Lecture Notes On

Synchronization From A Mathematical Point Of View

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Contents

1. Introduction to the course

1. **Practical information:**

   - All the lectures will be held on **Tuesdays, 15-17 (GMT+3)**.
   - All the lectures will be recorded.
   - All the necessary information and updates on the course (including the lecture notes and recorded videos) will be posted on the virtual learning system of Kadir Has university (Hub).

2. **Materials for further reading:**

   The present lecture notes together with the recorded videos of the lectures is sufficient for this course. However, if you are interested to study dynamical systems further and in more details, there are so many books available that you can use. Below are what we recommend.

   - If you're not into reading a full textbook and prefer something short, we recommend
     - S. Van Strien, *Lecture notes on ODEs.* available for free on the author's webpage.
   - If you want to read a textbook and have some background in mathematics (e.g. mathematical analysis, linear algebra and calculus), we recommend
     - L. Perko, *Differential equations and dynamical systems*, third edition.
     - M. W. Hirsch, S. Smale and R. L. Devaney, *Differential equations, dynamical systems and an introduction to chaos*, third edition.
   - If you want to read a textbook but you don't feel comfortable reading math literature or you prefer a textbook with more taste of applications, we recommend
     - S. Strogatz, *Nonlinear dynamics and chaos: with applications to physics, biology, chemistry, and engineering*, second edition.

- In the introductory session, we saw examples of synchronization in real world phenomena.
- A mathematical model for these phenomena is given by

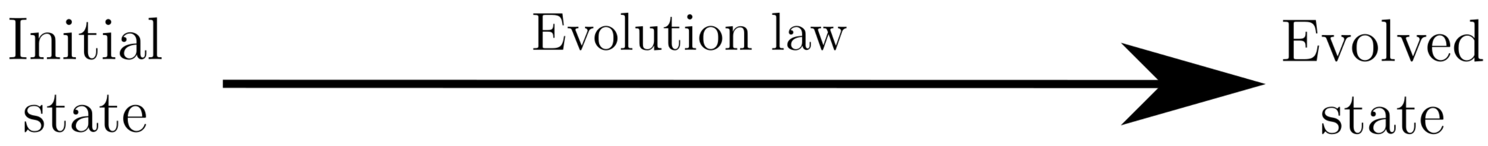
$$\frac{dx_i}{dt} = f_i\left(x_i\right) + \alpha \sum_{j=1}^{N} A_{ij} H_i\left(x_j - x_i\right), \qquad \forall i \in \{1, \dots, N\}, \tag{1.1}$$

  where $x_i \in \mathbb{R}^n$ $(n \geq 1)$, $A = (A_{ij})$ is the adjacency matrix of the network, and $f_i, H_i \in \mathcal{C}^2\left(\mathbb{R}^n\right)$.
- Our main goal in this course is to develop methods that help us to understand the dynamics of this mathematical model.

2. Introduction to dynamical systems

2.1. Definition

- Dynamical systems studies the evolution of a system.
  - A dynamical system is defined by a law of evolution which involves **time** and **state (position)**. For a given initial state, this evolution law describes how this state evolves as time passes.
  - This rule can be deterministic or random.
  - Time can vary continuously or discretely.
- In this course, we focus on deterministic[1] continuous(-time) systems.



[1] A system is called deterministic if the entire past and future of a state are uniquely determined by its state at the present time. [Arn92]

- Rigorous formulation:

Definition 2.1. Consider $\mathbb{R}^n$ $(n \geq 1)$*. Let* t *be real and* x *be a point in* $\mathbb{R}^n$*. A dynamical system is a function*

$$\begin{aligned} \phi :&\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n \\ &(t, x) \mapsto \phi(t, x) \end{aligned} \tag{2.1}$$

that satisfies

(i) $\phi(0, x) = x$ *for all* $x \in \mathbb{R}^n$*.*

(ii) $\phi\left(t_2, \phi\left(t_1, x\right)\right) = \phi\left(t_1 + t_2, x\right)$ *for all* $x \in \mathbb{R}^n$ *and for arbitrary* $t_1, t_2 \in \mathbb{R}$*.*

These two conditions are known as flow properties.

- The variable t is called the time variable. The variable x is called the phase or state variable. We also call $\mathbb{R}^n$ the phase space.

2.2. Visualization of dynamical systems

Suppose a dynamical system $\phi(t,x)$ is given. A standard way to visualize this dynamical system is that for all $x \in \mathbb{R}^n$, we draw the trajectory curve (path) that x takes as t varies. We show the direction of increasing in time by an arrow on this curve.

Example 2.2. *One can show (see Exercise 2.4) that the function* $\phi(t,x) = \left(e^{-t}x_1, e^{2t}x_2\right)$, *where* $x = (x_1, x_2) \in \mathbb{R}^2$ *is the phase variable, is a dynamical system. Consider an arbitrary point* $(c_1, c_2) \in \mathbb{R}^2$. *Suppose* (x_1, x_2) *is a point on the trajectory of* (c_1, c_2). *Thus, there exists* $t^* \in \mathbb{R}$ *such that*

$$(x_1, x_2) = \left(e^{-t^*}c_1, e^{2t^*}c_2\right). \tag{2.2}$$

In Example 2.2, if $c_1, c_2 \neq 0$, we have $\frac{x_1}{c_1} = e^{-t^*}$ and $\frac{x_2}{c_2} = e^{2t^*}$. Thus $\frac{x_2}{c_2} = \left(\frac{c_1}{x_1}\right)^2$. This implies that the curve paths through (c_1, c_2), where $c_1, c_2 \neq 0$, is given by

$$x_1^2 x_2 = c_1^2 c_2. \tag{2.3}$$

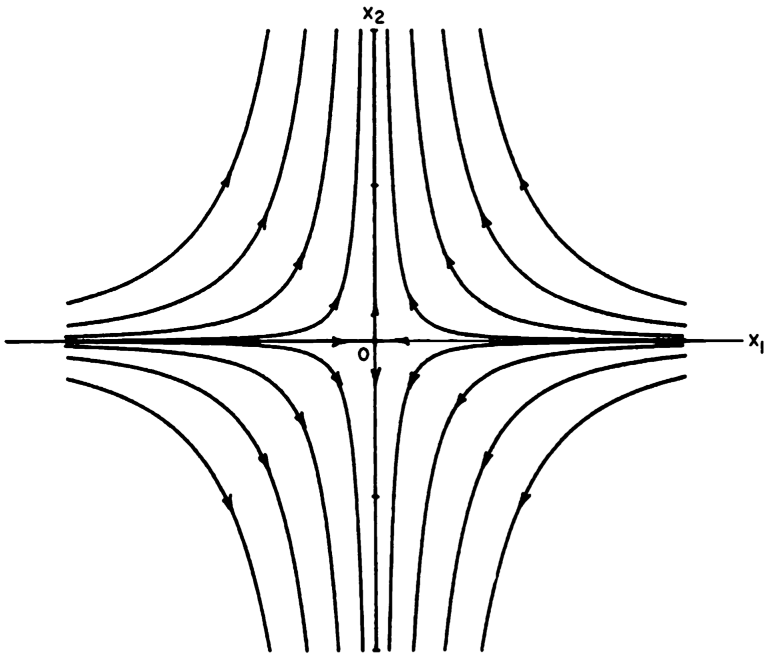


Figure 1: This figure shows how points in $\mathbb{R}^2$ move by $\phi\left(t, x\right) = \left(e^{-t}x_1, e^{2t}x_2\right)$

2.3. An example of dynamical systems

Example 2.3. Let a be a real number. Consider the function $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, defined by $\phi(t, x) = e^{at}x$. We show that ϕ satisfies the flow properties.

