4 Discrete tomography

In the previous chapters we discussed how to reconstruct an unknown picture function (which may represent different kinds of physical quantities) if know its line integrals along a finite set of lines. It was a continuous problem in the sense that the value of the unknown picture function could be any real number within a certain interval (e.g. $[0,1]$). Furthermore the support of the picture function (i.e. the set of those points of the picture region where the value of picture function is nonzero) contained infinitely many points and could have any shape. However, in some of the examples, we investigated the binary solutions of the problem, which are solutions that are either constant zero or constant one over each pixel of a digitized picture. Thus the range of the solution function was a finite subset of the reals, and finite sets are also called discrete sets. We deal with **discrete tomography** when we assume that the range (i.e. the set of possible values) of the unknown picture function is a given finite, or in other words discrete, set of the reals, and thus we are interested only in solutions that take values from this given finite set. The finite set can be for example the set $\{0,1\}$, and then we say that we have an unknown binary picture function, and we accept only binary solutions. The problem of the reconstruction of an unknown binary picture function, whose support can be any subset of the picture region, is part of the so called **geometric tomography**. The word geometric refers to the fact that the unknown picture function has binary values and thus we are only interested in the shape of its support, where it takes the nonzero values, as it characterizes the binary picture function. The assumption of a binary picture function is quite reasonable in some real life situations, when we try to reconstruct the inner structure of a homogeneous object completely made out of the same material, such as a turbine blade. In such situation we are interested in only whether there's a hidden crack or hole inside the object, which is just the lack of material in some positions, and that's determined only by the shape of the object.

Geometric tomography can be considered as part of discrete tomography for the range of the unknown picture function is discrete in its problems. However most authors, considering discrete tomography, also assumes that the picture region is a finite subset of a lattice, and thus the unknown picture function is defined over a finite set. This is a reasonable assumption for example in crystallography when we wish to reconstruct the arrangement of atoms in crystalline solids, and atoms may appear only in restricted positions, but this can reasonable even in computed tomography to simplify computations. It's very typical in science, that a simplified model is used to reduce the amount of computation necessary to answer a problem. For example, when computing the orbit of a satellite, all the mass of the Earth is considered to be concentrated into its center. This assumption naturally fails in real life, but it can be reasonable as long as the computations on the orbit remain valid (or at least acceptable). We need this simplification, because having a procedure, which requires the knowledge of the position of every single fly on Earth, would be quite useless. There's a similar thing in computed tomography. If the size of the pixels in a digitization of the picture region compared to the size of the picture region is small, then the different lengths of the intersection of a line with the different pixels are not that important anymore. What's really important is which pixels are intersected, but not the length of these intersections. Thus it's the same as assuming that the pixels are concentrated in their centers and we can only use lines that intersect at least one of the centers. The fact that any picture function can be well approximated by a digitized picture (if the number of pixels is large enough) gives the assumption that unknown picture function itself is digitized, and hence can be considered to be an unknown finite subset of a lattice (as the centers of the pixels form a lattice).

Let's discuss now discrete tomography in details. First of all, a **lattice** in the plane is the set of all linear combinations with integer coefficients of two linearly independent vectors of the plane. Since every lattice can be transformed into the lattice $\mathbb{Z}^2$ by a nonsingular linear transformation, we can always assume in discrete tomography, that we have the lattice $\mathbb{Z}^2$. Here $\mathbb{Z}^2$ is the set of all ordered pairs of integer numbers, which can be identified as the set of all points in the plane, whose Cartesian coordinates are integers. If $\underline{v}$ is a two dimensional vector with integer Cartesian coordinates (a, b), and these coordinates have a common divisor, which is larger than 1, then dividing both coordinates with the common divisor gives another vector with integer coordinates, which defines the same direction. Note that two vectors define the same direction if they are parallel to each other, or in other words when they are linearly dependent, which happens exactly, if they are proportional to each other. Furthermore if a vector is multiplied by -1, then it still defines the same direction. Thus, to characterize all possible directions in a lattice, it's enough to collect all those two-dimensional integer vectors, whose first component is non-negative and whose Cartesian coordinates are coprime numbers (i.e. integers with greatest common divisor equal to zero). Such vector is called **lattice direction**. This definition still includes both $(0,1)$ and $(0,-1)$ as lattice directions, despite they are parallel to each other. Hence we agree that only the vector $(0,1)$ is considered as lattice direction. A **lattice line** is a line, which is parallel to a lattice direction, and passes through at least one point of $\mathbb{Z}^2$. A **lattice set** is a finite subset of the lattice $\mathbb{Z}^2$, while the picture region is a rectangle $[a,b] \times [c,d]$, where a, b, c, d are all integers.

