1 Discrete Tomography

1.1 Reconstruction of binary matrices with prescribed row and column sums

Exercise 1.1.1 *Is there a binary matrix of size* 6×5 *with the row sum vector*

$$R = (r_1, r_2, r_3, r_4, r_5, r_6) = (0, 2, 4, 3, 3, 1)\,,$$

and column sum vector

$$S = (s_1, s_2, s_3, s_4, s_5) = (4, 3, 1, 2, 3)?$$

If yes find such matrix. Is the solution unique?

Solution. The vectors R and S are called compatible if

- no element of R is larger than the number of columns,
- no element of S is larger than the number of rows,
- if the sum of the elements of R equals to the sum of the elements of S.

If R and S are not compatible, then there's no solution of the problem. The number of rows is 6 and the number of columns is 5 now, hence we can clearly see that the first two properties hold, and furthermore

$$\sum_{i=1}^{6} r_i = 0 + 2 + 4 + 3 + 3 + 1 = 13, \quad \sum_{j=1}^{5} s_j = 4 + 3 + 1 + 2 + 3 = 13,$$

thus the vectors R and S are compatible. The maximal matrix corresponding to the row sum vector R is the matrix, whose first r_i elements of the i-th row equal to 1, while the further elements equal to 0, for each row index i. Now the maximal matrix is

$$\overline{A} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The column sum vector of the maximal matrix is

$$\overline{S} = (\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4, \overline{s}_5) = (5, 4, 3, 1, 0).$$

The non-increasing permutation of the column sum vector S is

$$S' = \left(s'_1, s'_2, s'_3, s'_4, s'_5\right) = (4, 3, 3, 2, 1).$$

There exists a binary matrix with row sum vector R and column sum vector S if and only if

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j \qquad \text{for all } k \in \{1, 2, \ldots, 5\}$$

Furthermore this binary matrix is unique if and only if all the above inequalities are satisfied with equalities. Now

$$\begin{aligned}
\sum_{j=1}^{1} s'_j = s'_1 = 4, &\quad \text{and} \quad \sum_{j=1}^{1} \overline{s}_j = \overline{s}_1 = 5, \\
\sum_{j=1}^{2} s'_j = s'_1 + s'_2 = 7, &\quad \text{and} \quad \sum_{j=2}^{1} \overline{s}_j = \overline{s}_1 + \overline{s}_2 = 9, \\
\sum_{j=1}^{3} s'_j = s'_1 + s'_2 + s'_3 = 10, &\quad \text{and} \quad \sum_{j=3}^{1} \overline{s}_j = \overline{s}_1 + \overline{s}_2 + \overline{s}_3 = 12, \\
\sum_{j=1}^{4} s'_j = s'_1 + s'_2 + s'_3 + s'_4 = 12, &\quad \text{and} \quad \sum_{j=4}^{1} \overline{s}_j = \overline{s}_1 + \overline{s}_2 + \overline{s}_3 + \overline{s}_4 = 13, \\
\sum_{j=1}^{5} s'_j = 13, &\quad \text{and} \quad \sum_{j=5}^{1} \overline{s}_j = 13.
\end{aligned}$$

Hence

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j$$

is satisfied for all $k \in \{1, 2, \ldots, 5\}$, and there exists a binary matrix of size 6×5 which has row sum vector R and column sum vector S. This matrix is not unique, because some of the above inequalities are strict. Now let's find a binary matrix with row sum vector R and column sum vector S.

First we start with the maximal matrix A and denote the elements of S' below the columns. These are the desired column sums. Then the steps of the method are demonstrated by the following sequence of matrices:

$$\overline{A} = \begin{array}{c}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 4 & 3 & 3 & 2 & 1 \end{matrix}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 4 & 3 & 3 & 2 & 1 \end{matrix}\end{array} \longrightarrow$$

$$\begin{array}{c}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 4 & 3 & 3 & 2 & 1 \end{matrix}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 4 & 3 & 3 & 2 & 1 \end{matrix}\end{array} \longrightarrow$$

$$\begin{array}{c}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 4 & 3 & 3 & 2 & 1 \end{matrix}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \\ \begin{matrix} 4 & 3 & 3 & 2 & 1 \end{matrix}\end{array} \longrightarrow$$

$$\begin{array}{c}\begin{pmatrix}0&0&0&0&0\\1&1&0&0&0\\1&1&1&0&1\\1&0&1&1&0\\1&0&1&1&0\\0&1&0&0&0\end{pmatrix}\\ \begin{matrix}4&3&3&2&1\end{matrix}\end{array}$$