(i) $\phi(0, x) = e^{a\times 0}x = x$.

(ii) For arbitrary real t_1 and t_2, we have $\phi\left(t_2, \phi\left(t_1, x\right)\right) = \phi\left(t_2, e^{at_1}x\right) = e^{at_2} \times e^{at_1}x = e^{a(t_1+t_2)}x = \phi\left(t_1 + t_2, x\right)$.

(a) Case $a < 0$. (b) Case $a > 0$.

Figure 2: Visualization of the dynamical system $\phi(t, x) = e^{at}x$

Exercise 2.4. Determine whether or not the following functions ϕ satisfy the flow properties.

1. $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *defined by* $\phi(t, x) = t + x$.
2. $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *defined by* $\phi(t, x) = t^2 + x$.
3. $\phi : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, *defined by* $\phi(t, x) = tx$.
4. $\phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}$, *defined by*

$$\phi\left(t, x\right) = \left(e^{at}x_1, e^{bt}x_2\right), \tag{2.4}$$

where a and b are real constants, and $x = (x_1, x_2) \in \mathbb{R}^2$ is the phase variable.

2.4. **Orbits**

- Fix a point x_0 in the phase space $\mathbb{R}^n$. The path that x_0 takes as time t varies is called the orbit or trajectory of x_0. More precisely, the orbit or trajectory of x_0 is the set

$$\{\phi(t, x_0) : \quad t \in \mathbb{R}\}. \tag{2.5}$$

  – Geometrically, an orbit is a curve in the phase space.

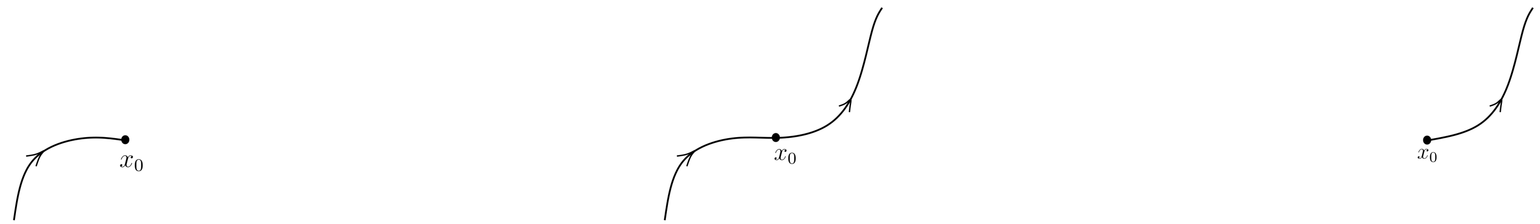
- The orbit of x_0 is defined for both positive and negative times t. However, for a given orbit, we can also focus only on positive or negative times:

  – The forward orbit or positive semi-orbit of a point $x_0 \in \mathbb{R}^n$ is the set

$$\{\phi(t, x_0) : \quad t \geq 0\}. \tag{2.6}$$

  – The backward orbit or negative semi-orbit of a point $x_0 \in \mathbb{R}^n$ is the set

$$\{\phi(t, x_0) : \quad t \leq 0\}. \tag{2.7}$$



(a) The backward orbit of x_0. (b) The orbit of x_0. (c) The forward orbit of x_0.

Example 2.5. *For the dynamical system given in Example 2.3, there are three orbits: (i)* $\{x : x > 0\}$ *(ii)* $\{0\}$ *(iii)* $\{x : x < 0\}$.

Exercise 2.6. *Consider the dynamical system given in Example 2.3 and let* $a = 0$. *How many orbits does this dynamical system have?*

Remark 2.7. *Orbits of a dynamical system never cross (here is why: assume the contrary. Thus, two different orbits* Γ_1 *and* Γ_2*, where* $\Gamma_1 \neq \Gamma_2$*, have a common point* p*. Then,* $\Gamma_1 = \{\phi(t,p) : t \in \mathbb{R}\} = \Gamma_2$*, which is a contradiction).*

- Some important examples of orbits:
  (i) Equilibria:
    - The orbit of a point x_0 is said to be constant if it contains only the point x_0 itself, i.e. the entire orbit is just the single point $\{x_0\}$.
    - We have $\phi(t, x_0) = x_0$ for all $t \in \mathbb{R}$. In other words, the point x_0 is steady; it does not move!
    - When the orbit of x_0 is constant, we call the point x_0 an equilibrium point or steady state (also called fixed point in some literatures).

  (ii) Periodic orbits
    - The orbit $\phi(t, x_0)$ of x_0 is said to be periodic if there exists $T > 0$ such that $\phi(t, x_0) = \phi(t + T, x_0)$.
    - The point x_0 comes back to itself after passing time T.
- We call the set of all the orbits of a dynamical system the *phase portrait* of that dynamical system. However, loosely speaking, by phase portrait we usually mean the visualization of that phase portrait, i.e. drawing figures like Figures 1 and 27.

2.5. **Time-t maps**

- Consider again the function

$$\begin{aligned} \phi :&\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \\ &(t, x) \mapsto \phi(t, x). \end{aligned} \tag{2.8}$$

- We can think of two particular scenarios here:

  1. We fix x and allow t to vary.
  2. We fix t and allow x to vary.

- Scenario 1:

  – This is the scenario that we considered before.

  – Let $x = x_0 \in \mathbb{R}^n$. In this case,

$$\begin{aligned} \phi :&\mathbb{R} \to \mathbb{R}^n, \\ &t \mapsto \phi(t, x_0). \end{aligned} \tag{2.9}$$

  – The function ϕ maps a real variable t to a point in $\mathbb{R}^n$. In particular, it maps 0 to x_0.

  – $\phi(t, x_0)$, as t varies in $\mathbb{R}$, describes the orbit of the point x_0.

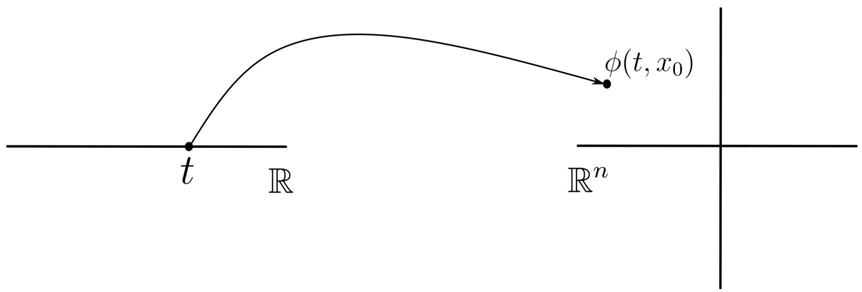


Figure 4: For a fixed $x = x_0$, the function ϕ maps $\mathbb{R}$ to $\mathbb{R}^n$.

- Scenario 2:
  - Let $t = t_0 \in \mathbb{R}$. In this case,

$$\begin{aligned} \phi :& \mathbb{R}^n \to \mathbb{R}^n, \\ & x \mapsto \phi(t_0, x). \end{aligned} \tag{2.10}$$

  - The function ϕ maps a point in $\mathbb{R}^n$ to a point in $\mathbb{R}^n$. In particular, when $t_0 = 0$, the function ϕ maps each point to itself, i.e. ϕ is the identity map.
  - When the time variable t is fixed, the function ϕ is called *time-t map*. For example, $x \mapsto \phi(1, x)$ is called *time*-1 *map*.
  - Time-t maps become important when we want to discretize a continuous-time system.