Definition 10 *Let $D = \left(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_q\right)$ a sequence of distinct lattice directions, and let $\mathcal{L}_k$ denote the set of all lattice lines parallel to $\underline{v}_k$ and intersecting at least one point of $\mathbb{Z}^2$ within a given picture region $R = [a,b] \times [c,d]$, for all $k = 1, 2, \ldots, q$. The* ***X-ray of the picture function*** *$f \colon R \cap \mathbb{Z}^2 \to \mathbb{R}$ parallel to $\underline{v}_k$ is the function*

$$P_{k,f} \colon \mathcal{L}_k \to \mathbb{R}, \qquad P_{k,f}(l) = \sum_{(x,y) \in l \cap R \cap \mathbb{Z}^2} f(x,y)$$

The ***X-ray of the lattice set*** *$F \subset R$ parallel to $\underline{v}_k$ is the function*

$$P_{k,F} \colon \mathcal{L}_k \to \mathbb{Z}, \qquad P_{k,F}(l) = |F \cap l|$$

where $|F \cap l|$ denotes the number of elements in the set $F \cap l$.

An example of a lattice set and line set is shown in Figure 4.1. The picture region is the rectangle $[2,8] \times [1,5]$. The points of the lattice set F are presented by solid dots, while the rest of the points of the integer lattice $\mathbb{Z}^2$ inside the picture region are presented by empty dots. The red lines show the line set $\mathcal{L} = \{l_1, l_2, \ldots, l_{17}\}$, where all lines are parallel to the lattice direction $\underline{v} = (1,2)$, and l_i passes through the point $\left(\frac{i-2}{2}, 0\right)$ for all $i = 1, 2, \ldots, 17$. The

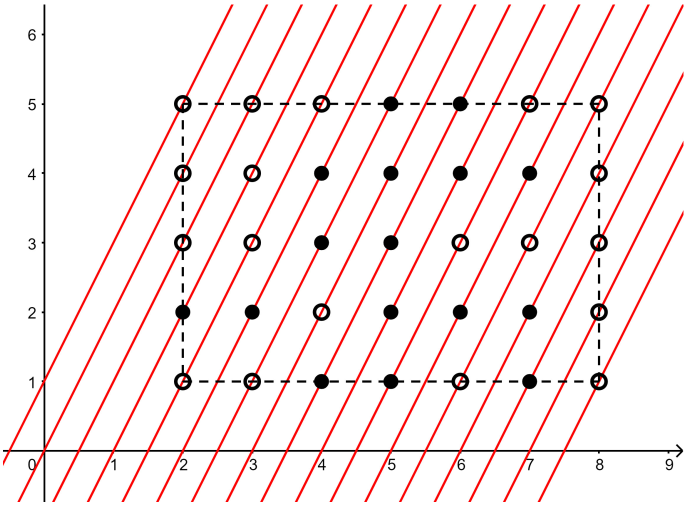


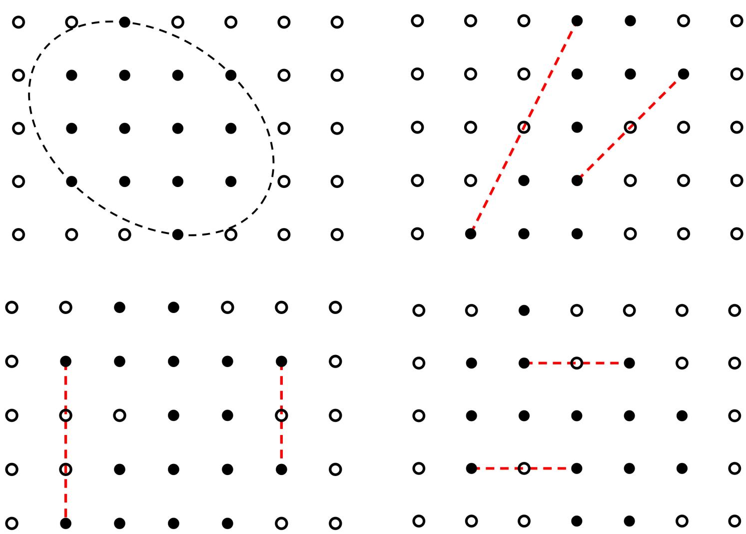
Figure 4.1

X-ray of the lattice set is the function $P_F\colon \mathcal{L} \to \mathbb{Z}$ with values

$$\begin{array}{ccccc}
P_F(l_1) = 0 & P_F(l_2) = 0 & P_F(l_3) = 0 & P_F(l_4) = 1 & P_F(l_5) = 1 \\
P_F(l_6) = 0 & P_F(l_7) = 2 & P_F(l_8) = 2 & P_F(l_9) = 1 & P_F(l_{10}) = 1 \\
P_F(l_{11}) = 3 & P_F(l_{12}) = 2 & P_F(l_{13}) = 1 & P_F(l_{14}) = 2 & P_F(l_{15}) = 1 \\
P_F(l_{16}) = 3 & P_F(l_{17}) = 2 & & &
\end{array}$$

