2.3 Systems of linear inequalities

A linear inequality for the unknowns (or variables) $x_1, x_2, x_3, \ldots x_n$ is inequaltilty, where a linear combination of $x_1, x_2, \ldots x_n$ is less than or larger than a constant. If the coefficients in the linear combination are $a_1, a_2, \ldots, a_n$, and the constant is b, then the linear inequality is

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n \leq b$$

or

$$a_1x_1 + a_2x_2 + a_3x_3 + \ldots + a_nx_n \geq b$$

For the sake of simplicity we deal only with inequalities where equality is allowed (i.e. less than or equal, larger or equal). Note that a greater-or-equal-type inequality can be transformed to a less-than-or-equal-type inequality by multiplying it with (-1). Thus it's enough to deal with linear inequalities where a linear combination of the unknowns is less than or equal to a constant. A system of m linear inequalities for the unknowns $x_1, x_2, x_3, \ldots x_n$ is of the following form.

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \leq b_2 \\ \vdots \qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \leq b_m \end{array}\right\}$$

The coefficients in the above system naturally define a matrix of size $m \times n$,

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

which is called the **coefficient matrix** of the system. If $\underline{x}$ denotes the column vector containing the unknowns and $\underline{b}$ denotes the column vector containing the the right-hand-side constants, then the matrix form of the system of inequalities is

$$A \cdot \underline{x} \leq \underline{b}$$

where the column vector $A \cdot \underline{x}$ is less than or equal to the column vector $\underline{b}$ if each entry of $A \cdot \underline{x}$ is less than or equal to the corresponding entry of

$\underline{b}$. Solving systems of linear inequalities can be quite challenging. Even the decision problem whether a given system has a solution or not is just complicated as a linear programming problem. The linear programming problem is to minimize (or maximize) a given linear combination of the unknowns under linear equality and inequality constraints. There are computationally efficient algorithms to solve linear programming problems, such as simplex method or interior point method, but the presentation of these algorithms is way beyond the scope of this text. Instead, we would like to introduce two methods which are much more simple to describe and can be used to solve small systems linear inequalities. These are the graphical solution method and Fourier-Motzkin elimination. Although the simplex method and interior point method are efficient for large system of inequalities, they are designed to find only one optimal solution and they're unable to find all the solutions of the system. An advantage of graphical solution method and Fourier-Motzkin elimination is that it's possible to give all solutions of the system with the help of them.

2.3.1 Graphical solution method

The graphical solution method can be applied to a system of linear inequalities for two variables. The unknowns in this situation are typically denoted by x and y instead of x_1 and x_2. The idea behind this method is that the set of points in the plane whose Cartesian coordinates (x, y) satisfy the linear equation $a_1x + a_2y = b$ is a line which is perpendicular to the vector (a_1, a_2). The set of points in the plane whose Cartesian coordinates (x, y) satisfy the linear inequality $a_1x + a_2y \leq b$ is a closed half-plane bounded by the line $a_1x + a_2y = b$. More precisely the linear inequality $a_1x + a_2y \leq b$ determines the opposite of the half-plane where the vector (a_1, a_2) points to. Thus the closed half-plane determined by $a_1x + a_2y \leq b$ can be presented in a figure by drawing the line $a_1x + a_2y = b$ and a vector (more precisely a directed line segment) with the same direction as $(-a_1, -a_2)$ and initial point on the line $a_1x + a_2y = b$.

The pair of real numbers (x, y) is a solution of the system of linear inequalities

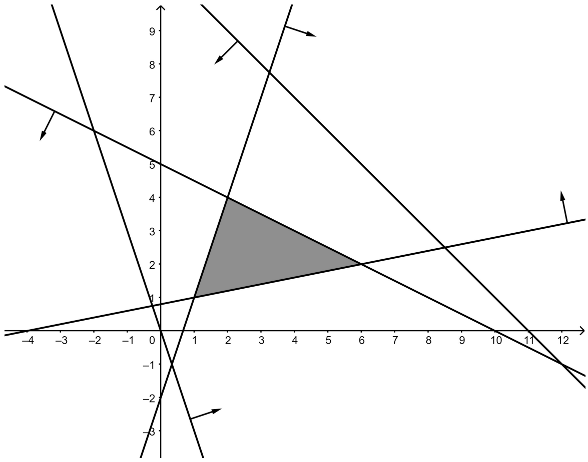
$$\left.\begin{aligned} a_{11}x + a_{12}y &\le b_1 \\ a_{21}x + a_{22}y &\le b_2 \\ \vdots \qquad\quad & \;\vdots \\ a_{m1}x + a_{m2}y &\le b_m \end{aligned}\right\}$$

if and only if the point in the plane with Cartesian coordinates is in the intersection of all half-planes determined by the inequalities in the system. Thus it's enough to present all the half-planes determined by the inequalities in the system and look for their intersection.

Example 1.

$$\left.\begin{aligned} x + 2y &\le 10 \\ x - 5y &\le -4 \\ -3x + y &\le -2 \\ x + y &\le 11 \\ -3x - y &\le 0 \end{aligned}\right\}$$

The line $x + 2y = 10$ can be drawn easily if we find two points of it. A point of the line can be found by substituting an arbitrary value to x or y and then solve the equation for the other variable. Let's choose now $x = 0$. Then y must satisfy $0 + 2y = 10$, which gives $y = 5$. This means $(0, 5)$ is a point of the line $x + 2y = 10$. If we choose $y = 0$, then x must satisfy $x + 2 \cdot 0 = 10$, which gives $x = 10$. This means $(10, 0)$ is a point of the line $x+2y = 10$. Thus $x+2y = 10$ is the line which passes through the points $(0, 5)$ and $(10, 0)$. The closed half-plane determined by $x + 2y \le 10$ can be presented by drawing (besides the corresponding line) a vector with the same direction as $(-1, -2)$ and initial point on the line $x + 2y = 10$. The rest of the half-planes are presented in a similar way.