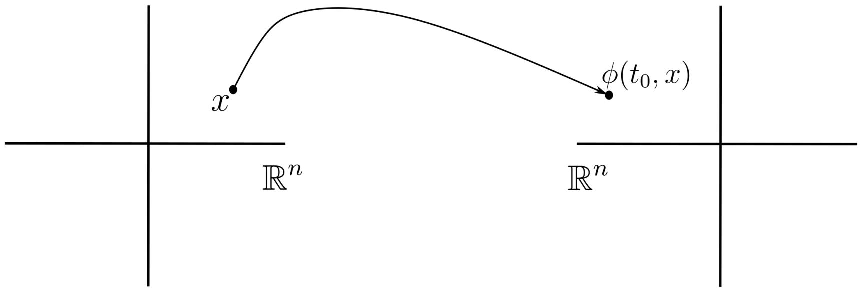


Figure 5: For a fixed $t = t_0$, the function ϕ maps $\mathbb{R}^n$ to $\mathbb{R}^n$.

2.6. **Invariance**

Consider a dynamical system $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ and let $A \neq \emptyset$ be a subset of $\mathbb{R}^n$.

- We say A is invariant with respect to ϕ if for every point x_0 in A, the entire orbit of x_0 lies in A, i.e. $\phi(t, x_0) \in A$ for all $t \in \mathbb{R}$.
  - The set A is invariant if and only if when we start from a point in A, moving forward and backward both, we remain in A and never leave it.

Example 2.8. *Let x_0 be an arbitrary point of the phase space. The orbit of x_0 is an invariant set.*

- We say A is positively invariant or forward invariant with respect to ϕ if for every point x_0 in A, the forward orbit of x_0 lies entirely in A, i.e. $\phi(t, x_0) \in A$ for all $t \geq 0$.
  - The set A is invariant if and only if when we start from a point in A and move forward, we remain in A and never leave it.
- We say A is negatively invariant or backward invariant with respect to ϕ if for every point x_0 in A, the backward orbit of x_0 lies entirely in A, i.e. $\phi(t, x_0) \in A$ for all $t \leq 0$.
  - The set A is invariant if and only if when we start from a point in A and move backward, we remain in A and never leave it.

Remark 2.9. *Every invariant set is forward and backward invariant as well. However, not every forward or backward invariant set is necessarily invariant.*

Exercise 2.10. *Determine whether or not the sets* $A_1 = (-2, 1)$ *and* $A_2 = (2, 3)$ *in* $\mathbb{R}$ *are (positively or negatively) invariant with respect to the dynamical system given by Example 2.3.*

3. **Introduction to ODEs**

3.1. **Vector fields**

Consider a function $f : \mathbb{R}^n \to \mathbb{R}^n$. One can think of f as

$$f \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \dots, x_n) \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) \end{pmatrix}, \tag{3.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \dots, n)$.

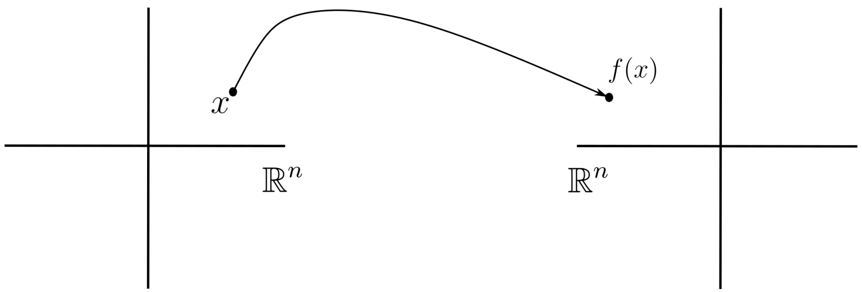


Figure 6: The function f takes the point $x \in \mathbb{R}^n$ and maps it to $f(x) \in \mathbb{R}^n$.

Example 3.1. *The followings are examples of* $f : \mathbb{R}^n \to \mathbb{R}^n$.

(i) $f(x) = x^2 + 1$. *Here,* $f : \mathbb{R}^1 \to \mathbb{R}^1$. *(ii)* $f\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x_1 + \sin x_2 \\ x_2 \end{smallmatrix}\right)$. *Here,* $f : \mathbb{R}^2 \to \mathbb{R}^2$.
(iii) $f\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \begin{pmatrix} g(x_1) + \alpha H(x_2 - x_1) \\ g(x_2) + \alpha H(x_1 - x_2) \end{pmatrix}$, *where* α *is a real constant, and* $g, H : \mathbb{R}^2 \to \mathbb{R}$. *Here,* $f : \mathbb{R}^2 \to \mathbb{R}^2$.
(iv) $f\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$. *Here,* $f : \mathbb{R}^3 \to \mathbb{R}^3$.
(v) $f(x) = \alpha g(x) + \beta h(x)$, *where* $x \in \mathbb{R}^n$, α *and* β *are real constants and* $g, h : \mathbb{R}^n \to \mathbb{R}^n$. *Here,* $f : \mathbb{R}^n \to \mathbb{R}^n$.

- In this course, we call a function $f : \mathbb{R}^n \to \mathbb{R}^n$ a vector field. Here is why:
- One way to visualize a function $f : \mathbb{R}^n \to \mathbb{R}^n$ is that for every point $x \in \mathbb{R}^n$, we draw the vector $f(x)$ starting at the point x and ending at $x + f(x)$ (see Figure 7).

Figure 7: For every point $x \in \mathbb{R}^n$, we draw the vector $f(x)$ starting at the point x and ending at $x + f(x)$.

Figure 8: The vector field $f(x_1, x_2) = (x_1, x_2 - x_1)$.

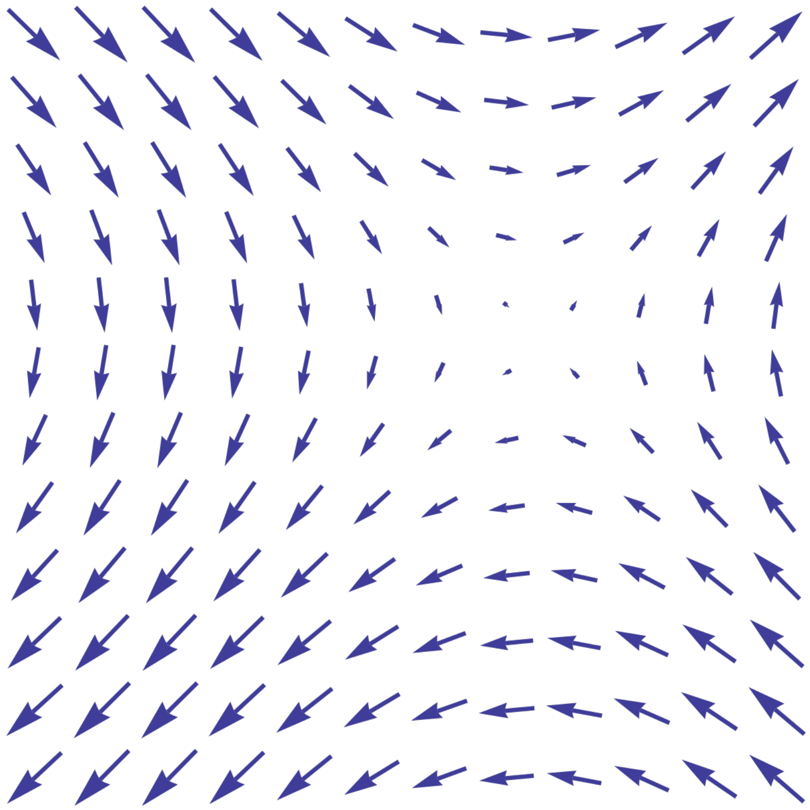


Figure 9: A portion of the vector field $f(x_1, x_2) = (\sin x_2, \sin x_1)$ on $\mathbb{R}^2$ (this figure is copied from Wikipedia).

