## 3.4 Series expansion method

The idea behind the series expansion method is that, given the picture region, we choose a set of **basis functions** $b_1, b_2, \ldots, b_J$, each of which is a picture function for the given picture region. These must be chosen such that for any picture function $f$ that we want to reconstruct, there exists a linear combination of the basis functions that we consider an adequate approximation of $f$. If the line integrals of an unknown function $f$ are given along a set of lines, then we choose a linear combination of the basis functions whose line integrals along the same lines are as close to the measurements of $f$ as possible.

There are many possible choices of the basis functions, such as the generalized Kaiser-Bessel window functions, also known as blobs, which are widely used in X-ray transmission tomography. However, given an $m \times n$ uniform partition of the picture region, another typical choice is the set of characteristic functions of the pixels. The characteristic function of the pixel $R_{i,j}$ is the function $r_{i,j}$ of two variables defined by

$$r_{i,j}(x,y) = \begin{cases} 1, & \text{if } (x,y) \in R_{i,j} \\ 0, & \text{if } (x,y) \notin R_{i,j} \end{cases}$$

Then a linear combination of these characteristic functions is the picture function

$$g = x_{i,j} \cdot r_{i,j}$$

where $x_{i,j} \in \mathbb{R}$. Note that $g$ is nothing but the picture function which takes the constant value $x_{i,j}$ over the pixel $R_{i,j}$ for each $i \in \{1, 2, \ldots, m\}$ and $j \in \{1, 2, \ldots, n\}$. Thus the line integral of such function $g$ along any line $l$ is a linear combination of the variables $x_{i,j}$, where the coefficient of $x_{i,j}$ is exactly the length of the intersection of $l$ with the pixel $R_{i,j}$. Making the measurements of an unknown function $f$ equal to the linear combinations of the variables $x_{i,j}$, defined by the line integrals along a given set of lines, we obtain a system of linear equations. Unfortunately not all picture function $f$ can be given as a linear combination of the characteristic functions $r_{i,j}$. Besides there are different sources of error during the data collection, hence the system may be unsolvable. In such situation we need to introduce an extra variable $e_k$ for each equation of the system, which presents difference between measurement corresponding to that equation and the line integral of the picture function provided by the variables $x_{i,j}$. Then our task is to find a solution of the system which minimizes the square sum of the extra variables, i.e. $(e_1)^2 + (e_2)^2 + \ldots + (e_K)^2$.

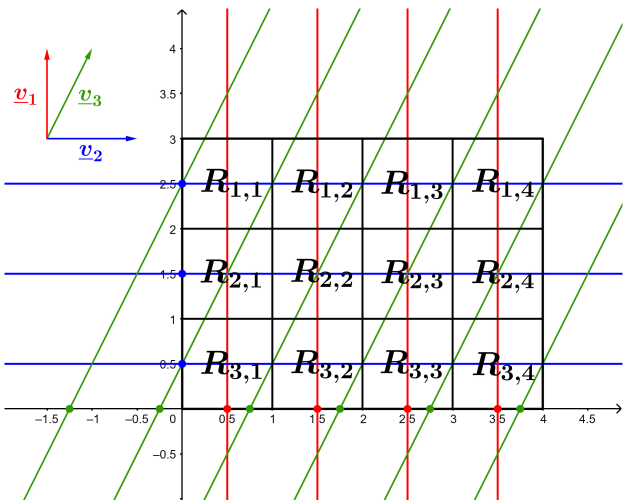
**Example 1**

Let $R = [0,4] \times [0,3]$, $m = 3$, $n = 4$. Then the uniform partition of $[0,4]$ is $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$ and the uniform partition of $[0,3]$ is $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$. These imply the pixels

$$\begin{array}{lll}
R_{1,1} = [0,1] \times [2,3] & R_{2,1} = [0,1] \times [1,2] & R_{3,1} = [0,1] \times [0,1] \\
R_{1,2} = [1,2] \times [2,3] & R_{2,2} = [1,2] \times [1,2] & R_{3,2} = [1,2] \times [0,1] \\
R_{1,3} = [2,3] \times [2,3] & R_{2,3} = [2,3] \times [1,2] & R_{3,3} = [2,3] \times [0,1] \\
R_{1,4} = [3,4] \times [2,3] & R_{2,4} = [3,4] \times [1,2] & R_{3,4} = [3,4] \times [0,1]
\end{array}$$

Consider the following set of lines:

