 + 4 + 2 + 3 + 3 = 17 \quad \text{and} \quad \sum_{j=1}^{6} s_j = 2 + 3 + 1 + 4 + 4 + 2 = 16$$

and hence the vectors R and S are not compatible.

Example 3

Let $R = (3, 5, 2, 3, 1)$ and $S = (0, 2, 4, 4, 4, 0)$. Then R and S are compatible and the maximal matrix corresponding to R is

$$\overline{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus $S' = (4,4,4,2,0,0)$ and $\overline{S} = (5,4,3,1,1,0)$. Here

$$\begin{aligned}
4 \le 5 &\implies s'_1 \le \overline{s}_1 \\
4+4 \le 5+4 &\implies s'_1 + s'_2 \le \overline{s}_1 + \overline{s}_2 \\
4+4+4 \le 5+4+3 &\implies s'_1 + s'_2 + s'_3 \le \overline{s}_1 + \overline{s}_2 + \overline{s}_3 \\
4+4+4+2 > 5+4+3+1 &\implies s'_1 + s'_2 + s'_3 + s'_4 > \overline{s}_1 + \overline{s}_2 + \overline{s}_3 + \overline{s}_4
\end{aligned}$$

Hence there's no binary matrix of size 5×6 which has row sum vector R and column sum vector S, because

$$\sum_{j=1}^{k} s'_j \le \sum_{j=1}^{k} \overline{s}_j$$

is not satisfied for all $k \in \{1, 2, \ldots, 6\}$.

Example 4

Let $R = (2,3,5,5,3)$ and $S = (1,3,4,5,4,1)$. Then R and S are compatible and the maximal matrix corresponding to R is

$$\overline{A} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Thus $S' = (5,4,4,3,1,1)$ and $\overline{S} = (5,5,4,2,2,0)$. Here

$$\begin{aligned}
5 &\le 5 \\
5+4 &\le 5+5 \\
5+4+4 &\le 5+5+4 \\
5+4+4+3 &\le 5+5+4+2 \\
5+4+4+3+1 &\le 5+5+4+2+2 \\
5+4+4+3+1+1 &\le 5+5+4+2+2+0
\end{aligned}$$

Hence

$$\sum_{j=1}^{k} s'_j \le \sum_{j=1}^{k} \overline{s}_j$$

is satisfied for all $k \in \{1, 2, \ldots, 6\}$, and there exists a binary matrix of size 5×6 which has row sum vector R and column sum vector S. This matrix is not unique, because some of the above inequalities are strict. Now let's find a binary matrix with row sum vector R and column sum vector S.

$$\begin{matrix}\begin{pmatrix}1&1&0&0&0&0\\1&1&1&0&0&0\\1&1&1&1&1&0\\1&1&1&1&1&0\\1&1&1&0&0&0\end{pmatrix}\\ \begin{matrix}5&4&4&3&1&1\end{matrix}\end{matrix} \longrightarrow \begin{matrix}\begin{pmatrix}1&1&0&0&0&0\\1&1&1&0&0&0\\1&1&1&1&1&0\\1&1&1&1&0&1\\1&1&1&0&0&0\end{pmatrix}\\ \begin{matrix}5&4&4&3&1&1\end{matrix}\end{matrix} \longrightarrow$$

$$\longrightarrow \begin{matrix}\begin{pmatrix}1&1&0&0&0&0\\1&1&1&0&0&0\\1&1&1&1&1&0\\1&1&1&1&0&1\\1&1&0&1&0&0\end{pmatrix}\\ \begin{matrix}5&4&4&3&1&1\end{matrix}\end{matrix} \longrightarrow \begin{matrix}\begin{pmatrix}1&1&0&0&0&0\\1&1&1&0&0&0\\1&1&1&1&1&0\\1&1&1&1&0&1\\1&0&1&1&0&0\end{pmatrix}\\ \begin{matrix}5&4&4&3&1&1\end{matrix}\end{matrix}$$

Now the last matrix has row sum vector R and column sum vector S'. Let's permute the columns to get column sum vector S.

$$\begin{matrix}\begin{pmatrix}1&1&0&0&0&0\\1&1&1&0&0&0\\1&1&1&1&1&0\\1&1&1&1&0&1\\1&0&1&1&0&0\end{pmatrix}\\ \begin{matrix}5&4&4&3&1&1\end{matrix}\end{matrix} \longrightarrow \begin{matrix}\begin{pmatrix}0&0&0&1&1&0\\0&0&1&1&1&0\\1&1&1&1&1&0\\0&1&1&1&1&1\\0&1&1&1&0&0\end{pmatrix}\\ \begin{matrix}1&3&4&5&4&1\end{matrix}\end{matrix}$$

The matrix on the right has row sum vector R and column sum vector S.

Example 5

Let $R = (6, 3, 4, 2, 2)$ and $S = (1, 2, 5, 5, 3, 1)$. Then R and S are compatible and the maximal matrix corresponding to R is

$$\overline{A} = \begin{pmatrix}1&1&1&1&1&1\\1&1&1&0&0&0\\1&1&1&1&0&0\\1&1&0&0&0&0\\1&1&0&0&0&0\end{pmatrix}$$

Thus $S' = (5,5,3,2,1,1)$ and $\overline{S} = (5,5,3,2,1,1)$. Here $S' = \overline{S}$, and hence

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j$$

is satisfied for all $k \in \{1,2,\ldots,6\}$, and there exists a binary matrix of size 5×6 which has row sum vector R and column sum vector S. This matrix is unique, because the above inequalities are all satisfied with equalities. To find a binary matrix with row sum vector R and column sum vector S it's enough to permute the columns of the maximal matrix.

$$\begin{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 5 & 5 & 3 & 2 & 1 & 1 \end{matrix} \end{matrix} \longrightarrow \begin{matrix} \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 1 & 2 & 5 & 5 & 3 & 1 \end{matrix} \end{matrix}$$

The matrix on the right has row sum vector R and column sum vector S.