We can see that the system is solvable and the set of solutions is a triangle (shaded area in the figure). We also found that the inequalities $x + y \leq 11$ and $-3x - y \leq 0$ are redundant, which means that the set of solutions doesn't change if we omit these inequalities from the system. The triangle is bounded from below by the line $x - 5y = -4$ and bounded from above by the lines $-3x + y = -2$ and $x + 2y = 10$. We can rearrange the inequality corresponding to the lower bounding line as $\frac{1}{5}x + \frac{4}{5} \leq y$ and we can rearrange the inequalities corresponding to the upper bounding lines as $y \leq 3x - 2$ and $y \leq -\frac{1}{2}x + 5$. Furthermore we can see that the first component of any point in the triangle is between 1 and 6. Thus the set of solutions can be algebraically characterized as the set of points in the plane whose Cartesian coordinates (x, y) satisfy

$$\left.\begin{array}{c} 1 \leq x \leq 6 \\ \frac{1}{5}x + \frac{4}{5} \leq y \leq \min\left\{3x - 2, -\frac{1}{2}x + 5\right\} \end{array}\right\}$$

We can go even further and say $3x - 2 \leq -\frac{1}{2}x + 5$ is satisfied if and only if $x \leq 2$, hence the set of solutions is the set of points in the plane whose Cartesian coordinates (x, y) satisfy

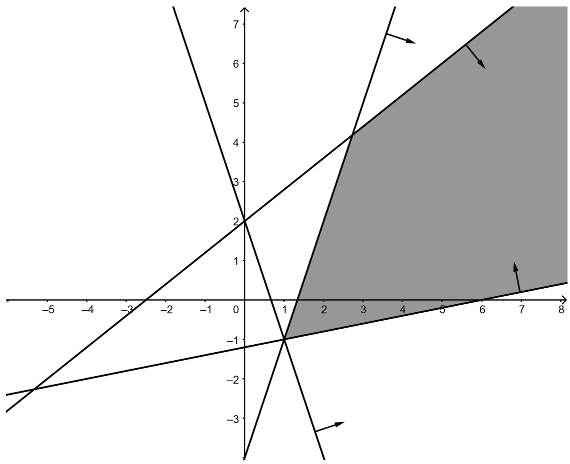
$$\left.\begin{array}{c} 1 \leq x \leq 2 \\ \frac{1}{5}x + \frac{4}{5} \leq y \leq 3x - 2 \end{array}\right\} \quad \text{or} \quad \left.\begin{array}{c} 2 \leq x \leq 6 \\ \frac{1}{5}x + \frac{4}{5} \leq y \leq -\frac{1}{2}x + 5 \end{array}\right\}$$

Geometrically this mean we cut the triangle into two pieces with the help of the vertical line $x = 2$ and say a point presents a solution if it's an element of the left-hand-part or an element of the right-hand-part.

Example 2.

$$\left.\begin{aligned} x-5y &\le 6 \\ -3x+y &\le -4 \\ -3x-y &\le -2 \\ -4x+5y &\le 10 \end{aligned}\right\}$$

The half-planes determined by the inequalities in the system are presented in a similar way as in Example 1. above.



We can see that the system is solvable, but this time the set of solutions is an unbounded area in the plane (shaded in the figure). We find that the inequality $-3x-y\le -2$ is redundant. The set of solutions is bounded from below by the line $x-5y=6$ and bounded from above by the lines $-3x+y=-4$ and $-4x+5y=10$. We can rearrange the inequality corresponding to the lower bounding line as $\frac{1}{5}x-\frac{6}{5}\le y$ and we can rearrange the inequalities corresponding to the upper bounding lines as $y\le 3x-4$ and $y\le \frac{4}{5}x+2$. The first component of any point in the set of solutions is larger than 1. Thus the set of solutions can be algebraically described as the set of points in the plane whose Cartesian coordinates (x,y) satisfy

$$\left.\begin{aligned} 1\le x &< +\infty \\ \tfrac{1}{5}x-\tfrac{6}{5}\le y &\le \min\left\{3x-4, \tfrac{4}{5}x+2\right\} \end{aligned}\right\}$$

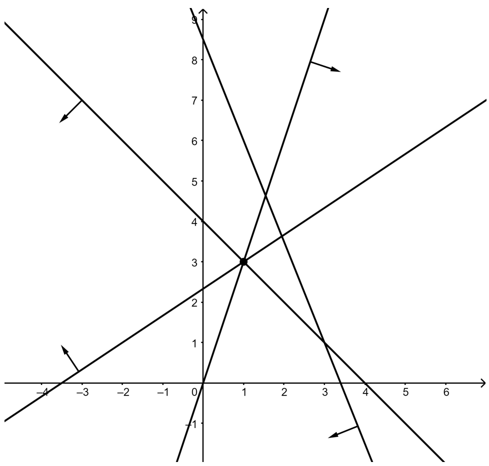
Here $3x - 4 \le \frac{4}{5}x + 2$ is satisfied if and only if $x \le \frac{30}{11}$, hence the set of solutions is the set of points in the plane whose Cartesian coordinates (x, y) satisfy

$$\left.\begin{array}{r} 1 \le x \le \frac{30}{11} \\ \frac{1}{5}x - \frac{6}{5} \le y \le 3x - 4 \end{array}\right\} \quad \text{or} \quad \left.\begin{array}{r} \frac{30}{11} \le x < +\infty \\ \frac{1}{5}x - \frac{6}{5} \le y \le \frac{4}{5}x + 2 \end{array}\right\}$$

Example 3.

$$\left.\begin{array}{r} x + y \le 4 \\ 2x - 3y \le -7 \\ 5x + 2y \le 17 \\ -3x + y \le 0 \end{array}\right\}$$

The half-planes determined by the inequalities in the system are presented in a similar way as in Example 1.