- $l_1, l_2, l_3, l_4$ are lines parallel to $\underline{v}_1 = (0,1)$ and passing through the points $P_{1,1} = \left(\frac{1}{2}, 0\right)$, $P_{1,2} = \left(\frac{3}{2}, 0\right)$, $P_{1,3} = \left(\frac{5}{2}, 0\right)$, $P_{1,4} = \left(\frac{7}{2}, 0\right)$ respectively.
- $l_5, l_6, l_7$ are lines parallel to $\underline{v}_2 = (1,0)$ and passing through the points $P_{2,1} = \left(0, \frac{5}{2}\right)$, $P_{2,2} = \left(0, \frac{3}{2}\right)$, $P_{2,3} = \left(0, \frac{1}{2}\right)$ respectively.
- $l_8, l_9, l_{10}, l_{11}, l_{12}, l_{13}$ are lines parallel to $\underline{v}_3 = (1,2)$ and passing through the points $P_{3,1} = \left(-\frac{5}{4}, 0\right)$, $P_{3,2} = \left(-\frac{1}{4}, 0\right)$, $P_{3,3} = \left(\frac{3}{4}, 0\right)$, $P_{3,4} = \left(\frac{7}{4}, 0\right)$, $P_{3,5} = \left(\frac{11}{4}, 0\right)$, $P_{3,6} = \left(\frac{15}{4}, 0\right)$ respectively.



Let $m_k$ be the line integral of an unknown function $f$ along line $l_k$ for $k = 1, 2, \ldots, 13$, where

$$\begin{aligned}
&m_1 = 1, \quad m_2 = 3, \quad m_3 = 0, \quad m_4 = 2 \\
&m_5 = 2, \quad m_6 = 2, \quad m_7 = 2, \\
&m_8 = \tfrac{\sqrt{5}}{4}, \; m_9 = \tfrac{\sqrt{5}}{2}, \; m_{10} = \sqrt{5}, \; m_{11} = \tfrac{\sqrt{5}}{4}, \; m_{12} = \tfrac{3\sqrt{5}}{4}, \; m_{13} = \tfrac{\sqrt{5}}{4}.
\end{aligned}$$

We try to find the values $x_{i,j}$, such that the function $g$, which takes the constant value $x_{i,j}$ on the pixel $R_{i,j}$ for all $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, has the line integrals along the lines $l_k$ equal to $m_k$ for all $k = 1, 2, \ldots 13$.

The line $l_1$ intersects the pixels $R_{1,1}$, $R_{2,1}$, $R_{3,1}$, the line $l_2$ intersects the pixels $R_{1,2}$, $R_{2,2}$, $R_{3,2}$, the line $l_3$ intersects the pixels $R_{1,3}$, $R_{2,3}$, $R_{3,3}$, and the line $l_4$ intersects the pixels $R_{1,4}$, $R_{2,4}$, $R_{3,4}$. Each time the length of the intersection is 1. Thus the line integrals of $g$ along the lines $l_1$, $l_2$, $l_3$, $l_4$ are

$$\begin{gathered}
x_{1,1} + x_{2,1} + x_{3,1} \\
x_{1,2} + x_{2,2} + x_{3,2} \\
x_{1,3} + x_{2,3} + x_{3,3} \\
x_{1,4} + x_{2,4} + x_{3,4}
\end{gathered}$$

respectively. The line $l_5$ intersects the pixels $R_{1,1}$, $R_{1,2}$, $R_{1,3}$, $R_{1,4}$, the line $l_6$ intersects the pixels $R_{2,1}$, $R_{2,2}$, $R_{2,3}$, $R_{2,4}$, and the line $l_7$ intersects the pixels $R_{3,1}$, $R_{3,2}$, $R_{3,3}$, $R_{3,4}$. Each time the length of the intersection equals to 1. Thus the line integrals of $g$ along the lines $l_5$, $l_6$, $l_7$ are

$$\begin{gathered}
x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} \\
x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} \\
x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4}
\end{gathered}$$

