

# 3 Computed tomography

## 3.1 Pictures, digitization and phantoms

A picture consists of two components: the picture region and the picture function. The **picture region** is a rectangle  $R$  whose sides are parallel to the axes of a Cartesian coordinate system in the plane. Such a rectangle can be easily given as the direct product of two intervals  $[a, b]$  and  $[c, d]$ .

$$R = \{(x, y) \mid x \in [a, b], y \in [c, d]\}$$

Here  $[a, b]$  and  $[c, d]$  are just the orthogonal projection of  $R$  onto the coordinate axes. The **picture function** is a function  $f$  of two variables, which is constant zero outside of the picture region. We often refer to the value of the picture at a point  $(x, y)$  as the **density** at  $(x, y)$ .

A partition of the interval  $[a, b]$  is a sequence of points  $a_0, a_1, \dots, a_n \in [a, b]$ , where  $a = a_0 < a_1 < \dots < a_{n-1} < a_n = b$ . A partition naturally divides the interval  $[a, b]$  into the union of the intervals  $[a_{i-1}, a_i]$ ,  $i \in \{1, 2, \dots, n\}$ . The uniform partition of  $[a, b]$  is when all the intervals  $[a_{i-1}, a_i]$  have the same length, that is when

$$a_i = a + i \cdot \frac{b - a}{n}, \quad i \in \{0, 1, \dots, n\}$$

Similarly a partition of the interval  $[c, d]$  is a sequence of points  $c_0, c_1, \dots, c_m \in [c, d]$ , where  $c = c_0 < c_1 < \dots < c_{m-1} < c_m = d$ . A partition naturally divides the interval  $[c, d]$  into the union of the intervals  $[c_{i-1}, c_i]$ ,  $i \in \{1, 2, \dots, m\}$ . The uniform partition of  $[c, d]$  is when

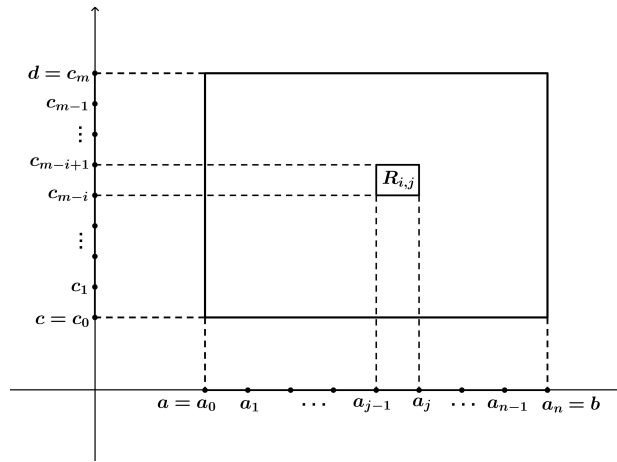
$$c_i = c + i \cdot \frac{d - c}{m}, \quad i \in \{0, 1, \dots, m\}$$

These two partitions together give a partition of the picture region  $R$ , which is the set of rectangles obtained by taking the direct product of each pair of

the intervals  $[a_{i-1}, a_i]$  and  $[c_{j-1}, c_j]$ ,  $i \in \{0, 1, \dots, n\}$  and  $j \in \{0, 1, \dots, m\}$ . We call the elements of the partition of the picture region  $R$  pixels if they're obtained by uniform partitions of  $[a, b]$  and  $[c, d]$ . We can refer to the pixels by double indexing in the same manner as in the case of matrices if

$$R_{i,j} = [a_{j-1}, a_j] \times [c_{m-i}, c_{m-i+1}], \quad i \in \{1, 2, \dots, m\}$$

Then we say  $R_{i,j}$  is the pixel of the picture region in the  $i$ -th row and  $j$ -th column.



An  $m \times n$  digitized picture is one whose value in the interior of any pixel of a uniform partition of the picture region is constant. The  $m \times n$  digitization of a picture is an  $m \times n$  digitized picture such that the double integral of the original picture over any pixel equals to the double integral of the digitized picture over the same pixel. In X-ray transmission computed tomography the picture region is the reconstruction region and the the density of the picture at a point  $(x, y)$  is the relative linear attenuation number at an effective energy of the tissue at the point  $(x, y)$ .