The steps performed above are the following:

1. The column sum of the last column of the first matrix is 0, while the desired values is 1. Thus, we look for the largest column index, which is before the last column and contains at least one nonzero element. This is the fourth column. Then we choose an element of the fourth column, which equals to 1 and the element of last column in the same row is 0. Now there's only one such element in the fourth column, but later if there are more options we prefer the 1 with the largest row index. Then we push this 1 from the fourth column to the last column, leaving it in the same row. This step doesn't change the rows sums, but makes the column sum of the last column equal to the desired number.

2. Now the last column of the second matrix, which doesn't have the desired column sum is the fourth column. Thus, we look for the largest column index, which is before the fourth column and contains at least one nonzero element. This is the third column. Then we choose an element of the third column, which equals to 1 and the element of fourth column in the same row is 0. Now there are three such elements, and we choose the last one in the fifth row. Then we push this 1 from the third column to the fourth column, leaving it in the same row. The result of this step is the third matrix.

3. In the third matrix the problem of the fourth column sum is not resolved yet, thus we repeat the previous step, but this time pushing the fourth element of the third column to the fourth column.

4. Now the last column of the fourth matrix, which doesn't have the desired column sum is the third column. Thus, we look for the largest column index, which is before the third column and contains at least one nonzero element. This is the second column. Then we choose an

element of the second column, which equals to 1 and the element of third column in the same row is 0. Now there are three such elements, and we choose the last one in the fifth row. Then we push this 1 from the second column to the third column, leaving it in the same row.

5. In the fifth matrix the problem of the third column sum is not resolved yet, thus we repeat the previous step, but this time pushing the fourth element of the second column to the third column.

6. The last column of the sixth matrix, which doesn't have the desired column sum is the second column. Then we choose an element of the first column, which equals to 1 and the element of second column in the same row is 0. There are three such elements, and we choose the last one in the sixth row. Then we push this 1 from the first column to the second column, leaving it in the same row. This step doesn't change the rows sums, but makes all the column sums of the matrix equal to the desired numbers.

Now the last matrix has row sum vector R and column sum vector S'. Let's permute the columns to get column sum vector S.

$$\begin{array}{c}\begin{pmatrix}0&0&0&0&0\\1&1&0&0&0\\1&1&1&0&1\\1&0&1&1&0\\1&0&1&1&0\\0&1&0&0&0\end{pmatrix}\\ \begin{matrix}4&3&3&2&1\end{matrix}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix}0&0&0&0&0\\1&1&0&0&0\\1&1&1&0&1\\1&0&0&1&1\\1&0&0&1&1\\0&1&0&0&0\end{pmatrix}\\ \begin{matrix}4&3&1&2&3\end{matrix}\end{array}$$

The matrix on the right has row sum vector R and column sum vector S. ■

Exercise 1.1.2 *Is there a binary matrix of size* 5×7 *with the row sum vector*

$$R = (r_1, r_2, r_3, r_4, r_5) = (4, 3, 5, 2, 4),$$

and column sum vector

$$S = (s_1, s_2, s_3, s_4, s_5, s_6, s_7) = (2, 4, 1, 4, 3, 2, 2)?$$

If yes find such matrix. Is the solution unique?