respectively. Furthermore the line $l_8$ intersects only the pixel $R_{1,1}$ in a line segment of length $\frac{\sqrt{5}}{4}$. The line $l_9$ intersects the pixels $R_{1,1}$, $R_{1,2}$, $R_{3,1}$ in line segments of length $\frac{\sqrt{5}}{4}$, and intersects $R_{2,1}$ in a line segment of length $\frac{\sqrt{5}}{2}$. The line $l_{10}$ intersects the pixels $R_{1,2}$, $R_{1,3}$, $R_{3,1}$, $R_{3,2}$ in line segments of length $\frac{\sqrt{5}}{4}$ and intersects $R_{2,2}$ in a line segment of length $\frac{\sqrt{5}}{2}$. The line $l_{11}$ intersects the pixels $R_{1,3}$, $R_{1,4}$, $R_{3,2}$, $R_{3,3}$ in line segments of length $\frac{\sqrt{5}}{4}$ and intersects $R_{2,3}$ in a line segment of length $\frac{\sqrt{5}}{2}$. The line $l_{12}$ intersects the pixels $R_{1,4}$, $R_{3,3}$, $R_{3,4}$ in line segments of length $\frac{\sqrt{5}}{4}$ and intersects $R_{2,4}$ in a line segment of length $\frac{\sqrt{5}}{2}$. Finally $l_{13}$ intersects only the pixel $R_{3,4}$ in a line segment of length $\frac{\sqrt{5}}{4}$. Thus the line integrals of $g$ along the lines $l_8$, $l_9$, $l_{10}$, $l_{11}$, $l_{12}$, $l_{13}$ are

$$\begin{array}{l}
\frac{\sqrt{5}}{4}\,x_{1,1} \\
\frac{\sqrt{5}}{4}\,x_{1,1} + \frac{\sqrt{5}}{4}x_{1,2} + \frac{\sqrt{5}}{2}x_{2,1} + \frac{\sqrt{5}}{4}x_{3,1} \\
\frac{\sqrt{5}}{4}\,x_{1,2} + \frac{\sqrt{5}}{4}x_{1,3} + \frac{\sqrt{5}}{2}x_{2,2} + \frac{\sqrt{5}}{4}x_{3,1} + \frac{\sqrt{5}}{4}x_{3,2} \\
\frac{\sqrt{5}}{4}\,x_{1,3} + \frac{\sqrt{5}}{4}x_{1,4} + \frac{\sqrt{5}}{2}x_{2,3} + \frac{\sqrt{5}}{4}x_{3,2} + \frac{\sqrt{5}}{4}x_{3,3} \\
\frac{\sqrt{5}}{4}\,x_{1,4} + \frac{\sqrt{5}}{2}x_{2,4} + \frac{\sqrt{5}}{4}x_{3,3} + \frac{\sqrt{5}}{4}x_{3,4} \\
\frac{\sqrt{5}}{4}\,x_{3,4}
\end{array}$$

respectively. Hence making all these line integrals equal to the corresponding line integrals of the unknown function $f$ yields the system of equations

$$\left.\begin{array}{r}
x_{1,1} + x_{2,1} + x_{3,1} = 1 \\
x_{1,2} + x_{2,2} + x_{3,2} = 3 \\
x_{1,3} + x_{2,3} + x_{3,3} = 0 \\
x_{1,4} + x_{2,4} + x_{3,4} = 2 \\
x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} = 2 \\
x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} = 2 \\
x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} = 2 \\
\frac{\sqrt{5}}{4}\,x_{1,1} = \frac{\sqrt{5}}{4} \\
\frac{\sqrt{5}}{4}\,x_{1,1} + \frac{\sqrt{5}}{4}x_{1,2} + \frac{\sqrt{5}}{2}x_{2,1} + \frac{\sqrt{5}}{4}x_{3,1} = \frac{\sqrt{5}}{2} \\
\frac{\sqrt{5}}{4}\,x_{1,2} + \frac{\sqrt{5}}{4}x_{1,3} + \frac{\sqrt{5}}{2}x_{2,2} + \frac{\sqrt{5}}{4}x_{3,1} + \frac{\sqrt{5}}{4}x_{3,2} = \sqrt{5} \\
\frac{\sqrt{5}}{4}\,x_{1,3} + \frac{\sqrt{5}}{4}x_{1,4} + \frac{\sqrt{5}}{2}x_{2,3} + \frac{\sqrt{5}}{4}x_{3,2} + \frac{\sqrt{5}}{4}x_{3,3} = \frac{\sqrt{5}}{4} \\
\frac{\sqrt{5}}{4}\,x_{1,4} + \frac{\sqrt{5}}{2}x_{2,4} + \frac{\sqrt{5}}{4}x_{3,3} + \frac{\sqrt{5}}{4}x_{3,4} = \frac{3\sqrt{5}}{4} \\
\frac{\sqrt{5}}{4}\,x_{3,4} = \frac{\sqrt{5}}{4}
\end{array}\right\}$$