3.2. **Solutions of ODEs**

- **Question:** Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is given. Let $x_0 \in \mathbb{R}^n$. Does there exist any function

$$\begin{aligned} x :&\mathbb{R} \to \mathbb{R}^n \\ &t \mapsto x(t) = (x_1(t), \ldots, x_n(t)) \end{aligned} \tag{3.2}$$

  such that $\frac{dx(t)}{dt} = f(x(t))$ and $x(0) = x_0$? If it exists, is it unique?

- By $\frac{dx(t)}{dt} = f(x(t))$, we mean

$$\begin{aligned} \frac{dx_1(t)}{dt} &= f_1(x_1(t), x_2(t), \ldots, x_n(t)), \\ \frac{dx_2(t)}{dt} &= f_2(x_1(t), x_2(t), \ldots, x_n(t)), \\ \vdots \quad &\vdots \qquad\qquad \vdots \\ \frac{dx_n(t)}{dt} &= f_n(x_1(t), x_2(t), \ldots, x_n(t)), \end{aligned} \tag{3.3}$$

  where $f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1,x_2,\ldots,x_n) \\ \vdots \\ f_n(x_1,x_2,\ldots,x_n) \end{pmatrix}$.

- Some terminologies and notations:
  - We call the equation $\frac{dx(t)}{dt} = f(x(t))$, i.e. equation (3.3), a system of ordinary differential equations.
  - The condition $x(0) = x_0$ is called an initial condition.
  - The equation $\frac{dx(t)}{dt} = f(x(t))$ together with the initial condition $x(0) = x_0$ is called an initial value problem (I.V.P).
  - Such a function $x(t)$, if it exists, is called a solution of the initial value problem $\frac{dx(t)}{dt} = f(x(t))$ and $x(0) = x_0$.
  - In this course, for simplicity, we use dot to show derivative with respect to time. For example, $\dot{x} := \frac{dx(t)}{dt}$.

Example 3.2. The initial value problem

$$\dot{x} = 3x^{\frac{2}{3}}, \qquad \textit{and} \quad x(0) = 0 \tag{3.4}$$

has two different solutions $x(t) = t^3$ *and* $x(t) = 0$.

Example 3.3. The initial value problem

$$\begin{aligned} \dot{x}_1 &= -4x_2, \\ \dot{x}_2 &= x_1, \qquad \textit{and} \quad x(0) = (c_1, c_2), \end{aligned} \tag{3.5}$$

where (c_1, c_2) *is an arbitrary point in* $\mathbb{R}^2$ *has at least one solution (we will see later that this is the only solution) defined for* $t \in \mathbb{R}$, *given by*

$$\begin{aligned} x_1(t) &= c_1 \cos 2t - 2c_2 \sin 2t, \\ x_2(t) &= \frac{c_1}{2} \sin 2t + c_2 \cos 2t. \end{aligned} \tag{3.6}$$

Example 3.4. The initial value problem

$$\dot{x} = x^2, \qquad \textit{and} \quad x(0) = 1 \tag{3.7}$$

has the solutions $x(t) = \frac{1}{1-t}$, *which is defined for* $t \in (-\infty, 1)$. *Notice that* $x(t) = \frac{1}{1-t}$ *satisfies* $\dot{x} = x^2$ *for* $t \in (1, \infty)$, *however the initial condition is not satisfied since* $0 \notin (1, \infty)$.

Example 3.5. The initial value problem

$$\dot{x} = f(x), \qquad \textit{and} \quad x(0) = 0, \tag{3.8}$$

where

$$f(x) = \begin{cases} 1 & \textit{when } x < 0 \\ -1 & \textit{when } x \geq 0 \end{cases} \tag{3.9}$$

has no solutions. Can you see why? Hint: if $x(t)$ *is a solution then it needs to be differentiable at every* t, *particularly at* $t = 0$.

- Remember the question that we asked earlier: Does the I.V.P $\dot{x} = f(x)$ and $x(0) = x_0$ have solution? Uniqueness?
  - **Quick Answer:** As the examples that we just reviewed suggest:

    In general, NO! For an arbitrary vector field f and arbitrary initial point $x_0 \in \mathbb{R}^n$, the solutions neither need to exist nor be unique; even if they exist, they are not necessarily defined for all $t \in \mathbb{R}$.

- Before we proceed to an elegant answer to our question, let's see what the geometrical/physical meaning of a solution is. Suppose that there is a unique solution $x(t)$ for the I.V.P $\dot{x} = f(x)$ and $x(0) = x_0$.
  - The solution $x(t)$ describes how x_0 moves in $\mathbb{R}^n$ as t varies.
  - Define $\Gamma := \{x(t) : t \in \mathbb{R}\}$. Geometrically, Γ is a curve in $\mathbb{R}^n$. Let $t^* \in \mathbb{R}$ and $x^* := x(t^*)$. The tangent vector to the curve Γ is given by $\frac{dx}{dt}(t^*)$. However, $\frac{dx}{dt}(t^*) = f(x(t^*)) = f(x^*)$. This means that at every point x on the curve Γ, the vector $f(x)$ is tangent to Γ.
  - Having in mind that $x(t)$ describes the movement of x_0, the vector $f(x^*)$ is the velocity vector at time t^*.

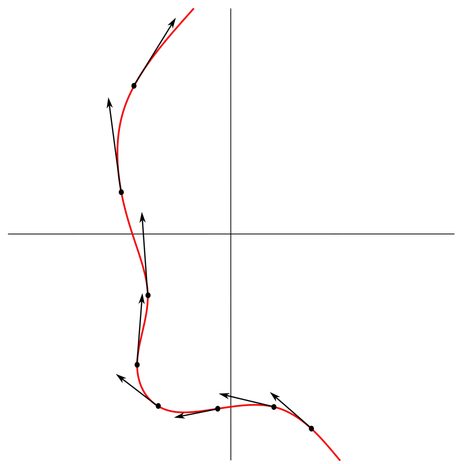


Figure 10: At every point x_0, the solution curve of $\dot{x} = f(x)$ passing through x_0 is tangent to the vector $f(x_0)$.

THEOREM 3.6. *Let $f \in \mathcal{C}^1(\mathbb{R}^n)$*[2]*, and $x_0 \in \mathbb{R}^n$. Then, there exists a open interval $I_{x_0} = (\alpha(x_0), \beta(x_0))$, where $\alpha(x_0) < 0 < \beta(x_0)$, such that the initial value problem*

$$\begin{aligned} \dot{x} &= f(x) \\ x(0) &= x_0 \end{aligned} \tag{3.10}$$

has a unique solution $x(t)$ on I_{x_0}. Moreover, the interval I_{x_0} is maximal in the sense that if $x^*(t)$ is a solution of (3.10) defined on an interval J, then $J \subset I_{x_0}$ and $x^*(t) = x(t)$ on J.