We can see that the system is solvable, but this time the set of solutions contains only the single point $(1, 3)$. We find that the inequality $5x + 2y \le 17$ is redundant. To prove algebraically that there's no solution besides the point $(1, 3)$, let's express y from all the relevant (i.e. non-redundant) inequalities. Then we have

$$\left.\begin{array}{r} y \le -x + 4 \\ \frac{2}{3}x + \frac{7}{3} \le y \qquad \\ y \le 3x \qquad \end{array}\right\}$$

This shows that

$$\frac{2}{3}x + \frac{7}{3} \leq y \leq \min\{-x+4, 3x\}$$

but this can satisfied only if the following two inequalities hold.

$$\left.\begin{array}{l} \frac{2}{3}x + \frac{7}{3} \leq -x + 4 \\ \frac{2}{3}x + \frac{7}{3} \leq 3x \end{array}\right\}$$

These can be rearranged as

$$\left.\begin{array}{r} \frac{5}{3}x \leq \frac{5}{3} \\ -\frac{7}{3}x \leq -\frac{7}{3} \end{array}\right\}$$

and then

$$\left.\begin{array}{r} x \leq 1 \\ x \geq 1 \end{array}\right\}$$

The only solution of the above system is $x = 1$. Substituting this back into the last system which contained y gives

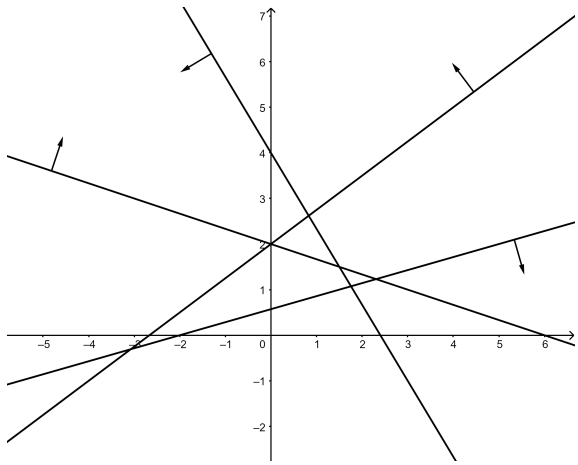
$$\left.\begin{array}{r} y \leq 3 \\ 3 \leq y \\ y \leq 3 \end{array}\right\}$$

which shows that $y = 3$. Thus the only solution is clearly $(x, y) = (1, 3)$.

Example 4.

$$\left.\begin{array}{r} 3x - 4y \leq -8 \\ -x - 3y \leq -6 \\ 5x + 3y \leq 12 \\ -2x + 7y \leq 4 \end{array}\right\}$$

The half-planes determined by the inequalities in the system are presented in the following figure.



We can see that the system is has no solution, as no point of the intersection of the half-planes determined by the first three inequalities is an element of the half-plane determined by the last inequality. To see this algebraically, let's express y from all inequalities. Then we have

$$\left.\begin{array}{r l} \frac{3}{4}x+2 \leq y & \\ -\frac{1}{3}x+2 \leq y & \\ & y \leq -\frac{5}{3}x+4 \\ & y \leq \frac{2}{7}x+\frac{4}{7} \end{array}\right\}$$

This shows that

$$\max\left\{\frac{3}{4}x+2, -\frac{1}{3}x+2\right\} \leq y \leq \min\left\{-\frac{5}{3}x+4, \frac{2}{7}x+\frac{4}{7}\right\}$$

but this can satisfied only if the following four inequalities hold.

$$\left.\begin{array}{r} \frac{3}{4}x+2 \leq -\frac{5}{3}x+4 \\ \frac{3}{4}x+2 \leq \frac{2}{7}x+\frac{4}{7} \\ -\frac{1}{3}x+2 \leq -\frac{5}{3}x+4 \\ -\frac{1}{3}x+2 \leq \frac{2}{7}x+\frac{4}{7} \end{array}\right\}$$

These can be rearranged as

$$\left.\begin{aligned} \tfrac{29}{12}x &\le 2 \\ \tfrac{13}{28}x &\le -\tfrac{10}{7} \\ \tfrac{4}{3}x &\le 2 \\ -\tfrac{13}{21}x &\le -\tfrac{10}{7} \end{aligned}\right\}$$

and then

$$\left.\begin{aligned} x &\le \tfrac{24}{29} \\ x &\le \tfrac{40}{13} \\ x &\le \tfrac{3}{2} \\ x &\ge \tfrac{30}{13} \end{aligned}\right\}$$

Here $\frac{24}{29} < 1$ and $1 < \frac{30}{13}$, hence there's no $x \in \mathbb{R}$ which could satisfy the above inequalities.