Multiplying each of the last 6 equations by $\frac{4}{\sqrt{5}}$ gives

$$\left.\begin{aligned}
x_{1,1} + x_{2,1} + x_{3,1} &= 1 \\
x_{1,2} + x_{2,2} + x_{3,2} &= 3 \\
x_{1,3} + x_{2,3} + x_{3,3} &= 0 \\
x_{1,4} + x_{2,4} + x_{3,4} &= 2 \\
x_{1,1} + x_{1,2} + x_{1,3} + x_{1,4} &= 2 \\
x_{2,1} + x_{2,2} + x_{2,3} + x_{2,4} &= 2 \\
x_{3,1} + x_{3,2} + x_{3,3} + x_{3,4} &= 2 \\
x_{1,1} &= 1 \\
x_{1,1} + x_{1,2} + 2x_{2,1} + x_{3,1} &= 2 \\
x_{1,2} + x_{1,3} + 2x_{2,2} + x_{3,1} + x_{3,2} &= 4 \\
x_{1,3} + x_{1,4} + 2x_{2,3} + x_{3,2} + x_{3,3} &= 1 \\
x_{1,4} + 2x_{2,4} + x_{3,3} + x_{3,4} &= 3 \\
x_{3,4} &= 1
\end{aligned}\right\}$$

Here the unknowns have double index, so before we write the matrix form of the above system we need to fix an ordering of the above unknowns. This can be for example the lexicographic order, where $x_{i,j} \leq x_{k,l}$ holds if $i < k$, or $i = k$ and $j \leq l$. Hence the ordering is

$$(x_{1,1}, x_{1,2}, x_{1,3}, x_{1,4}, x_{2,1}, x_{2,2}, x_{2,3}, x_{2,4}, x_{3,1}, x_{3,2}, x_{3,3}, x_{3,4})$$

Then the above system of equations in matrix form is $A \cdot \underline{x} = \underline{b}$, where

$$A = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

and

$$\underline{x} = (x_{1,1}\ x_{1,2}\ x_{1,3}\ x_{1,4}\ x_{2,1}\ x_{2,2}\ x_{2,3}\ x_{2,4}\ x_{3,1}\ x_{3,2}\ x_{3,3}\ x_{3,4})^\top$$

$$\underline{b} = (1\ 3\ 0\ 2\ 2\ 2\ 2\ 1\ 2\ 4\ 1\ 3\ 1)^\top$$

The extended coefficient matrix is

$$\left(\begin{array}{cccccccccccc|c}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 3 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 2 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 2 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\
0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 0 & 4 \\
0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 1 & 1 & 3 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{array}\right)$$

This can transformed to reduced row echelon form with Gauss elimination. The reduced row echelon form is

$$\left(\begin{array}{cccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

This shows that we have two free variables $x_{3,2}$ and $x_{3,3}$. The rest of the variables can be given as

$$\begin{pmatrix} x_{1,1} \\ x_{1,2} \\ x_{1,3} \\ x_{1,4} \\ x_{2,1} \\ x_{2,2} \\ x_{2,3} \\ x_{2,4} \\ x_{3,1} \\ x_{3,4} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & -1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ -1 & -1 \\ 0 & -1 \\ -1 & -1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_{3,2} \\ x_{3,3} \end{pmatrix} + \begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

This can be arranged in the same manner as in the case of a matrix:

$$\begin{array}{llll} x_{1,1} = 1 & x_{1,2} = 2 - x_{3,2} - x_{3,3} & x_{1,3} = x_{3,2} - 1 & x_{1,4} = x_{3,3} \\ x_{2,1} = x_{3,2} + x_{3,3} - 1 & x_{2,2} = x_{3,3} + 1 & x_{2,3} = 1 - x_{3,2} - x_{3,3} & x_{2,4} = 1 - x_{3,3} \\ x_{3,1} = 1 - x_{2,3} - x_{3,3} & x_{3,2} = x_{3,2} & x_{3,3} = x_{3,3} & x_{3,4} = 1 \end{array}$$

Now let's find the non-negative solutions. Then we have the following system of linear inequalities:

$$\left.\begin{aligned} 2 - x_{3,2} - x_{3,3} &\geq 0 \\ x_{3,2} - 1 &\geq 0 \\ x_{3,3} &\geq 0 \\ x_{3,2} + x_{3,3} - 1 &\geq 0 \\ x_{3,3} + 1 &\geq 0 \\ 1 - x_{3,2} - x_{3,3} &\geq 0 \\ 1 - x_{3,3} &\geq 0 \\ x_{3,2} &\geq 0 \end{aligned}\right\}$$