Proof. See [VS18], the proof of the existence-uniqueness theorem (Theorem 3.6) and the discussion on the maximal solutions (Chapter 5). □

Remark 3.7. *This theorem guarantees that if the vector field is $\mathcal{C}^1$-smooth, then the solution of the I.V.P exists and is defined on some maximal interval $I \subseteq \mathbb{R}$. However, as Example 3.4 shows, this interval is not necessarily equal to $\mathbb{R}$; although this theorem guarantees the existence and uniqueness of the solution, it does not guarantee the solution to exist for all $t \in \mathbb{R}$. In this course, we assume*[3] *that the solution $x(t)$ of the I.V.P (3.10) exists for all $t \in \mathbb{R}$, i.e. $I_{x_0} = \mathbb{R}$.*

Remark 3.8. *In system (3.10), the function f does not depend directly on t. Such systems are called autonomous. Nonautonomous systems are those where t is an independent variable of the function f; a nonautonomous system is written as $\dot{x} = f(t, x)$, where $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$. Theorem 3.6 holds for nonautonomous case too (see [VS18], Theorem 3.6). In this course, our focus is on autonomous systems.*

Exercise 3.9. *Can you say why Theorem 3.6 cannot guarantee the existence and uniqueness of solutions in Examples 3.2 and 3.5? What can this theorem say about Example 3.3?*

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[2]In general, we say $f : \mathbb{R}^n \to \mathbb{R}^m$, given by $f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, \ldots, x_n) \\ \vdots \\ f_m(x_1, x_2, \ldots, x_n) \end{pmatrix}$, is $\mathcal{C}^1$-smooth, denoted by $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$, if for all $1 \le i \le m$ and $1 \le j \le n$, the partial derivative $\frac{\partial f_i}{\partial x_j}(x_1, \cdots, x_n)$ exists and is continuous. When $n = m$, we write $f \in \mathcal{C}^1(\mathbb{R}^n)$. The functions f in Example 3.1 are smooth (assuming g, h and H are smooth).

[3]This assumption is not that much strong. Indeed, for any arbitrary system $\dot{x} = f(x)$, there exists a system of ODEs which is topologically equivalent to $\dot{x} = f(x)$ and its solutions are defined on whole $\mathbb{R}$ (see [Per01], Section 3.1).

In general, finding explicit solutions of ODEs is not possible. Even when the explicit solutions are available, they can be very difficult to deal with. The aim of this course is not solving ODEs. In this course, we develop methods that can be used to analyze ODEs without necessarily solving them.

3.3. **Dynamical systems defined by ODEs**

Theorem 3.6 states that when $f \in \mathcal{C}^1(\mathbb{R}^n)$, for any arbitrary $x_0 \in \mathbb{R}^n$, the I.V.P $\dot{x} = f(x)$ and $x(0) = x_0$ has a unique solution on I_{x_0}. Denote this solution by $\phi_t(x_0)$.

Solving this I.V.P for every $x_0 \in \mathbb{R}^n$, we obtain a family of solutions $\phi_t(x_0)$. Define

$$\phi(t,x) := \phi_t(x). \tag{3.11}$$

Then

THEOREM 3.10. *The function $\phi : (t,x) \mapsto \phi(t,x)$ defined by (3.11) satisfies the flow properties*

(i) $\phi(0,x) = x$ for all $x \in \mathbb{R}^n$.

(ii) $\phi(t_2, \phi(t_1,x)) = \phi(t_1+t_2,x)$ for all $x \in \mathbb{R}^n$ and for arbitrary $t_1, t_2 \in \mathbb{R}$.

Exercise 3.11. *Prove Theorem 3.10.*

Remark 3.12. *Assuming $I_{x_0} = \mathbb{R}$ for every $x_0 \in \mathbb{R}^n$, Theorem 3.10 implies that the function $\phi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by (3.11) is a dynamical system (see Definition 2.1). We call the function $\phi(t,x)$ the flow*[4] *generated by the system $\dot{x} = f(x)$.*

Remark 3.13 (Smooth dependence on initial condition). *When f is $\mathcal{C}^1$-smooth, the associated flow $\phi(t,x)$ is a $\mathcal{C}^1$-smooth function of (t,x) (see e.g. [HSD12], Section 17.6). This means that not only we can think of $\frac{\partial \phi(t,x)}{\partial t}$, but also we can think of the expression $\frac{\partial \phi(t,x)}{\partial x}$. Later, in this course, we will discuss this further.*

[4]More precisely, flow is the family of all time-t maps defined on the phase space that satisfies the flow properties. This family with the composition operator is a group. In this course, we treat the flow as the function $\phi(t,x)$ generated by a system of ODEs.

3.4. **Equilibria of ODEs**

The point $x_0 \in \mathbb{R}^n$ is an equilibrium for the $\mathcal{C}^1$-smooth vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ if $x(t) = x_0$, for all $t \in \mathbb{R}$, satisfies

$$\dot{x} = f(x). \tag{3.12}$$

Proposition 3.14. *The point $x_0 \in \mathbb{R}^n$ is an equilibrium for the $\mathcal{C}^1$-smooth $f : \mathbb{R}^n \to \mathbb{R}^n$ if and only if $f(x_0) = 0$.*

Proof. Suppose $x(t) = x_0$ is an equilibrium. Then, $x(t) = x_0$ satisfies $\dot{x} = f(x)$. Thus, $f(x_0) = f(x(t)) = \frac{dx(t)}{dt} = \frac{dx_0}{dt} = 0$. Now, assume x_0 is a point such that $f(x_0) = 0$. Observe that $x(t) = x_0$ satisfies the I.V.P $\dot{x} = f(x)$ and $x(0) = x_0$. However, by Theorem 3.6, this is the solution of this I.V.P. □

Remark 3.15. *Proposition 3.14 is intuitively obvious: the only scenario in which the point x_0 does not move is that the velocity vector at x_0 is zero, i.e. $f(x_0) = 0$. On the other hand, if the velocity vector at the point is zero, this means that the point does not move.*

Example 3.16. *Consider the system $\dot{x} = x^2 + 1$, where $x \in \mathbb{R}$. The function $f(x) = x^2 + 1$ has no roots in $\mathbb{R}$. Thus, this system has no equilibrium points. Notice that $x = i$ and $x = -i$, where $i = \sqrt{-1}$, are not equilibria since our system is defined on $\mathbb{R}$.*

Example 3.17. *Consider the system*

$$\begin{aligned} \dot{x} &= x^2 + y^2, \\ \dot{y} &= xz + y, \\ \dot{z} &= z^2 - x^3 - 1. \end{aligned} \tag{3.13}$$

To find the equilibria of this system, we need to find all points $(x_0, y_0, z_0) \in \mathbb{R}^3$ *such that* $x_0^2 + y_0^2 = 0$, $x_0 z_0 + y_0 = 0$ *and* $z_0^2 - x_0^3 - 1 = 0$. *From* $x_0^2 + y_0^2 = 0$, *we obtain that if* (x_0, y_0, z_0) *is an equilibrium, then* $x_0 = y_0 = 0$. *Substituting this into the equation* $z_0^2 - x_0^3 - 1 = 0$, *we obtain that either* $z_0 = -1$ *or* $z_0 = 1$. *However, both of these values of* z_0 *together with* $x_0 = y_0 = 0$ *satisfies* $x_0 z_0 + y_0 = 0$. *This means that system (3.13) has two equilibria* $(0, 0, -1)$ *and* $(0, 0, 1)$.