While the aim of X-ray transmission computed tomography is the reconstruction of real objects from their actual x-ray projections, the theoretical development of CT was based on experiments on mathematically described objects from computer simulated projection data. The basic reason for this that computer simulation enables us to investigate individually various effects that can't be separated physically. Such mathematically described objects

are called **phantoms**. A test phantom is nothing but a picture on which we wish to test reconstruction algorithms or data collection methods. A simple phantom can be defined for example by specifying the constant values of the picture function over each pixel of a digitized picture. Such phantom can be specified with the help of a matrix  $A = (a_{ij})$  where  $a_{ij}$  determines the constant value over the pixel  $R_{i,j}$ . Another type of simple phantom is a polygon inside the picture region and specifying the picture function equal to 1 inside the polygon and equal to zero outside the polygon.

### 3.2 Data collection

Let a picture with picture region  $R = [a, b] \times [c, d]$  and picture function  $f$  be given, which acts as a phantom. A line  $l$  can be specified by giving a point and a direction vector of the line. The parametric equation of the line  $l$  passing through the point  $P$  and parallel to the vector  $\underline{v}$  is

$$l = \{P + t \cdot \underline{v} \mid t \in \mathbb{R}\}$$

The parametrization of the line is not unique, however the value of the line integral is independent of the choice of the parametrization. If the parametric equation of the  $l$  is given, then the line integral of the picture function  $f$  can be computed as

$$\int_l f = \int_{-\infty}^{\infty} f(P + t \cdot \underline{v}) \cdot |\underline{v}| dt$$

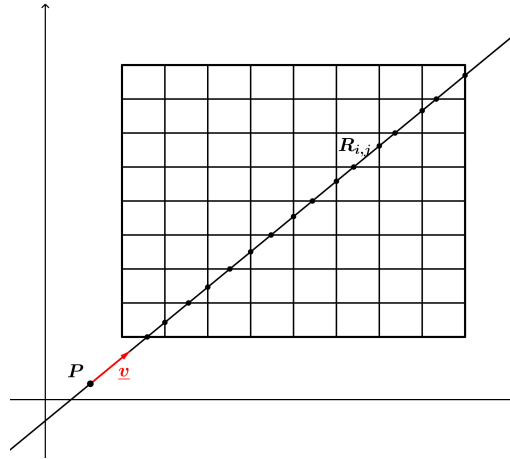
where  $|\underline{v}|$  denotes the Euclidean length of the vector  $\underline{v}$ . If  $\underline{v} = (v_1, v_2)$ , then  $|\underline{v}| = \sqrt{(v_1)^2 + (v_2)^2}$ . Despite the integral is taken over an unbounded interval, the picture function is zero outside of a finite domain (the picture region), thus it's enough to take the above integral over a finite interval which depends on position of the line to the picture region and depends on the parametrization.

$$\int_l f = \int_{l \cap R} f$$

If a uniform partition of the picture region is also given with pixels  $R_{i,j}$ , then the line integral of  $f$  along the line  $l$  can be computed as the sum of the line

integrals along the intersections of  $l$  with each pixel.

$$\int_{l \cap R} f = \sum_{i=1}^m \sum_{j=1}^n \int_{l \cap R_{i,j}} f$$



The line integral of a constant function over a bounded domain equals to the value of the function multiplied by the length of intersection of the line and the domain. Thus if  $f$  is simple phantom which takes the constant value  $a_{i,j}$  over the pixel  $R_{i,j}$  of an  $m \times n$  uniform partition of the picture region  $R$  then

$$\int_l f = \sum_{i=1}^m \sum_{j=1}^n \int_{l \cap R_{i,j}} f = \sum_{i=1}^m \sum_{j=1}^n a_{i,j} \cdot \lambda_1(l \cap R_{i,j})$$

where  $\lambda_1$  denotes the one-dimensional Lebesgue measure (i.e. length).