Solution. No element of R is larger than 7 and no element of S is larger than 5. Furthermore

$$\sum_{i=1}^{5} r_i = 4+3+5+2+4 = 18, \quad \sum_{j=1}^{7} s_j = 2+4+1+4+3+2+2 = 18,$$

thus the vectors R and S are compatible. The maximal matrix corresponding to the row sum vector R is

$$\overline{A} = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The column sum vector of the maximal matrix is

$$\overline{S} = (\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4, \overline{s}_5, \overline{s}_6, \overline{s}_7) = (5,5,4,3,1,0,0).$$

The non-increasing permutation of the column sum vector S is

$$S' = \left(s'_1, s'_2, s'_3, s'_4, s'_5, s'_6, s'_7\right) = (4,4,3,2,2,2,1).$$

There exists a binary matrix with row sum vector R and column sum vector S if and only if

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j \qquad \text{for all } k \in \{1,2,\ldots,7\}$$

Furthermore this binary matrix is unique if and only if all the above inequalities are satisfied with equalities. Now

$$\begin{aligned}
\sum_{j=1}^{1} s'_j = s'_1 = 4, &\quad \text{and} \quad \sum_{j=1}^{1} \overline{s}_j = \overline{s}_1 = 5,\\
\sum_{j=1}^{2} s'_j = 8, &\quad \text{and} \quad \sum_{j=2}^{1} \overline{s}_j = 10,\\
\sum_{j=1}^{3} s'_j = 11, &\quad \text{and} \quad \sum_{j=3}^{1} \overline{s}_j = 14,\\
\sum_{j=1}^{4} s'_j = 13, &\quad \text{and} \quad \sum_{j=4}^{1} \overline{s}_j = 17,\\
\sum_{j=1}^{5} s'_j = 15, &\quad \text{and} \quad \sum_{j=5}^{1} \overline{s}_j = 18,\\
\sum_{j=1}^{5} s'_j = 17, &\quad \text{and} \quad \sum_{j=5}^{1} \overline{s}_j = 18,\\
\sum_{j=1}^{5} s'_j = 18, &\quad \text{and} \quad \sum_{j=5}^{1} \overline{s}_j = 18.
\end{aligned}$$

Hence

$$\sum_{j=1}^{k} s'_j \le \sum_{j=1}^{k} \overline{s}_j$$

is satisfied for all $k \in \{1, 2, \ldots, 7\}$, and there exists a binary matrix of size 5×7, which has row sum vector R and column sum vector S. This matrix is not unique, because some of the above inequalities are strict. Now let's find a binary matrix with row sum vector R and column sum vector S.

First we start with the maximal matrix A and denote the elements of S' below the columns. These are the desired column sums. Then the steps of the method are demonstrated by the following sequence of matrices:

$$\overline{A} = \begin{array}{c}\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&0&0&0\\1&1&1&1&1&0&0\\1&1&0&0&0&0&0\\1&1&1&1&0&0&0\end{pmatrix}\\ \begin{matrix}4&4&3&2&2&2&1\end{matrix}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&0&0&0\\1&1&1&1&0&0&1\\1&1&0&0&0&0&0\\1&1&1&1&0&0&0\end{pmatrix}\\ \begin{matrix}4&4&3&2&2&2&1\end{matrix}\end{array} \longrightarrow$$

$$\begin{array}{c}\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&0&0&0\\1&1&1&1&0&0&1\\1&1&0&0&0&0&0\\1&1&1&0&0&1&0\end{pmatrix}\\ \begin{array}{ccccccc}4&4&3&2&2&2&1\end{array}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix}1&1&1&1&0&0&0\\1&1&1&0&0&0&0\\1&1&1&0&0&1&1\\1&1&0&0&0&0&0\\1&1&1&0&0&1&0\end{pmatrix}\\ \begin{array}{ccccccc}4&4&3&2&2&2&1\end{array}\end{array} \longrightarrow$$

$$\begin{array}{c}\begin{pmatrix}1&1&1&0&1&0&0\\1&1&1&0&0&0&0\\1&1&1&0&0&1&1\\1&1&0&0&0&0&0\\1&1&1&0&0&1&0\end{pmatrix}\\ \begin{array}{ccccccc}4&4&3&2&2&2&1\end{array}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix}1&1&1&0&1&0&0\\1&1&1&0&0&0&0\\1&1&1&0&0&1&1\\1&1&0&0&0&0&0\\1&1&0&0&1&1&0\end{pmatrix}\\ \begin{array}{ccccccc}4&4&3&2&2&2&1\end{array}\end{array} \longrightarrow$$