2.3.2 Fourier-Motzkin elimination

The graphical solution method is a good tool to visualize the set of solutions of a system of linear inequalities. We can also easily find the redundant inequalities in the system with the help of it. However sometimes we need to apply an algebraic approach for example when the lines corresponding to the inequalities intersect each other in points which are very close to each other. In some cases it's hard to make a decision upon the graphical image whether an intersection point of two lines is an element of a third line or not. Furthermore the graphical solution method can be applied to systems with only two unknowns. We could develop an extended version for the case of three unknowns since inequalities can be visualized then as half-spaces of the three dimensional coordinate space, but the shape of the intersection of such half-spaces can be very complicated and hard to find a good projection onto the plane. Besides there's no hope to visualize inequalities when the number unknowns is larger than 3, since these could be presented only in higher dimensional spaces.

Now we introduce an algebraical approach to find all the solutions of a system of linear inequalities. This method was first developed by French mathematician J.B.J. Fourier in 1826 and later T. S. Motzkin rediscovered it in 1936 to solve linear programming problems, and hence the method is named after them. The Fourier-Motzkin elimination method eliminates the unknowns from the system one-by-one until only one unknown is left. Then it can be easily told whether the system is solvable or not, and what's the possible range of that single unknown when the system is solvable. After that the possible values of the variables can be substituted back step-by-step to find all the solutions of the system. Now we first show how to eliminate the last unknown from any system and then this can be repeated until it's necessary and only one unknown is left.

Consider the following system of linear inequalities:

$$\left.\begin{array}{r} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \le b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \le b_2 \\ \vdots \qquad\qquad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \le b_m \end{array}\right\}$$

We can collect here first those inequalities where the coefficient of x_n is positive, then collect the those inequalities where the coefficient of x_n is negative, and finally collect those where the coefficient of x_n is zero (i.e. x_n is missing). Assume that the above system is already in this form, that is there are non-negative integers $0 \le k \le l \le m$ such that

$$\begin{array}{lll} a_{in} > 0 & \text{for all} & i \in \{1, 2, \ldots, k\} \\ a_{in} < 0 & \text{for all} & i \in \{k+1, k+2, \ldots, l\} \\ a_{in} = 0 & \text{for all} & i \in \{l+1, l+2, \ldots, m\} \end{array}$$

In other words the coefficient of x_n is positive in the first k inequalities, negative in the next $l-k$ inequalities, and zero in the rest of the inequalities.

$$\left.\begin{array}{r} a_{1\,1}x_1 + a_{1\,2}x_2 + \ldots + a_{1\,n}x_n \le b_1 \\ a_{2\,1}x_1 + a_{2\,2}x_2 + \ldots + a_{2\,n}x_n \le b_2 \\ \vdots \qquad\qquad\quad \vdots \\ a_{k\,1}x_1 + a_{k\,2}x_2 + \ldots + a_{k\,n}x_n \le b_k \\ a_{k+1\,1}x_1 + a_{k+1\,2}x_2 + \ldots + a_{k+1\,n}x_n \le b_{k+1} \\ \vdots \qquad\qquad\quad \vdots \\ a_{l\,1}x_1 + a_{l\,2}x_2 + \ldots + a_{l\,n}x_n \le b_l \\ a_{l+1\,1}x_1 + a_{l+1\,2}x_2 + \ldots + a_{l+1\,n}x_n \le b_{l+1} \\ \vdots \qquad\qquad\quad \vdots \\ a_{m\,1}x_1 + a_{m\,2}x_2 + \ldots + a_{m\,n}x_n \le b_m \end{array}\right\}$$

This can be rearranged then as

$$\left.\begin{array}{r}
x_n \le \frac{b_1}{a_{1\,n}} - \frac{a_{1\,1}}{a_{1\,n}}x_1 - \frac{a_{1\,2}}{a_{1\,n}}x_2 - \ldots - \frac{a_{1\,n-1}}{a_{1\,n}}x_{n-1} \\
x_n \le \frac{b_2}{a_{2\,n}} - \frac{a_{2\,1}}{a_{2\,n}}x_1 - \frac{a_{2\,2}}{a_{2\,n}}x_2 - \ldots - \frac{a_{2\,n-1}}{a_{2\,n}}x_{n-1} \\
\vdots \qquad\qquad\qquad \vdots \qquad\qquad \\
x_n \le \frac{b_k}{a_{k\,n}} - \frac{a_{k\,1}}{a_{k\,n}}x_1 - \frac{a_{k\,2}}{a_{k\,n}}x_2 - \ldots - \frac{a_{k\,n-1}}{a_{k\,n}}x_{n-1} \\
\frac{b_{k+1}}{a_{k+1\,n}} - \frac{a_{k+1\,1}}{a_{k+1\,n}}x_1 - \frac{a_{k+1\,2}}{a_{k+1\,n}}x_2 - \ldots - \frac{a_{k+1\,n-1}}{a_{k+1\,n}}x_{n-1} \le x_n \\
\vdots \qquad\qquad \vdots \\
\frac{b_l}{a_{l\,n}} - \frac{a_{l\,1}}{a_{l\,n}}x_1 - \frac{a_{l\,2}}{a_{l\,n}}x_2 - \ldots - \frac{a_{l\,n-1}}{a_{l\,n}}x_{n-1} \le x_n \\
a_{l+1\,1}x_1 + a_{l+1\,2}x_2 + \ldots + a_{l+1\,n-1}x_{n-1} \le b_{l+1} \\
\vdots \qquad\qquad \vdots \\
a_{m\,1}x_1 + a_{m\,2}x_2 + \ldots + a_{m\,n-1}x_{n-1} \le b_m
\end{array}\right\}$$

Let's see the following cases.