Here the fourth and sixth inequalities imply that $x_{3,2} + x_{3,3} - 1 = 0$, and hence $x_{3,3} = 1 - x_{3,2}$. This can be substituted into the rest of the inequalities.

$$\left.\begin{aligned} 1 &\geq 0 \\ x_{3,2} - 1 &\geq 0 \\ 1 - x_{3,2} &\geq 0 \\ 2 - x_{3,2} &\geq 0 \\ x_{3,2} &\geq 0 \\ x_{3,2} &\geq 0 \end{aligned}\right\}$$

Here the second and third inequalities together imply that $x_{3,2} - 1 = 0$, that is $x_{3,2} = 1$ and then $x_{3,3} = 1 - x_{3,2} = 0$. It's easy to check that $x_{3,2} = 1$ and $x_{3,3} = 0$ is a solution of the above system of inequalities. Thus substituting $x_{3,2} = 1$ and $x_{3,3} = 0$ into the solutions of the system of equations gives that the only non-negative solution is

$$\begin{array}{llll} x_{1,1} = 1 & x_{1,2} = 1 & x_{1,3} = 0 & x_{1,4} = 0 \\ x_{2,1} = 0 & x_{2,2} = 1 & x_{2,3} = 0 & x_{2,4} = 1 \\ x_{3,1} = 0 & x_{3,2} = 1 & x_{3,3} = 0 & x_{3,4} = 1 \end{array}$$

**Example 2**

Let $R = [0,4] \times [0,3]$, $m = 3$, $n = 4$. Then the uniform partition of $[0,4]$ is $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 4$ and the uniform partition of $[0,3]$ is $c_0 = 0$, $c_1 = 1$, $c_2 = 2$, $c_3 = 3$. These imply the pixels

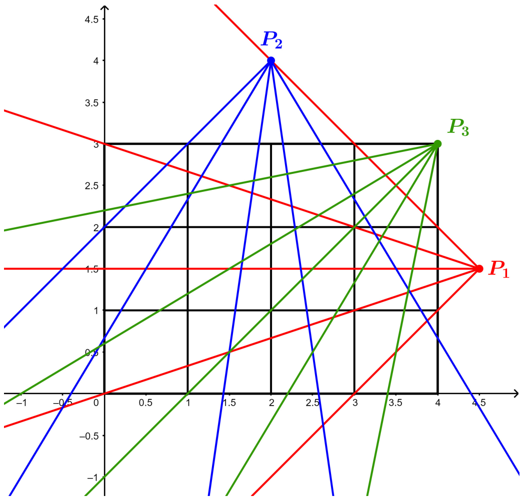
$$\begin{array}{lll} R_{1,1} = [0,1] \times [2,3] & R_{2,1} = [0,1] \times [1,2] & R_{3,1} = [0,1] \times [0,1] \\ R_{1,2} = [1,2] \times [2,3] & R_{2,2} = [1,2] \times [1,2] & R_{3,2} = [1,2] \times [0,1] \\ R_{1,3} = [2,3] \times [2,3] & R_{2,3} = [2,3] \times [1,2] & R_{3,3} = [2,3] \times [0,1] \\ R_{1,4} = [3,4] \times [2,3] & R_{2,4} = [3,4] \times [1,2] & R_{3,4} = [3,4] \times [0,1] \end{array}$$

Consider the following set of lines:

- $l_1, l_2, l_3, l_4, l_5$ are lines passing through the point $P_1 = \left(\frac{9}{2}, \frac{3}{2}\right)$ and parallel to the vectors $\underline{v}_{1,1} = (1,-1)$, $\underline{v}_{1,2} = (3,-1)$, $\underline{v}_{1,3} = (1,0)$, $\underline{v}_{1,4} = (3,1)$, $\underline{v}_{1,5} = (1,1)$ respectively.

- $l_6, l_7, l_8, l_9, l_{10}$ are lines passing through the point $P_2 = (2,4)$ and parallel to the vectors $\underline{v}_{2,1} = (1,1)$, $\underline{v}_{2,2} = (3,5)$, $\underline{v}_{2,3} = (1,7)$, $\underline{v}_{2,4} = (1,-7)$, $\underline{v}_{2,5} = (3,-5)$ respectively.