Remark 3.18. *Suppose that* x_0 *is an equilibrium point of* $\dot{x} = f(x)$. *Sometimes it is more convenient if we shift the equilibrium point to the origin. In this case, we can define the change of variable* $y = x - x_0$. *Then,* $\dot{y} = \dot{x} = f(x) = f(y + x_0)$. *Thus, if we define* $g(y) = f(y + x_0)$, *then* $\dot{y} = g(y)$ *and* $g(0) = f(x_0) = 0$.

Example 3.19. *Consider the system*

$$\begin{aligned}\dot{x}_1 &= 0,\\ \dot{x}_2 &= 0,\end{aligned} \tag{3.14}$$

where $x_1, x_2 \in \mathbb{R}$. *Any arbitrary point on the plane is an equilibrium for this system.*

Exercise 3.20. *Find all the equilibria of the system* $\dot{x} = f(x)$, *where* f *is*

(i) $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = x^3 - 5x^2 + x$.

(ii) $f : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $f\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x^3 - 5x^2 + x \\ 1 \end{smallmatrix}\right)$.

(iii) $f : \mathbb{R} \to \mathbb{R}$ *given by* $f(x) = x^3 - \alpha x$, *where* α *is a real constant.*

PROPOSITION 3.21. *Consider a smooth system* $\dot{x} = f(x)$ *defined on* $\mathbb{R}^n$ *and let* $\phi(t, x_0)$ *be the orbit of a point* $x_0 \in \mathbb{R}^n$. *Assume that the solution* $\phi(t, x_0)$ *of* $\dot{x} = f(x)$ *converges to a single point as* $t \to \infty$. *More precisely, there exists* $p \in \mathbb{R}^n$ *such that* $\lim_{t\to\infty} \phi(t, x_0) = p$. *Then,* p *is an equilibrium point.*

Proof. By the assumption, we have $\lim_{t\to\infty} \phi(t, x_0) = \phi(0, p) = p$. Let $\tau \in \mathbb{R}$. Then, due to the the continuity of ϕ (note that by Remark 3.13, ϕ is smooth and so continuous), we have

$$\lim_{t\to\infty} \phi\left(\tau, \phi\left(t, x_0\right)\right) = \phi\left(\tau, \phi(0, p)\right). \tag{3.15}$$

Taking the flow properties into account,

$$\lim_{t\to\infty} \phi\left(\tau + t, x_0\right) = \phi\left(\tau, p\right). \tag{3.16}$$

However, $\lim_{t\to\infty} \phi\left(\tau + t, x_0\right) = \lim_{t\to\infty} \phi\left(t, x_0\right) = p$. This implies that $\phi(\tau, p) = p$. But, τ is arbitrary. This means that p is an equilibrium. □

3.5. **One-dimensional ODEs**

Let $f : \mathbb{R} \to \mathbb{R}$ be $\mathcal{C}^1$-smooth and consider the system

$$\dot{x} = f(x). \tag{3.17}$$

The equilibria of this system are zeros (roots) of the function f. The function f can have different numbers of equilibria; from zero (e.g. when $f(x) > 0$ for all x) to infinitely many equilibria (e.g. $f(x) = 0$ for all x). Consider the case that f has at least two equilibria. Namely, p_1 and p_2, i.e. $f(p_1) = f(p_2) = 0$, and $p_1 < p_2$. Assume that f has no other roots on the interval (p_1, p_2). Therefore, either $f(x) > 0$ for all $x \in (p_1, p_2)$ or $f(x) < 0$ for all $x \in (p_1, p_2)$. Consider an arbitrary point x_0 in (p_1, p_2) and let $x(t)$ be the solution of the I.V.P $\dot{x} = f(x)$ and $x(0) = x_0$. Thus, $\frac{dx(t)}{dt} = f\left(x\left(t\right)\right)$. This implies that $x(t)$ is a strictly increasing function of t if $f > 0$ on (p_1, p_2), and a strictly decreasing function of t if $f < 0$ on (p_1, p_2).

Consider the case that $x(t)$ is strictly increasing (the decreasing case is analogous). Note that $x(t) < p_2$ for all t. This is simply because that $\{x(t) : t \in \mathbb{R}\}$ and $\{p_2\}$ are distinct orbits and so they do not cross each other (see Remark 2.7). Therefore, $x : t \mapsto x(t)$ is an increasing bounded real-valued function defined on $\mathbb{R}$. Thus $\lim_{t\to\infty} x\left(t\right)$ exists. However, it follows from Proposition 3.21 that this limit must be an equilibrium. Since, by assumption, there is no equilibrium in the interval (p_1, p_2), we have $\lim_{t\to\infty} x\left(t\right) = p_2$. The case that $x(t)$ is decreasing is similar; in this case, we have $\lim_{t\to\infty} x\left(t\right) = p_1$.

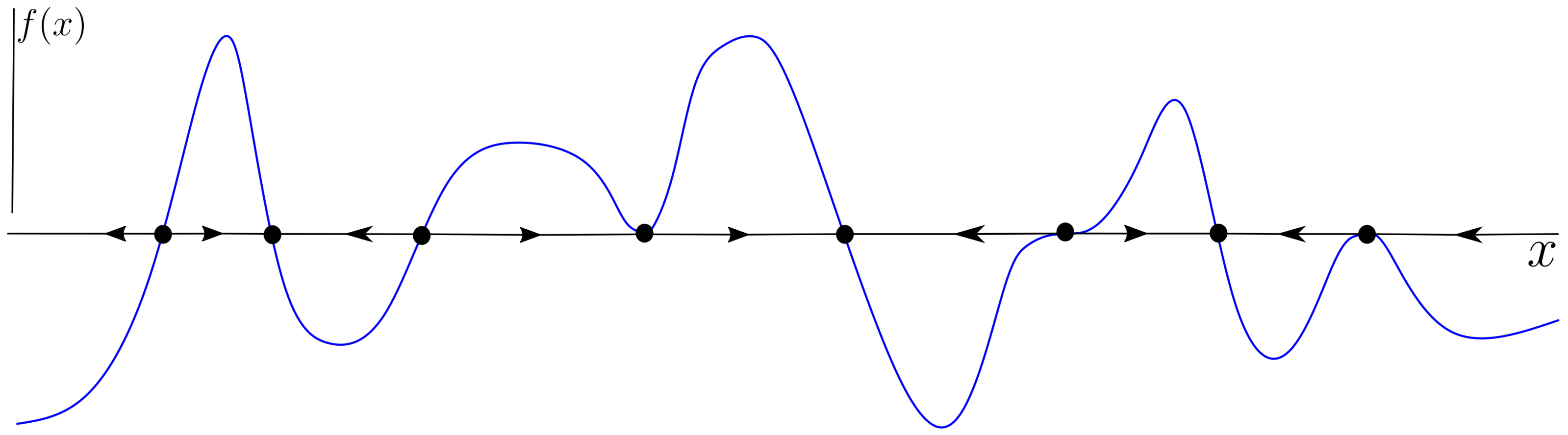


Figure 11: Dynamics of $\dot{x} = f(x)$, where $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{C}^1$-smooth.

Proposition 3.22. *Let p be an equilibrium point of system (3.17).*

(i) If $f'(p) < 0$, there exists an open interval J containing p such that for any $x_0 \in J$, we have $\lim_{t\to\infty} x(t) = p$, where $x(t)$ is the solution of (3.17) satisfying $x(0) = x_0$.

(ii) If $f'(p) > 0$, there exists an open interval J containing p such that for any $x_0 \in J$, we have $\lim_{t\to-\infty} x(t) = p$, where $x(t)$ is the solution of (3.17) satisfying $x(0) = x_0$.