### Example

Let  $R = [0, 4] \times [0, 3]$ ,  $m = 3$ ,  $n = 4$ . Then the uniform partition of  $[0, 4]$  is  $a_0 = 0$ ,  $a_1 = 1$ ,  $a_2 = 2$ ,  $a_3 = 3$ ,  $a_4 = 4$  and the uniform partition of  $[0, 3]$  is  $c_0 = 0$ ,  $c_1 = 1$ ,  $c_2 = 2$ ,  $c_3 = 3$ . These imply the pixels

$$\begin{array}{lll} R_{1,1} = [0, 1] \times [2, 3] & R_{2,1} = [0, 1] \times [1, 2] & R_{3,1} = [0, 1] \times [0, 1] \\ R_{1,2} = [1, 2] \times [2, 3] & R_{2,2} = [1, 2] \times [1, 2] & R_{3,2} = [1, 2] \times [0, 1] \\ R_{1,3} = [2, 3] \times [2, 3] & R_{2,3} = [2, 3] \times [1, 2] & R_{3,3} = [2, 3] \times [0, 1] \\ R_{1,4} = [3, 4] \times [2, 3] & R_{2,4} = [3, 4] \times [1, 2] & R_{3,4} = [3, 4] \times [0, 1] \end{array}$$

Let  $f$  be the simple phantom defined by the matrix

$$A = \begin{pmatrix} \frac{1}{4} & \frac{2}{3} & \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{2} & 1 & 0 \end{pmatrix}$$

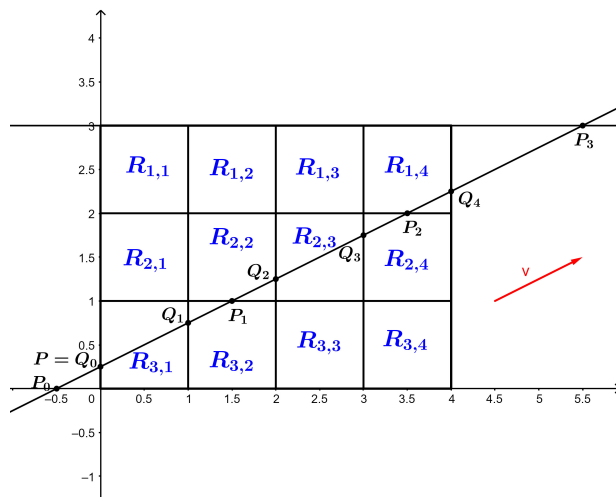
and let  $l$  be the line passing through the point  $P = (0, \frac{1}{4})$  and parallel to the vector  $\underline{v} = (2, 1)$ . The parametrization of  $l$  is

$$l = \{P + t \cdot \underline{v} \mid t \in \mathbb{R}\}$$

This means every point of the line  $l$  can be written in the form

$$P + t \cdot \underline{v} = \left(0, \frac{1}{4}\right) + t \cdot (2, 1) = \left(0 + 2t, \frac{1}{4} + t\right) = \left(2t, \frac{1}{4} + t\right)$$

where  $t \in \mathbb{R}$ .



Let  $P_0, P_1, P_2, P_3$  denote the intersection points of  $l$  with the horizontal lines  $y = 0, y = 1, y = 2, y = 3$  respectively, and let  $Q_0, Q_1, Q_2, Q_3, Q_4$  denote the intersection points of  $l$  with the vertical lines  $x = 0, x = 1, x = 2, x = 3, x = 4$  respectively. Then the corresponding parameters of  $P_0, P_1, P_2, P_3$  can

be computed by making  $\frac{1}{4} + t$  equal to 0, 1, 2, 3, and then the first components of  $P_0, P_1, P_2, P_3$  are obtained by substituting the parameter to  $2t$ . That is

$$\begin{aligned}\frac{1}{4} + t = 0 &\Rightarrow t = -\frac{1}{4} \Rightarrow 2t = -\frac{1}{2} \Rightarrow P_0 = \left(-\frac{1}{2}, 0\right) \\ \frac{1}{4} + t = 1 &\Rightarrow t = \frac{3}{4} \Rightarrow 2t = \frac{3}{2} \Rightarrow P_1 = \left(\frac{3}{2}, 1\right) \\ \frac{1}{4} + t = 2 &\Rightarrow t = \frac{7}{4} \Rightarrow 2t = \frac{7}{2} \Rightarrow P_2 = \left(\frac{7}{2}, 2\right) \\ \frac{1}{4} + t = 3 &\Rightarrow t = \frac{11}{4} \Rightarrow 2t = \frac{11}{2} \Rightarrow P_3 = \left(\frac{11}{2}, 3\right)\end{aligned}$$