1. If all coefficients of x_n are zero (i.e. $k = l = 0$ and x_n is missing from each inequality), then x_n may take any value independently of the values of the other unknowns, however the system can be still unsolvable.

2. If the coefficients of x_n are positive in all inequalities of the system (i.e. $k = l = m$), then we have only upper bounds for x_n. In this situation we can choose arbitrary values for $x_1, x_2, \ldots, x_{n-1}$ and then x_n must be not larger than the minimum of the upper bounds, which can be computed after substituting the values of $x_1, x_2, \ldots, x_{n-1}$.

3. If the coefficients of x_n are negative in all inequalities of the system (i.e. $k = 0$ and $l = m$), then we have only lower bounds for x_n. In this situation we can choose arbitrary values for $x_1, x_2, \ldots, x_{n-1}$ and then x_n must be not less than the maximum of the lower bounds, which can be computed after substituting the values of $x_1, x_2, \ldots, x_{n-1}$.

4. If the coefficients of x_n are positive or zero (but not negative) in all inequalities of the system (i.e. $k = l < m$), then we have to find the set

of solutions of the system

$$\left.\begin{array}{r} a_{l\,1}x_1 + a_{l\,2}x_2 + \ldots + a_{l\,n-1}x_{n-1} \le b_l \\ a_{l+1\,1}x_1 + a_{l+1\,2}x_2 + \ldots + a_{l+1\,n-1}x_{n-1} \le b_{l+1} \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m\,1}x_1 + a_{m\,2}x_2 + \ldots + a_{m\,n-1}x_{n-1} \le b_m \end{array}\right\}$$

If there's a solution, then x_n must be not larger than the minimum of the upper bounds, which can computed with the help of the solutions of the above system.

5. If the coefficients of x_n are negative or zero (but not positive) in all inequalities of the system (i.e. $k = 0$ and $0 < l < m$), then we have to find the set of solutions of the system

$$\left.\begin{array}{r} a_{l\,1}x_1 + a_{l\,2}x_2 + \ldots + a_{l\,n-1}x_{n-1} \le b_l \\ a_{l+1\,1}x_1 + a_{l+1\,2}x_2 + \ldots + a_{l+1\,n-1}x_{n-1} \le b_{l+1} \\ \vdots \qquad\qquad\qquad \vdots \\ a_{m\,1}x_1 + a_{m\,2}x_2 + \ldots + a_{m\,n-1}x_{n-1} \le b_m \end{array}\right\}$$

   If there's a solution, then x_n must be not less than the maximum of the lower bounds, which can computed with the help of the solutions of the above system.

6. If there are both positive and negative coefficients of x_n in the system (i.e. $0 < k < l \le m$), then we have both upper and lower bounds for x_n. Thus x_n must be not larger than the minimum of the upper bounds and not less than the maximum of the lower bounds. We can choose a value for x_n if and only if the maximum of the lower bounds is less then or equal to the minimum of the upper bounds. This property holds exactly when every lower bound for x_n is not larger than any of the upper bounds. This yields to a system of inequalities together with those where the coefficient of x_n is zero if there's any. This new system contains only the unknowns $x_1, x_2, \ldots, x_{n-1}$. A solution of that system can be substituted back to get the lower and upper bound for x_n.

Now we discuss what to do when only one unknown is left. Assume we have system

$$\left.\begin{array}{r@{\;\le\;}l} a_{1\,1}x_1 & b_1 \\ a_{2\,1}x_1 & b_2 \\ \vdots\quad & \;\vdots \\ a_{k\,1}x_1 & b_k \\ a_{k+1\,1}x_1 & b_{k+1} \\ \vdots\quad & \;\vdots \\ a_{l\,1}x_1 & b_k \\ a_{l+1\,1}x_1 & b_{l+1} \\ \vdots\quad & \;\vdots \\ a_{m\,1}x_1 & b_m \end{array}\right\}$$

where

$$\begin{array}{ll} a_{i1} > 0 & \text{if } \; i \in \{1, 2, \ldots, k\} \\ a_{i1} < 0 & \text{if } \; i \in \{k+1, k+2, \ldots, l\} \\ a_{i1} = 0 & \text{if } \; i \in \{l+1, l+2, \ldots, m\} \end{array}$$

The later means we possibly have inequalities of the form $0 \le b_i$. Such inequalities can be produced during the preceding steps. If there's at least one $i \in \{l+1, l+2, \ldots, m\}$ such that $b_i < 0$, then the system has no solution. Otherwise those inequalities are irrelevant and it's enough to treat the first l inequalities.

1. If the coefficients of x_1 are positive in all inequalities of the system, then we have only upper bounds for x_n. In this situation the system is solvable and the only restriction for x_1 is that it must be not larger than the minimum of the upper bounds.

2. If the coefficients of x_1 are negative in all inequalities of the system, then we have only lower bounds for x_1. In this situation the system is solvable and the only restriction for x_1 is that it must be not less than the maximum of the lower bounds.

3. If there are both positive and negative coefficients of x_1 in the system, then we have both upper and lower bounds for x_1. Thus x_1 must be not larger than the minimum of the upper bounds and not less than the maximum of the lower bounds. We can choose a value for x_1 if and only if the maximum of the lower bounds is less then or equal to the minimum of the upper bounds. Otherwise the system is unsolvable.

Now can solve any system of linear inequalities with the help of the above procedure. The drawback of the Fourier-Motzkin elimination method is that the number of inequalities can exponentially grow during the procedure, thus we can use it for systems with relatively few inequalities.