- $l_{11}, l_{12}, l_{13}, l_{14}, l_{15}$ are lines passing through the point $P_3 = (4,3)$ and parallel to the vectors $\underline{v}_{3,1} = (5,1)$, $\underline{v}_{3,2} = (5,3)$, $\underline{v}_{3,3} = (1,1)$, $\underline{v}_{3,4} = (3,5)$, $\underline{v}_{3,5} = (1,5)$ respectively.




Let $m_k$ be the line integral of an unknown function $f$ along line $l_k$ for $k = 1, 2, \ldots, 15$, where

$$\begin{array}{lllll} m_1 = 0, & m_2 = \frac{2\sqrt{10}}{3}, & m_3 = 3, & m_4 = \frac{2\sqrt{10}}{3} & m_5 = \sqrt{2} \\ m_6 = 0, & m_7 = \frac{\sqrt{34}}{5}, & m_8 = \frac{10\sqrt{2}}{7}, & m_9 = \frac{5\sqrt{2}}{7} & m_{10} = \frac{4\sqrt{34}}{15} \\ m_{11} = \frac{\sqrt{26}}{5}, & m_{12} = \frac{2\sqrt{34}}{5}, & m_{13} = \sqrt{2}, & m_{14} = \frac{\sqrt{34}}{5}, & m_{15} = \frac{2\sqrt{26}}{5} \end{array}$$

We try to find the values $x_{i,j}$, such that the function $g$, which takes the constant value $x_{i,j}$ on the pixel $R_{i,j}$ for all $i = 1, 2, 3$ and $j = 1, 2, 3, 4$, has the line integrals along the lines $l_k$ equal to $m_k$ for all $k = 1, 2, \ldots 15$.

- The line $l_1$ intersects only the pixel $R_{1,4}$ and the length of the intersection is $\sqrt{2}$.
- The line $l_2$ intersects the pixels $R_{1,1}$, $R_{1,2}$, $R_{1,3}$, $R_{2,4}$. The length of the intersection equals to $\frac{\sqrt{10}}{3}$ in each case.
- The line $l_3$ intersects the pixels $R_{2,1}$, $R_{2,2}$, $R_{2,3}$, $R_{2,4}$. The length of the intersection equals to 1 in each case.
- The line $l_4$ intersects the pixels $R_{3,1}$, $R_{3,2}$, $R_{3,3}$, $R_{2,4}$. The length of the intersection equals to $\frac{\sqrt{10}}{3}$ in each case.

- The line $l_5$ intersects only the pixel $R_{3,4}$ and the length of the intersection is $\sqrt{2}$.

- The line $l_6$ intersects only the pixel $R_{1,1}$ and the length of the intersection is $\sqrt{2}$.

- The line $l_7$ intersects the pixels $R_{1,1}$, $R_{1,2}$, $R_{2,1}$, $R_{3,1}$. The lengths of the intersections are $\frac{\sqrt{34}}{15}$, $\frac{2\sqrt{34}}{15}$, $\frac{\sqrt{34}}{5}$, $\frac{\sqrt{34}}{15}$, respectively.

- The line $l_8$ intersects the pixels $R_{1,2}$, $R_{2,2}$, $R_{3,2}$. The length of the intersection equals to $\frac{5\sqrt{2}}{7}$ in each case.

- The line $l_9$ intersects the pixels $R_{1,3}$, $R_{2,3}$, $R_{3,3}$. The length of the intersection equals to $\frac{5\sqrt{2}}{7}$ in each case.

- The line $l_{10}$ intersects the pixels $R_{1,3}$, $R_{1,4}$, $R_{2,4}$, $R_{3,4}$. The lengths of the intersections are $\frac{\sqrt{34}}{15}$, $\frac{2\sqrt{34}}{15}$, $\frac{\sqrt{34}}{5}$, $\frac{\sqrt{34}}{15}$, respectively.

- The line $l_{11}$ intersects the pixels $R_{1,1}$, $R_{1,2}$, $R_{1,3}$, $R_{2,4}$. The length of the intersection equals to $\frac{\sqrt{26}}{5}$ in each case.