$$\begin{array}{c}\begin{pmatrix}1&1&1&0&1&0&0\\1&1&1&0&0&0&0\\1&1&0&1&0&1&1\\1&1&0&0&0&0&0\\1&1&0&0&1&1&0\end{pmatrix}\\ \begin{array}{ccccccc}4&4&3&2&2&2&1\end{array}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix}1&1&1&0&1&0&0\\1&1&0&1&0&0&0\\1&1&0&1&0&1&1\\1&1&0&0&0&0&0\\1&1&0&0&1&1&0\end{pmatrix}\\ \begin{array}{ccccccc}4&4&3&2&2&2&1\end{array}\end{array} \longrightarrow$$

$$\begin{array}{c}\begin{pmatrix}1&1&1&0&1&0&0\\1&1&0&1&0&0&0\\1&1&0&1&0&1&1\\1&1&0&0&0&0&0\\1&0&1&0&1&1&0\end{pmatrix}\\ \begin{array}{ccccccc}4&4&3&2&2&2&1\end{array}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix}1&1&1&0&1&0&0\\1&1&0&1&0&0&0\\1&1&0&1&0&1&1\\1&0&1&0&0&0&0\\1&0&1&0&1&1&0\end{pmatrix}\\ \begin{array}{ccccccc}4&4&3&2&2&2&1\end{array}\end{array} \longrightarrow$$

$$\begin{array}{c}\begin{pmatrix}1&1&1&0&1&0&0\\1&1&0&1&0&0&0\\1&1&0&1&0&1&1\\1&0&1&0&0&0&0\\0&1&1&0&1&1&0\end{pmatrix}\\ \begin{array}{ccccccc}4&4&3&2&2&2&1\end{array}\end{array}$$

Now the last matrix has row sum vector R and column sum vector S'. Let's permute the columns to get column sum vector S.

$$\begin{array}{c}\begin{pmatrix}1&1&1&0&1&0&0\\1&1&0&1&0&0&0\\1&1&0&1&0&1&1\\1&0&1&0&0&0&0\\0&1&1&0&1&1&0\end{pmatrix}\\ \begin{matrix}4&4&3&2&2&2&1\end{matrix}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix}0&1&0&1&1&0&1\\1&1&0&1&0&0&0\\1&1&1&1&0&1&0\\0&0&0&1&1&0&0\\0&1&0&0&1&1&1\end{pmatrix}\\ \begin{matrix}2&4&1&4&3&2&2\end{matrix}\end{array}$$

■

Exercise 1.1.3 *Is there a binary matrix of size* 4×6 *with the row sum vector*

$$R = (r_1, r_2, r_3, r_4) = (4, 6, 3, 2)\,,$$

and column sum vector

$$S = (s_1, s_2, s_3, s_4, s_5, s_6) = (3, 2, 4, 3, 3, 0)?$$

If yes find such matrix. Is the solution unique?

Solution. No element of R is larger than 6 and no element of S is larger than 4. Furthermore

$$\sum_{i=1}^{5} r_i = 4+6+3+2 = 15, \quad \sum_{j=1}^{7} s_j = 3+2+4+3+3+0 = 15,$$

thus the vectors R and S are compatible. The maximal matrix corresponding to the row sum vector R is

$$\overline{A} = \begin{pmatrix}1&1&1&1&0&0\\1&1&1&1&1&1\\1&1&1&0&0&0\\1&1&0&0&0&0\end{pmatrix}$$

The column sum vector of the maximal matrix is

$$\overline{S} = (\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4, \overline{s}_5, \overline{s}_6) = (4, 4, 3, 2, 1, 1).$$

The non-increasing permutation of the column sum vector S is

$$S' = \left(s'_1, s'_2, s'_3, s'_4, s'_5, s'_6\right) = (4, 3, 3, 3, 2, 0).$$

There exists a binary matrix with row sum vector R and column sum vector S if and only if