Example 1.

$$\left.\begin{aligned} 3x_1 - x_2 + 2x_3 &\le -2 \\ x_1 - 3x_2 + x_3 &\le 6 \\ 4x_1 + x_2 - x_3 &\le -1 \\ -x_1 + 2x_2 - 3x_3 &\le 0 \end{aligned}\right\}$$

Here the coefficients of x_3 are positive in the first and second inequalities, and negative in the third and fourth inequalities. Now we rearrange them to get lower and upper bounds for x_3.

$$\left.\begin{aligned} & x_3 \le -\tfrac{3}{2}x_1 + \tfrac{1}{2}x_2 - 1 \\ & x_3 \le -x_1 + 3x_2 + 6 \\ 4x_1 + x_2 + 1 \le x_3 & \\ -\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 \le x_3 & \end{aligned}\right\}$$

Thus

$$\max\left\{4x_1 + x_2 + 1, -\frac{1}{3}x_1 + \frac{2}{3}x_2\right\} \le x_3 \le \min\left\{-\frac{3}{2}x_1 + \frac{1}{2}x_2 - 1, -x_1 + 3x_2 + 6\right\}$$

The system may have a solution only if each lower bound for x_3 is not larger than any of the upper bounds. Hence we need to find the set of solutions of the system

$$\left.\begin{aligned} 4x_1 + x_2 + 1 &\le -\tfrac{3}{2}x_1 + \tfrac{1}{2}x_2 - 1 \\ 4x_1 + x_2 + 1 &\le -x_1 + 3x_2 + 6 \\ -\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 &\le -\tfrac{3}{2}x_1 + \tfrac{1}{2}x_2 - 1 \\ -\tfrac{1}{3}x_1 + \tfrac{2}{3}x_2 &\le -x_1 + 3x_2 + 6 \end{aligned}\right\}$$

Let's transform it into standard form.

$$\left.\begin{aligned} \tfrac{11}{2}x_1 + \tfrac{1}{2}x_2 &\le -2 \\ 5x_1 - 2x_2 &\le 5 \\ \tfrac{7}{6}x_1 + \tfrac{1}{6}x_2 &\le -1 \\ \tfrac{2}{3}x_1 - \tfrac{7}{3}x_2 &\le 6 \end{aligned}\right\}$$

Collect the those inequalities first where the coefficient of x_2 is positive.

$$\left.\begin{array}{r}\frac{11}{2}x_1+\frac{1}{2}x_2\leq -2\\ \frac{7}{6}x_1+\frac{1}{6}x_2\leq -1\\ 5x_1-2x_2\leq 5\\ \frac{2}{3}x_1-\frac{7}{3}x_2\leq 6\end{array}\right\}$$

Here the coefficients of x_2 are positive in the first and second inequalities, and negative in the third and fourth inequalities. Rearrange them to get lower and upper bounds for x_2.

$$\left.\begin{array}{rl} & x_2\leq -11x_1-4\\ & x_2\leq -7x_1-6\\ \frac{5}{2}x_1-\frac{5}{2}\leq x_2 & \\ \frac{2}{7}x_1-\frac{18}{7}\leq x_2 & \end{array}\right\}$$

Thus

$$\max\left\{\frac{5}{2}x_1-\frac{5}{2},\frac{2}{7}x_1-\frac{18}{7}\right\}\leq x_2\leq\min\left\{-11x_1-4,-7x_1-6\right\}$$

The system may have a solution only if each lower bound for x_2 is not larger than any of the upper bounds. Hence we need to find the set of solutions of the system

$$\left.\begin{array}{r}\frac{5}{2}x_1-\frac{5}{2}\leq -11x_1-4\\ \frac{5}{2}x_1-\frac{5}{2}\leq -7x_1-6\\ \frac{2}{7}x_1-\frac{18}{7}\leq -11x_1-4\\ \frac{2}{7}x_1-\frac{18}{7}\leq -7x_1-6\end{array}\right\}$$

Let's transform it into standard form.

$$\left.\begin{array}{r}\frac{27}{2}x_1\leq -\frac{3}{2}\\ \frac{19}{2}x_1\leq -\frac{7}{2}\\ \frac{79}{7}x_1\leq -\frac{10}{7}\\ \frac{51}{7}x_1\leq -\frac{24}{7}\end{array}\right\}$$

Here all coefficients of x_1 are positive, hence we have only upper bound for x_1, but no lower bound. The upper bounds are

$$\left.\begin{array}{l} x_1 \leq -\frac{1}{9} \\ x_1 \leq -\frac{7}{19} \\ x_1 \leq -\frac{10}{79} \\ x_1 \leq -\frac{8}{17} \end{array}\right\}$$

Thus

$$x_1 \leq \min\left\{-\frac{1}{9}, -\frac{7}{19}, -\frac{10}{79}, -\frac{8}{17}\right\} = -\frac{8}{17}$$

Finally we see that the solutions of the system are those triples (x_1, x_2, x_3) which satisfy

$$\left.\begin{array}{c} -\infty < x_1 \leq -\frac{8}{17} \\ \max\left\{\frac{5}{2}x_1 - \frac{5}{2}, \frac{2}{7}x_1 - \frac{18}{7}\right\} \leq x_2 \leq \min\left\{-11x_1 - 4, -7x_1 - 6\right\} \\ \max\left\{4x_1 + x_2 + 1, -\frac{1}{3}x_1 + \frac{2}{3}x_2\right\} \leq x_3 \leq \min\left\{-\frac{3}{2}x_1 + \frac{1}{2}x_2 - 1, -x_1 + 3x_2 + 6\right\} \end{array}\right\}$$

To give one solution let's choose $x_1 = -1$ first. Then

$$\max\left\{-5, -\frac{20}{7}\right\} \leq x_2 \leq \min\left\{7, 1\right\},$$

that is

$$-\frac{20}{7} \leq x_2 \leq 1$$

Let's choose $x_2 = -1$. Then

$$\max\left\{-4, -\frac{1}{3}\right\} \leq x_3 \leq \min\left\{0, 4\right\},$$

that is

$$-\frac{1}{3} \leq x_1 \leq 0$$

Finally choose $x_3 = 0$. This shows that $(-1, -1, 0)$ is a solution of the system.