- The line $l_{12}$ intersects the pixels $R_{1,3}$, $R_{1,4}$, $R_{2,1}$, $R_{2,2}$, $R_{2,3}$, $R_{3,1}$. The lengths of the intersections are $\frac{2\sqrt{34}}{15}$, $\frac{\sqrt{34}}{5}$, $\frac{\sqrt{34}}{15}$, $\frac{\sqrt{34}}{5}$, $\frac{\sqrt{34}}{15}$, $\frac{2\sqrt{34}}{15}$, respectively.

- The line $l_{13}$ intersects the pixels $R_{1,4}$, $R_{2,3}$, $R_{3,2}$. The length of the intersection equals to $\sqrt{2}$ in each case.

- The line $l_{14}$ intersects the pixels $R_{1,4}$, $R_{2,3}$, $R_{2,4}$, $R_{3,3}$. The lengths of the intersections are $\frac{\sqrt{34}}{5}$, $\frac{\sqrt{34}}{15}$, $\frac{2\sqrt{34}}{15}$, $\frac{\sqrt{34}}{5}$ respectively.

- The line $l_{15}$ intersects the pixels $R_{1,4}$, $R_{2,4}$, $R_{3,4}$. The length of the intersection equals to $\frac{\sqrt{26}}{5}$ in each case.

Hence making all the line integrals of $g$ equal to the corresponding line integrals of the unknown function $f$ yields the system of equations

$$\left.\begin{aligned}
\sqrt{2}x_{1,4} &= 0\\
\tfrac{\sqrt{10}}{3}x_{1,1}+\tfrac{\sqrt{10}}{3}x_{1,2}+\tfrac{\sqrt{10}}{3}x_{1,3}+\tfrac{\sqrt{10}}{3}x_{2,4} &= \tfrac{2\sqrt{10}}{3}\\
x_{2,1}+x_{2,2}+x_{2,3}+x_{2,4} &= 3\\
\tfrac{\sqrt{10}}{3}x_{3,1}+\tfrac{\sqrt{10}}{3}x_{3,2}+\tfrac{\sqrt{10}}{3}x_{3,3}+\tfrac{\sqrt{10}}{3}x_{2,4} &= \tfrac{2\sqrt{10}}{3}\\
\sqrt{2}x_{3,4} &= \sqrt{2}\\
\sqrt{2}x_{1,1} &= 0\\
\tfrac{\sqrt{34}}{15}x_{1,1}+\tfrac{2\sqrt{34}}{15}x_{1,2}+\tfrac{\sqrt{34}}{5}x_{2,1}+\tfrac{\sqrt{34}}{15}x_{3,1} &= \tfrac{\sqrt{34}}{5}\\
\tfrac{5\sqrt{2}}{7}x_{1,2}+\tfrac{5\sqrt{2}}{7}x_{2,2}+\tfrac{5\sqrt{2}}{7}x_{3,2} &= \tfrac{10\sqrt{2}}{7}\\
\tfrac{5\sqrt{2}}{7}x_{1,3}+\tfrac{5\sqrt{2}}{7}x_{2,3}+\tfrac{5\sqrt{2}}{7}x_{3,3} &= \tfrac{5\sqrt{2}}{7}\\
\tfrac{\sqrt{34}}{15}x_{1,3}+\tfrac{2\sqrt{34}}{15}x_{1,4}+\tfrac{\sqrt{34}}{5}x_{2,4}+\tfrac{\sqrt{34}}{15}x_{3,4} &= \tfrac{4\sqrt{34}}{15}\\
\tfrac{\sqrt{26}}{5}x_{1,1}+\tfrac{\sqrt{26}}{5}x_{1,2}+\tfrac{\sqrt{26}}{5}x_{1,3}+\tfrac{\sqrt{26}}{5}x_{2,4} &= \tfrac{\sqrt{26}}{5}\\
\tfrac{2\sqrt{34}}{15}x_{1,3}+\tfrac{\sqrt{34}}{5}x_{1,4}+\tfrac{\sqrt{34}}{15}x_{2,1}+\tfrac{\sqrt{34}}{5}x_{2,2}+\tfrac{\sqrt{34}}{15}x_{2,3}+\tfrac{2\sqrt{34}}{15}x_{3,1} &= \tfrac{2\sqrt{34}}{5}\\
\sqrt{2}x_{1,4}+\sqrt{2}x_{2,3}+\sqrt{2}x_{3,2} &= \sqrt{2}\\
\tfrac{\sqrt{34}}{5}x_{1,4}+\tfrac{\sqrt{34}}{15}x_{2,3}+\tfrac{2\sqrt{34}}{15}x_{2,4}+\tfrac{\sqrt{34}}{5}x_{3,3} &= \tfrac{\sqrt{34}}{5}\\
\tfrac{\sqrt{26}}{5}x_{1,4}+\tfrac{\sqrt{26}}{5}x_{2,4}+\tfrac{\sqrt{26}}{5}x_{3,4} &= \tfrac{2\sqrt{26}}{5}
\end{aligned}\right\}$$