Similarly the corresponding parameters of  $Q_0, Q_1, Q_2, Q_3, Q_4$  can be computed by making  $2t$  equal to 0, 1, 2, 3, 4, and then the second components of  $Q_0, Q_1, Q_2, Q_3, Q_4$  are obtained by substituting the parameter to  $t + \frac{1}{4}$ . That is

$$\begin{aligned}2t = 0 &\Rightarrow t = 0 \Rightarrow t + \frac{1}{4} = \frac{1}{4} \Rightarrow Q_0 = \left(0, \frac{1}{4}\right) \\ 2t = 1 &\Rightarrow t = \frac{1}{2} \Rightarrow t + \frac{1}{4} = \frac{3}{4} \Rightarrow Q_1 = \left(1, \frac{3}{4}\right) \\ 2t = 2 &\Rightarrow t = 1 \Rightarrow t + \frac{1}{4} = \frac{5}{4} \Rightarrow Q_2 = \left(2, \frac{5}{4}\right) \\ 2t = 3 &\Rightarrow t = \frac{3}{2} \Rightarrow t + \frac{1}{4} = \frac{7}{4} \Rightarrow Q_3 = \left(3, \frac{7}{4}\right) \\ 2t = 4 &\Rightarrow t = 2 \Rightarrow t + \frac{1}{4} = \frac{9}{4} \Rightarrow Q_4 = \left(4, \frac{9}{4}\right)\end{aligned}$$

Thus the line  $l$  intersects only the pixels  $R_{3,1}, R_{3,2}, R_{2,2}, R_{2,3}, R_{2,4}, R_{1,4}$ . The intersection with the pixel  $R_{3,1}$  is the line segment  $\overline{Q_0Q_1}$ , which can be shortly written as  $l \cap R_{3,1} = \overline{Q_0Q_1}$ . Furthermore

$$\begin{aligned}l \cap R_{3,2} &= \overline{Q_1P_1} & l \cap R_{2,2} &= \overline{P_1Q_2} & l \cap R_{2,3} &= \overline{Q_2Q_3} \\ l \cap R_{2,4} &= \overline{Q_3P_2} & l \cap R_{1,4} &= \overline{P_2Q_4}\end{aligned}$$

The length of any line segment  $\overline{PQ}$  connecting the points  $P = (p_1, p_2)$  and  $Q = (q_1, q_2)$  is computed by the formula  $\sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2}$ . Hence