Proof. It is a simple consequence of the discussion above. $\square$

3.6. Higher-order ODEs

In this course, we mainly study first-order ODEs; our systems involve with the first order derivative with respect to t, not higher orders derivatives. For instance, the equation

$$\frac{d^3x(t)}{dt^3} + 13\frac{d^2x(t)}{dt^2} \cdot x(t) - 5\frac{dx(t)}{dt} + 3[x(t)]^2 = 0 \tag{3.18}$$

is a third-order ODE.

Consider the n-order differential equation of the form

$$x^{(n)} - F\left(x^{(n-1)}, x^{(n-2)}, \ldots, x^{(1)}, x\right) = 0, \tag{3.19}$$

where F is some function and $x^{(k)} = \frac{d^k x(t)}{dt^k}$. Define

$$y_1 := x(t),\ y_2 := \frac{dx(t)}{dt},\ y_3 := \frac{d^2x(t)}{dt^2},\ \ldots,\ y_n := \frac{d^{n-1}x(t)}{dt^{n-1}}. \tag{3.20}$$

Then,

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ &\vdots \\ \dot{y}_n &= F(y_{n-1}, y_{n-2}, \ldots, y_1). \end{aligned} \tag{3.21}$$

This trick allows us to reduce higher-order ODEs of the form (3.19) to first-order systems of the form (3.21).

Example 3.23. *Consider system (3.18), i.e.*

$$\frac{d^3x(t)}{dt^3} + 13\frac{d^2x(t)}{dt^2} \cdot x(t) - 5\frac{dx(t)}{dt} + 3\left[x(t)\right]^2 = 0. \tag{3.22}$$

Define

$$y_1 := x(t)\,,\; y_2 := \frac{dx(t)}{dt}, \;\; \textit{and} \;\; y_3 := \frac{d^2x(t)}{dt^2}. \tag{3.23}$$

Then, we can rewrite system (3.18) as

$$\begin{aligned} \dot{y}_1 &= y_2, \\ \dot{y}_2 &= y_3, \\ \dot{y}_3 &= -13y_3y_1 + 5y_2 - 3y_1^2. \end{aligned} \tag{3.24}$$

4. Linear systems

In this section, we study the solutions of systems of the form $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is a linear function of x. In other words, let A be an $n \times n$ real matrix, i.e. $A \in \mathbb{R}^{n\times n}$. We are interested in initial value problems of the form

$$\begin{aligned} \dot{x} &= Ax, \\ x(0) &= x_0, \end{aligned} \tag{4.1}$$

where $x_0 \in \mathbb{R}^n$.

Remark 4.1 (Equilibria of linear systems)*. It follows from Proposition 3.14 that x^* is an equilibrium point of the system $\dot{x} = Ax$ if and only if $Ax^* = 0$, i.e. $x^* \in \mathrm{Null}\,(A)$, where* Null (A) *is the null space of A. Thus,*

- *the origin is always an equilibrium point of the system $\dot{x} = Ax$.*
- *the linear system $\dot{x} = Ax$ either has a unique equilibrium at the origin (this is the case that* $\det(A) \neq 0$*) or has uncountably many equilibria (this is the case that* $\det(A) = 0$*, or equivalently,* $\dim(\mathrm{Null}\,(A)) \geq 1$*).*

4.1. Matrix exponentials: solutions of linear ODEs

- For any $x \in \mathbb{R}$, the series

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + \frac{x}{1} + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \frac{x^5}{120} + \frac{x^6}{720} + \cdots \tag{4.2}$$

  is well-defined, i.e. the series converges. We denote the value of this series by e^x, i.e. $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$[5]. This allows us to define the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ by $\exp(x) := e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

- The constant e, called the Euler's number, is an irrational real number which is approximately equal to 2.71828. For instance, using a calculator, we can find that $\exp(3) = e^3 \approx 20.0855$, $\exp(-1) = e^{-1} \approx 0.3678$, $\exp(4.017) = e^{4.017} \approx 55.5342$, $\exp(7) = e^7 \approx 1096.6331$, $\exp(0) = e^0 \approx 1$ and $\exp(e) = e^e \approx 15.1542$.

- The idea of the exponential of real numbers can be generalized to matrices too. Let $A \in \mathbb{R}^{n\times n}$, and consider

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = I + A + \frac{A^2}{2} + \frac{A^3}{6} + \frac{A^4}{24} + \frac{A^5}{120} + \frac{A^6}{720} + \cdots, \tag{4.3}$$

  where I is the identity matrix. Then

THEOREM 4.2. *For any real $n \times n$ matrix A, series (4.3) is convergent, i.e. the matrix e^A is well-defined.*

Proof. See [VS18], Lemma 4.3. □

Definition 4.3. *For any $A \in \mathbb{R}^{n\times n}$, we define* $\exp(A) := e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$.

[5]For a given positive integer n, we define $n!$ (read it n factorial) by $n! := 1 \times 2 \times 3 \times \cdots \times (n-1) \times n$. For example, $1! = 1$, $6! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720$, and $10! = 1 \times 2 \times \cdots \times 9 \times 10 = 3628800$. We also have the agreement $0! = 1$.

Example 4.4. Let λ and μ be arbitrary real numbers and consider the matrix $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$. Then

$$A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \qquad A^2 = \begin{pmatrix} \lambda^2 & 0 \\ 0 & \mu^2 \end{pmatrix}, \qquad A^3 = \begin{pmatrix} \lambda^3 & 0 \\ 0 & \mu^3 \end{pmatrix}, \qquad A^4 = \begin{pmatrix} \lambda^4 & 0 \\ 0 & \mu^4 \end{pmatrix} \tag{4.4}$$

and similarly, for all integer $k \geq 0$, we have

$$A^k = \begin{pmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{pmatrix}. \tag{4.5}$$

Therefore,

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{A^k}{k!} &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} + \frac{1}{2!}\begin{pmatrix} \lambda^2 & 0 \\ 0 & \mu^2 \end{pmatrix} + \frac{1}{3!}\begin{pmatrix} \lambda^3 & 0 \\ 0 & \mu^3 \end{pmatrix} + \frac{1}{4!}\begin{pmatrix} \lambda^4 & 0 \\ 0 & \mu^4 \end{pmatrix} + \cdots \\ &= \begin{pmatrix} 1 + \lambda + \frac{1}{2!}\lambda^2 + \frac{1}{3!}\lambda^3 + \cdots & 0 \\ 0 & 1 + \mu + \frac{1}{2!}\mu^2 + \frac{1}{3!}\mu^3 + \cdots \end{pmatrix} \\ &= \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{pmatrix}. \end{aligned} \tag{4.6}$$

Theorem 4.5. *Let A be an $n \times n$ real matrix and x_0 be an arbitrary point in $\mathbb{R}^n$. Then, the solution of the initial value problem*

$$\begin{aligned} \dot{x} &= Ax, \\ x(0) &= x_0, \end{aligned} \tag{4.7}$$

is given by $x(t) = e^{At}x_0$.