Here we multiply the the first equation by $\frac{1}{\sqrt{2}}$, the second equation by $\frac{3}{\sqrt{10}}$, the fourth equation by $\frac{3}{\sqrt{10}}$, the fifth and sixth equations by $\frac{1}{\sqrt{2}}$, the seventh equation by $\frac{15}{\sqrt{34}}$, the eighth and ninth equations by $\frac{7}{5\sqrt{2}}$, the tenth equation by $\frac{15}{\sqrt{34}}$, the eleventh equation by $\frac{5}{\sqrt{26}}$, the twelfth equation by $\frac{15}{\sqrt{34}}$, the thirteenth equation by $\frac{1}{\sqrt{2}}$, the fourteenth equation by $\frac{15}{\sqrt{34}}$, and the fifteenth equation by $\frac{5}{\sqrt{26}}$. Then all coefficients of the system are integers.

$$\left.\begin{aligned}
x_{1,4} &= 0\\
x_{1,1}+x_{1,2}+x_{1,3}+x_{2,4} &= 2\\
x_{2,1}+x_{2,2}+x_{2,3}+x_{2,4} &= 3\\
x_{3,1}+x_{3,2}+x_{3,3}+x_{2,4} &= 2\\
x_{3,4} &= 1\\
x_{1,1} &= 0\\
x_{1,1}+2x_{1,2}+3x_{2,1}+x_{3,1} &= 3\\
x_{1,2}+x_{2,2}+x_{3,2} &= 2\\
x_{1,3}+x_{2,3}+x_{3,3} &= 1\\
x_{1,3}+2x_{1,4}+3x_{2,4}+x_{3,4} &= 4\\
x_{1,1}+x_{1,2}+x_{1,3}+x_{2,4} &= 1\\
2x_{1,3}+3x_{1,4}+x_{2,1}+3x_{2,2}+x_{2,3}+2x_{3,1} &= 6\\
x_{1,4}+x_{2,3}+x_{3,2} &= 1\\
3x_{1,4}+x_{2,3}+2x_{2,4}+3x_{3,3} &= 3\\
x_{1,4}+x_{2,4}+x_{3,4} &= 2
\end{aligned}\right\}$$

Assuming lexicographic order of the variables the extended coefficient matrix is

$$\left(A\middle|\underline{b}\right)\left(\begin{array}{cccccccccccc|c}
0&0&0&1&0&0&0&0&0&0&0&0&0\\
1&1&1&0&0&0&0&1&0&0&0&0&2\\
0&0&0&0&1&1&1&1&0&0&0&0&3\\
0&0&0&0&0&0&0&1&1&1&1&0&2\\
0&0&0&0&0&0&0&0&0&0&0&1&1\\
1&0&0&0&0&0&0&0&0&0&0&0&0\\
1&2&0&0&3&0&0&0&1&0&0&0&3\\
0&1&0&0&0&1&0&0&0&1&0&0&2\\
0&0&1&0&0&0&1&0&0&0&1&0&1\\
0&0&2&1&0&0&0&3&0&0&0&1&4\\
1&1&1&1&0&0&0&0&0&0&0&0&1\\
0&0&2&3&1&3&1&0&2&0&0&0&6\\
0&0&0&1&0&0&1&0&0&1&0&0&1\\
0&0&0&3&0&0&1&2&0&0&3&0&3\\
0&0&0&1&0&0&0&1&0&0&0&1&2
\end{array}\right)$$

If we transform it into reduced row echelon form with the help of Gaussian elimination, then we get

$$\left(\begin{array}{cccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right)$$

Each column contains a pivot element except the last one, which means the system is solvable, and has a unique solution. This solution is

$$\begin{array}{llll}
x_{1,1} = 0 & x_{1,2} = 1 & x_{1,3} = 0 & x_{1,4} = 0 \\
x_{2,1} = 0 & x_{2,2} = 1 & x_{2,3} = 1 & x_{2,4} = 1 \\
x_{3,1} = 1 & x_{3,2} = 0 & x_{3,3} = 0 & x_{3,4} = 1
\end{array}$$