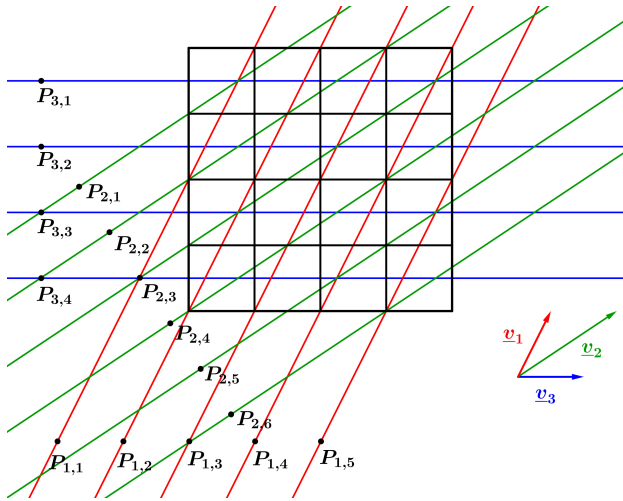
$$\begin{aligned}\lambda_1(l \cap R_{3,1}) &= \lambda_1(\overline{Q_0Q_1}) = \sqrt{(0 - 1)^2 + \left(\frac{1}{4} - \frac{3}{4}\right)^2} = \frac{\sqrt{5}}{2} \\ \lambda_1(l \cap R_{3,2}) &= \lambda_1(\overline{Q_1P_1}) = \sqrt{\left(1 - \frac{3}{2}\right)^2 + \left(\frac{3}{4} - 1\right)^2} = \frac{\sqrt{5}}{4} \\ \lambda_1(l \cap R_{2,2}) &= \lambda_1(\overline{P_1Q_2}) = \sqrt{\left(\frac{3}{2} - 2\right)^2 + \left(1 - \frac{5}{4}\right)^2} = \frac{\sqrt{5}}{4} \\ \lambda_1(l \cap R_{2,3}) &= \lambda_1(\overline{Q_2Q_3}) = \sqrt{(2 - 3)^2 + \left(\frac{5}{4} - \frac{7}{4}\right)^2} = \frac{\sqrt{5}}{2} \\ \lambda_1(l \cap R_{2,4}) &= \lambda_1(\overline{Q_3P_2}) = \sqrt{\left(3 - \frac{7}{2}\right)^2 + \left(\frac{7}{4} - 2\right)^2} = \frac{\sqrt{5}}{4} \\ \lambda_1(l \cap R_{1,4}) &= \lambda_1(\overline{P_2Q_4}) = \sqrt{\left(\frac{7}{2} - 4\right)^2 + \left(2 - \frac{9}{4}\right)^2} = \frac{\sqrt{5}}{4}\end{aligned}$$

Now we can calculate the line integral of the phantom  $f$  along the line  $l$ .

$$\begin{aligned}
 \int_l f &= \sum_{i=1}^m \sum_{j=1}^n a_{i,j} \cdot \lambda_1(l \cap R_{i,j}) = \\
 &= a_{3,1} \cdot \lambda_1(l \cap R_{3,1}) + a_{3,2} \cdot \lambda_1(l \cap R_{3,2}) + a_{2,2} \cdot \lambda_1(l \cap R_{2,2}) + \\
 &+ a_{2,3} \cdot \lambda_1(l \cap R_{2,3}) + a_{2,4} \cdot \lambda_1(l \cap R_{2,4}) + a_{1,4} \cdot \lambda_1(l \cap R_{1,4}) = \\
 &= \frac{1}{3} \cdot \frac{\sqrt{5}}{2} + \frac{1}{2} \cdot \frac{\sqrt{5}}{4} + \frac{1}{3} \cdot \frac{\sqrt{5}}{4} + \frac{1}{2} \cdot \frac{\sqrt{5}}{2} + \frac{1}{3} \cdot \frac{\sqrt{5}}{4} + \frac{1}{2} \cdot \frac{\sqrt{5}}{4} = \frac{5}{6} \sqrt{5}
 \end{aligned}$$

### 3.3 Typical line sets for data collection

Basically there are two types of line sets which are commonly used for data collection in CT: one related to parallel beam x-rays and the other one related to divergent beam x-rays. We talk about parallel beam x-ray, when there's a finite set of directions  $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_k$ , and for each direction  $\underline{v}_i$  there are points  $P_{i,1}, P_{i,2}, \dots, P_{i,s_i}$  along a line which is not parallel to  $\underline{v}_i$ , and adjacent points have the same distance from each other. Then the set of lines used for measurements is  $\{l_{i,j} \mid i \in \{1, 2, \dots, k\}, j \in \{1, 2, \dots, s_i\}\}$ , where  $l_{i,j}$  is the line passing through  $P_{i,j}$  and parallel to  $\underline{v}_i$ .



We talk about divergent beam x-rays when there's a finite set of points  $P_1, P_2, \dots, P_k$ , and for each point  $P_i$  there are directions  $\underline{v}_{i,1}, \underline{v}_{i,2}, \dots, \underline{v}_{i,s_i}$ . Then the set of lines used for measurements is  $\{l_{i,j} \mid i \in \{1, 2, \dots, k\}, j \in \{1, 2, \dots, s_i\}\}$ , where  $l_{i,j}$  is the line passing through  $P_i$  and parallel to  $\underline{v}_{i,j}$ .