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j \qquad \text{for all } k \in \{1, 2, \ldots, 6\}$$

Furthermore this binary matrix is unique if and only if all the above inequalities are satisfied with equalities. Now

$$\begin{aligned}
\sum_{j=1}^{1} s'_j = s'_1 = 4, &\quad \text{and} \quad \sum_{j=1}^{1} \overline{s}_j = \overline{s}_1 = 4, \\
\sum_{j=1}^{2} s'_j = 7, &\quad \text{and} \quad \sum_{j=2}^{1} \overline{s}_j = 8, \\
\sum_{j=1}^{3} s'_j = 10, &\quad \text{and} \quad \sum_{j=3}^{1} \overline{s}_j = 11, \\
\sum_{j=1}^{4} s'_j = 13, &\quad \text{and} \quad \sum_{j=4}^{1} \overline{s}_j = 13, \\
\sum_{j=1}^{5} s'_j = 15, &\quad \text{and} \quad \sum_{j=5}^{1} \overline{s}_j = 14, \\
\sum_{j=1}^{6} s'_j = 15, &\quad \text{and} \quad \sum_{j=6}^{1} \overline{s}_j = 15,
\end{aligned}$$

Hence

$$\sum_{j=1}^{5} s'_j > \sum_{j=5}^{1} \overline{s}_j$$

and there's no binary matrix of size 4×6, which has row sum vector R and column sum vector S. ■

Exercise 1.1.4 *Is there a binary matrix of size* 6×6 *with the row sum vector*

$$R = (r_1, r_2, r_3, r_4, r_5, r_6) = (3, 2, 4, 6, 1, 3),$$

and column sum vector

$$S = (s_1, s_2, s_3, s_4, s_5, s_6) = (2, 1, 5, 6, 1, 4)?$$

If yes find such matrix. Is the solution unique?

Solution. No element of R is larger than 6 and no element of S is larger than 6. Furthermore

$$\sum_{i=1}^{6} r_i = 3+2+4+6+1+3 = 19, \quad \sum_{j=1}^{6} s_j = 2+1+5+6+1+4 = 19,$$

thus the vectors R and S are compatible. The maximal matrix corresponding to the row sum vector R is

$$\overline{A} = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

The column sum vector of the maximal matrix is

$$\overline{S} = (\overline{s}_1, \overline{s}_2, \overline{s}_3, \overline{s}_4, \overline{s}_5, \overline{s}_6) = (6,5,4,2,1,1).$$

The non-increasing permutation of the column sum vector S is

$$S' = \left(s'_1, s'_2, s'_3, s'_4, s'_5, s'_6\right) = (6,5,4,2,1,1).$$

There exists a binary matrix with row sum vector R and column sum vector S if and only if

$$\sum_{j=1}^{k} s'_j \leq \sum_{j=1}^{k} \overline{s}_j \qquad \text{for all } k \in \{1,2,\ldots,6\}$$

Furthermore this binary matrix is unique if and only if all the above inequalities are satisfied with equalities. Now we see that the vectors $\overline{S}$ and S' are identical, thus the above inequalities are all satisfied with equalities. Hence there exists a binary matrix of size 6×6 which has row sum vector R and column sum vector S and this matrix is unique. Since the column sum vector of the maximal matrix is S', we only need to permute the columns of the maximal matrix to find the binary matrix with column sum vector S.

$$\begin{array}{c}\begin{pmatrix}1 & 1 & 1 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 1 & 1 & 1\\ 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0\end{pmatrix}\\ \begin{matrix}6 & 5 & 4 & 2 & 1 & 1\end{matrix}\end{array} \longrightarrow \begin{array}{c}\begin{pmatrix}0 & 0 & 1 & 1 & 0 & 1\\ 0 & 0 & 1 & 1 & 0 & 0\\ 1 & 0 & 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 1 & 1 & 1\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 1 & 0 & 1\end{pmatrix}\\ \begin{matrix}2 & 1 & 5 & 6 & 1 & 4\end{matrix}\end{array}$$

$\blacksquare$