Example 2.

$$\left.\begin{array}{r} x_1 + 2x_2 - 3x_3 + 4x_4 \leq -7 \\ x_1 + x_2 + 2x_3 - x_4 \leq 8 \\ x_1 + x_2 + x_3 - 3x_4 \leq 4 \\ -x_1 - 2x_2 - x_4 \leq 1 \\ 3x_1 + 2x_2 + x_3 \leq 4 \\ -x_1 + 2x_2 \leq 5 \end{array}\right\}$$

Here the coefficient of x_4 is positive in the first inequality, they are negative in the second, third and fourth inequalities, and zero in the fifth and sixth inequalities. Rearrange the first four inequalities to get lower and upper bounds for x_4.

$$\left.\begin{array}{r} x_4 \leq -\frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{3}{4}x_3 - \frac{7}{4} \\ x_1 + x_2 + 2x_3 - 8 \leq x_4 \\ \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 - \frac{4}{3} \leq x_4 \\ -x_1 - 2x_2 \leq x_4 \\ 3x_1 + 2x_2 + x_3 \leq 4 \\ -x_1 + 2x_2 \leq 5 \end{array}\right\}$$

Thus

$$\max\left\{x_1 + x_2 + 2x_3 - 8, \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 - \frac{4}{3}, -x_1 - 2x_2 - 1\right\} \leq x_4$$

and

$$x_4 \leq -\frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{3}{4}x_3 - \frac{7}{4}$$

The system may have a solution only if each lower bound for x_4 is not larger than any of the upper bounds. Hence we need to find the set of solutions of the system

$$\left.\begin{array}{rl} x_1 + x_2 + 2x_3 - 8 \leq & -\frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{3}{4}x_3 - \frac{7}{4} \\ \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}x_3 - \frac{4}{3} \leq & -\frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{3}{4}x_3 - \frac{7}{4} \\ -x_1 - 2x_2 - 1 \leq & -\frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{3}{4}x_3 - \frac{7}{4} \\ 3x_1 + 2x_2 + x_3 \leq & 4 \\ -x_1 + 2x_2 \leq & 5 \end{array}\right\}$$

Let's transform it into standard form.

$$\left.\begin{array}{r}\frac{5}{4}x_1+\frac{3}{2}x_2+\frac{5}{4}x_3\le\ \frac{25}{4}\\ \frac{7}{12}x_1+\frac{5}{6}x_2-\frac{5}{12}x_3\le-\frac{5}{12}\\ -\frac{3}{4}x_1-\frac{3}{2}x_2-\frac{3}{4}x_3\le-\frac{3}{4}\\ 3x_1+2x_2+x_3\le 4\\ -x_1+2x_2\le 5\end{array}\right\}$$

Collect the those equations first where the coefficient of x_3 is positive.

$$\left.\begin{array}{r}\frac{5}{4}x_1+\frac{3}{2}x_2+\frac{5}{4}x_3\le\ \frac{25}{4}\\ 3x_1+2x_2+x_3\le 4\\ \frac{7}{12}x_1+\frac{5}{6}x_2-\frac{5}{12}x_3\le-\frac{5}{12}\\ -\frac{3}{4}x_1-\frac{3}{2}x_2-\frac{3}{4}x_3\le-\frac{3}{4}\\ -x_1+2x_2\le 5\end{array}\right\}$$

Here the coefficients of x_3 are positive in the first and second inequalities, negative in the third and fourth inequalities, and zero in the fifth inequality. Rearrange them to get lower and upper bounds for x_3.

$$\left.\begin{array}{rl} & x_3\le -x_1-\frac{6}{5}x_2+5\\ & x_3\le -3x_1-2x_2+4\\ \frac{7}{5}x_1+2x_2+1\le x_3 & \\ -x_1-2x_2+1\le x_3 & \\ -x_1+2x_2\le\ 5 & \end{array}\right\}$$

Thus

$$\max\left\{\frac{7}{5}x_1+2x_2+1,-x_1-2x_2+1\right\}\le x_3\le\min\left\{-x_1-\frac{6}{5}x_2+5,-3x_1-2x_2+4\right\}$$

The system may have a solution only if each lower bound for x_3 is not larger than any of the upper bounds. Hence we need to find the set of solutions of the system

$$\left.\begin{array}{l}\frac{7}{5}x_1+2x_2+1\le -x_1-\frac{6}{5}x_2+5\\ \frac{7}{5}x_1+2x_2+1\le -3x_1-2x_2+4\\ -x_1-2x_2+1\le -x_1-\frac{6}{5}x_2+5\\ -x_1-2x_2+1\le -3x_1-2x_2+4\\ \quad -x_1+2x_2\le 5\end{array}\right\}$$