Note that both the X-ray of a picture function and the X-ray of a lattice set are functions defined on the set of lines L_k. Given a line l, the X-ray of the picture function f, defined on the lattice points of the picture region, assigns the sum of the values of the picture function along the line l. The X-ray of the lattice set F assigns the number of points of F, which lie on the line l. Hence the X-ray of a lattice set F is just the X-ray of the picture function, which equals to 1 in the points of F and equals to 0 everywhere else. It shows that the problem of reconstruction of an unknown lattice set from its X-rays is just a special case of the problem of reconstruction of an unknown picture function from its X-rays, however it's a very important case, which can be much harder to solve. To illustrate this just think about that solving the equation $x^2 + y^2 = z^3$ for the unknowns x, y, z over the set of the real numbers is easy. Just choose arbitrary values of x and y, and then take $z = \sqrt[3]{x^2 + y^2}$, but to give all the integer solutions of the above equations is much more challenging. It's a similar situation with the reconstruction of lattice sets, which requires more advanced techniques to solve, than the reconstruction of arbitrary picture functions defined over lattice sets.

Now we define some classes of lattice sets. First a set H (not necessarily discrete) in the plane is called **convex** if, for any two points $A, B \in H$, all the points of the line segment AB are points of H. A solid disc and a solid rectangle are a convex sets, while there are non-convex quadrilaterals. A lattice set F is called convex if there exists a convex set H in the plane such that $F = H \cap \mathbb{Z}^2$. In other words a lattice set F is convex if, for any two points $A, B \in F$, all the points of the line segment AB intersected by the lattice $\mathbb{Z}^2$ are elements of F. Given a Cartesian coordinate system in the plane, a set H is called **vertically convex** if for any two points $A, B \in H$ with equal first components, all the points of the line segment AB are points of H. A set H is called **horizontally convex** if for any two points $A, B \in H$ with equal second components, all the points of the line segment AB are points of H. A set, which is horizontally and vertically convex at the same time, is shortly called **hv-convex**. Horizontally convex, vertically convex and hv-convex lattice sets can be defined in a similar manner as in the case of convex lattice sets. It's clear from the definition, that every convex (lattice) set is hv-convex, but there are hv-convex sets, which are not convex. We show a few examples below.



The lattice set in the upper left corner is convex, the one in the upper right corner is hv-convex, but not convex. The example in the lower left corner is horizontally convex, but not vertically convex, while the set in the lower right corner is vertically convex, but not horizontally convex. Further types of lattice sets can be defined such as 4-connected and 8-connected sets, but we don't need these. Convex and hv-convex set are also presented only to have an idea what classes of lattice sets can be defined.

Let $D = \left(\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_q\right)$ a sequence of distinct lattice directions. We say that two lattice sets F and F' are **tomographically equivalent** with respect to the directions D, if the X-rays of F and F' are the same for all $k = 1, 2, \ldots q$. This means we can't distinguish F and F' upon their X-rays parallel to the directions in D. Let $\mathcal{E}$ be a class of lattice sets in the plane, such as the class of convex sets, hv-convex set, or 4-connected sets. Then the lattice set $F \in \mathcal{E}$ is **determined** by its X-rays parallel to D in the class $\mathcal{E}$, if there's no other set in $\mathcal{E}$, which is tomographically equivalent to F with respect to the directions in D. The role of the class $\mathcal{E}$ is important, since a convex lattice set may be determined in the class of convex sets, despite having a tomographically equivalent non-convex alternative, and hence not being determined in the class of all lattice sets.

Let $\mathcal{L}_k$ denote the sets of lines just as in Definition 10 for all $k = 1, 2, \ldots, q$. There are three basic problems which are heavily investigated in discrete tomography: consistency, uniqueness and reconstruction. The **problem of consistency** is about to find an answer to the question: Given the functions $p_k \colon \mathcal{L}_k \to \mathbb{Z}$, is there a lattice set $F \in \mathcal{E}$, such that the X-ray of F parallel to $\underline{v}_k$ equals to p_k for all $k = 1, 2, \ldots, q$? The **problem of uniqueness** is about to find an answer to the question: Is the lattice set $F \in \mathcal{E}$ determined by its X-rays parallel to D in the class $\mathcal{E}$? Given the functions $p_k \colon \mathcal{L}_k \to \mathbb{Z}$, the **problem of reconstruction** is about to construct a lattice set $F \in \mathcal{E}$, such that the X-ray of F parallel to $\underline{v}_k$ equals to p_k for all $k = 1, 2, \ldots, q$.

All these three problems can be answered easily if the number of distinct directions is $q = 2$, but they are much harder if $q \geq 3$. In the next section we present an elegant solution to the three basic problems, if the two X-rays parallel to the coordinate directions are taken. Later a network-flow algorithm is used to solve them in the case of two arbitrary directions.