Let's transform it into standard form.

$$\left.\begin{aligned} \tfrac{12}{5}x_1 + \tfrac{16}{5}x_2 &\leq 4 \\ \tfrac{22}{5}x_1 + 4x_2 &\leq 3 \\ -\tfrac{4}{5}x_2 &\leq 4 \\ 2x_1 &\leq 3 \\ -x_1 + 2x_2 &\leq 5 \end{aligned}\right\}$$

Collect the equations upon the coefficients of x_2.

$$\left.\begin{aligned} \tfrac{12}{5}x_1 + \tfrac{16}{5}x_2 &\leq 4 \\ \tfrac{22}{5}x_1 + 4x_2 &\leq 3 \\ -x_1 + 2x_2 &\leq 5 \\ -\tfrac{4}{5}x_2 &\leq 4 \\ 2x_1 &\leq 3 \end{aligned}\right\}$$

Here the coefficients of x_2 are positive in the first three inequalities, negative in the fourth inequality, and zero in the fifth. Rearrange them to get lower and upper bounds for x_2.

$$\left.\begin{aligned} x_2 &\leq -\tfrac{3}{4}x_1 + \tfrac{5}{4} \\ x_2 &\leq -\tfrac{11}{10}x_1 + \tfrac{3}{4} \\ x_2 &\leq \tfrac{1}{2}x_1 + \tfrac{5}{2} \\ -5 &\leq x_2 \\ 2x_1 &\leq 3 \end{aligned}\right\}$$

Thus

$$-5 \leq x_2 \leq \min\left\{-\frac{3}{4}x_1 + \frac{5}{4}, -\frac{11}{10}x_1 + \frac{3}{4}, \frac{1}{2}x_1 + \frac{5}{2}\right\}$$

The system may have a solution only if each lower bound for x_2 is not larger than any of the upper bounds. Hence we need to find the set of solutions of the system

$$\left.\begin{aligned} -5 &\leq -\tfrac{3}{4}x_1 + \tfrac{5}{4} \\ -5 &\leq -\tfrac{11}{10}x_1 + \tfrac{3}{4} \\ -5 &\leq \tfrac{1}{2}x_1 + \tfrac{5}{2} \\ 2x_1 &\leq 3 \end{aligned}\right\}$$

Let's transform it into standard form.

$$\left.\begin{array}{r} \frac{3}{4}x_1 \leq \frac{25}{4} \\ \frac{11}{10}x_1 \leq \frac{23}{4} \\ -\frac{1}{2}x_1 \leq \frac{15}{2} \\ 2x_1 \leq 3 \end{array}\right\}$$

Collect the equations upon the coefficients of x_1.

$$\left.\begin{array}{r} \frac{3}{4}x_1 \leq \frac{25}{4} \\ \frac{11}{10}x_1 \leq \frac{23}{4} \\ 2x_1 \leq 3 \\ -\frac{1}{2}x_1 \leq \frac{15}{2} \end{array}\right\}$$

Rearrange the inequalities to get lower and upper bounds for x_1.

$$\left.\begin{array}{r} x_1 \leq \frac{25}{3} \\ x_1 \leq \frac{115}{22} \\ x_1 \leq \frac{3}{2} \\ -15 \leq x_1 \qquad \end{array}\right\}$$

Thus

$$-15 \leq x_2 \leq \min\left\{\frac{25}{3}, \frac{115}{22}, \frac{3}{2}\right\} = \frac{3}{2}$$

Finally we see that the solutions of the system are those (x_1, x_2, x_3, x_4) which satisfy

$$\left.\begin{array}{r} -15 \leq x_1 \leq \frac{3}{2} \\ -5 \leq x_2 \leq \min\left\{-\frac{3}{4}x_1 + \frac{5}{4}, -\frac{11}{10}x_1 + \frac{3}{4}, \frac{1}{2}x_1 + \frac{5}{2}\right\} \\ \max\left\{\frac{7}{5}x_1 + 2x_2 + 1, -x_1 - 2x_2 + 1\right\} \leq x_3 \leq \min\left\{-x_1 - \frac{6}{5}x_2 + 5, -3x_1 - 2x_2 + 4\right\} \\ \max\left\{x_1 + x_2 + 2x_3 - 8, \frac{1}{3}x_1 + \frac{1}{3}x_2 + \frac{1}{3}\ x_3 - \frac{4}{3}, -x_1 - 2x_2 - 1\right\} \leq x_4 \qquad\qquad\qquad\quad \\ x_4 \leq -\frac{1}{4}x_1 - \frac{1}{2}x_2 + \frac{3}{4}x_3 - \frac{7}{4} \qquad\qquad\qquad\quad \end{array}\right\}$$

To give one solution let's choose $x_1 = 1$ first. Then

$$-5 \leq x_2 \leq \min\left\{\frac{1}{2}, -\frac{7}{20}, 3\right\},$$

that is

$$-5 \leq x_2 \leq -\frac{7}{20}$$

Let's choose $x_2 = -1$. Then

$$\max\left\{\frac{2}{5}, 2\right\} \leq x_3 \leq \min\left\{\frac{26}{5}, 3\right\},$$

that is

$$2 \leq x_3 \leq 3$$

Let's choose $x_3 = 2$. Then

$$\max\left\{-4, -\frac{2}{3}, 0\right\} \leq x_4 \leq 0,$$

that is

$$0 \leq x_4 \leq 0$$

Now we don't have a choice for x_4, since it must equal to zero. This shows that $(1, -1, 2, 0)$ is a solution of the system.