Remark 4.6. *By definition, we have*

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} = I + tA + \frac{t^2 A^2}{2!} + \frac{t^3 A^3}{3!} + \frac{t^4 A^4}{4!} + \frac{t^5 A^5}{5!} + \cdots . \tag{4.8}$$

Proof. Observe that $e^{0 \times A} = I$ (see Corollary 4.13). Thus, $x(0) = e^{0A}x_0 = Ix_0 = x_0$, and so the initial condition is satisfied. We now need to show that $x(t) = e^{At}x_0$ satisfies $\dot{x} = Ax$. We have

$$\frac{de^{tA}}{dt} = \lim_{h \to 0} \frac{e^{(t+h)A} - e^{tA}}{h} \overset{hA \text{ and } tA \text{ commute}}{=\!=\!=\!=\!=\!=\!=\!=} \lim_{h \to 0} \frac{e^{tA}e^{hA} - e^{tA}}{h} = e^{tA} \lim_{h \to 0} \frac{e^{hA} - I}{h}. \tag{4.9}$$

On the other hand, $e^{hA} - I = \frac{hA}{1!} + \frac{h^2A^2}{2!} + \frac{h^3A^3}{3!} + \cdots$. This gives

$$\lim_{h \to 0} \frac{e^{hA} - I}{h} = \lim_{h \to 0} \lim_{k \to \infty} A + \frac{hA^2}{2!} + \frac{h^2A^3}{3!} + \cdots \frac{h^{k-1}A^k}{k!} = A. \tag{4.10}$$

Thus, by (4.9), we have $\frac{de^{tA}}{dt} = Ae^{tA}$, as desired. $\square$

Remark 4.7. *Following Theorem 4.2, the solution $x(t) = e^{At}x_0$ is defined for all $t \in \mathbb{R}$.*

Example 4.8. *Let λ and μ be arbitrary real numbers and consider the I.V.P*

$$\begin{aligned} &\dot{x}_1 = \lambda x_1, \\ &\dot{x}_2 = \mu x_2, \qquad \text{and} \qquad (x_1(0), x_2(0)) = (x_{10}, x_{20}), \end{aligned} \tag{4.11}$$

where (x_{10}, x_{20}) is an arbitrary point in $\mathbb{R}^2$. We can write (4.11) in the form

$$\begin{aligned} \dot{x} &= Ax, \\ x(0) &= (x_{10}, x_{20}), \end{aligned} \tag{4.12}$$

where $A = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ and $x = (x_1, x_2)$. According to Theorem 4.2, the solution of I.V.P (4.12) is given by $x(t) = e^{At}\left(\begin{smallmatrix} x_{10} \\ x_{20} \end{smallmatrix}\right)$. However, it follows from Example 4.4 that $e^{At} = \left(\begin{smallmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{smallmatrix}\right)$. Thus,

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{At}\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix}\begin{pmatrix} x_{10} \\ x_{20} \end{pmatrix} = \begin{pmatrix} e^{\lambda t}x_{10} \\ e^{\mu t}x_{20} \end{pmatrix}. \tag{4.13}$$

Exercise 4.9. *Solve the initial value problem $\dot{x} = ax$ and $x(0) = x_0$, where a is a real constant, $x \in \mathbb{R}$ and x_0 is an arbitrary point in $\mathbb{R}$. Compare your findings with Example 2.3.*

4.2. **Matrix exponentials: properties and examples**

Proposition 4.10. *Let A and B be real $n \times n$ matrices. Then*

(i) if $AB = BA$, then $e^A B = B e^A$.

(ii) if $AB = BA$, then $e^{A+B} = e^A e^B$.

(iii) $\left(e^A\right)^{-1} = e^{-A}$.

Exercise 4.11. *Prove Proposition 4.10.*

Example 4.12. *Consider a diagonal $n \times n$ real matrix $A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$, where $\lambda_1, \dots, \lambda_n \in \mathbb{R}$. Similar conclusion as in Example 4.4 gives*

$$e^A = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix} \tag{4.14}$$

Corollary 4.13. *If follows from Example 4.12 (take $\lambda_1 = \cdots = \lambda_n = 0$) that if A is the zero matrix, then $e^A = I$, where I is the identity matrix.*

Example 4.14. *Let λ and γ be real numbers and consider*

$$A = \begin{pmatrix} \lambda & \gamma \\ 0 & \lambda \end{pmatrix}. \tag{4.15}$$

In this example, we show that

$$e^A = e^\lambda \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}. \tag{4.16}$$

Write $A = \lambda I + M$, where I is the identity matrix and $M = \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)$. The matrices M and λI commute (we say two matrices P and Q commute if $PQ = QP$). By Proposition 4.10, we have $e^A = e^{\lambda I + M} = e^{\lambda I} e^M$.

In Example 4.4, we have shown that

$$e^{\lambda I} = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix}. \tag{4.17}$$

On the other hand, we have $M^2 = 0$, and therefore $M^k = 0$ for all integer $k \geq 0$. This yields

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} = I + M = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}. \tag{4.18}$$

We have

$$e^A = e^{\lambda I} e^M = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} = e^\lambda \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}. \tag{4.19}$$

Example 4.15. *Let a and b be real numbers and consider*

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{4.20}$$

We show that

$$e^A = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}. \tag{4.21}$$

Let $\lambda = a + bi$, where $i = \sqrt{-1}$. Thus

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,(\lambda) & -\mathrm{Im}\,(\lambda) \\ \mathrm{Im}\,(\lambda) & \mathrm{Re}\,(\lambda) \end{pmatrix}. \tag{4.22}$$

Note that, $\lambda^2 = (a + bi)^2 = a^2 - b^2 + 2abi$. Therefore

$$A^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\left(\lambda^2\right) & -\mathrm{Im}\left(\lambda^2\right) \\ \mathrm{Im}\left(\lambda^2\right) & \mathrm{Re}\left(\lambda^2\right) \end{pmatrix}. \tag{4.23}$$

Inductively, for any integer $k > 0$, we can show that

$$A^k = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^k = \begin{pmatrix} \mathrm{Re}\left(\lambda^k\right) & -\mathrm{Im}\left(\lambda^k\right) \\ \mathrm{Im}\left(\lambda^k\right) & \mathrm{Re}\left(\lambda^k\right) \end{pmatrix}. \tag{4.24}$$

We have

$$\begin{aligned} e^A &= \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \begin{pmatrix} \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) & -\mathrm{Im}\left(\frac{\lambda^k}{k!}\right) \\ \mathrm{Im}\left(\frac{\lambda^k}{k!}\right) & \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) \end{pmatrix} = \begin{pmatrix} \sum_{k=0}^{\infty} \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) & -\sum_{k=0}^{\infty} \mathrm{Im}\left(\frac{\lambda^k}{k!}\right) \\ \sum_{k=0}^{\infty} \mathrm{Im}\left(\frac{\lambda^k}{k!}\right) & \sum_{k=0}^{\infty} \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Re}\left(e^{\lambda}\right) & -\mathrm{Im}\left(e^{\lambda}\right) \\ \mathrm{Im}\left(e^{\lambda}\right) & \mathrm{Re}\left(e^{\lambda}\right) \end{pmatrix} = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}. \end{aligned} \tag{4.25}$$

Note that, in the last equality, we used the Euler's formula: for any real number x, we have $e^{ix} = \cos x + i \sin x$. Thus, for $\lambda = a + ib$, we get $e^{\lambda} = e^a e^{ib} = e^a \left(\cos b + i \sin b\right)$.

4.3. **Matrix exponentials: the key idea of calculation**

A natural question that may arise here is that how we can calculate e^A for an arbitrary matrix A. The key idea is as follows. Let P be an invertible matrix, and consider $B := P^{-1}AP$. Then, for any integer $k > 0$, we have

$$B^k = \left(P^{-1}AP\right)^k = \overbrace{\left(P^{-1}AP\right)\cdots\left(P^{-1}AP\right)}^{k \text{ times}} = P^{-1}A\cancel{P}\cancel{P^{-1}}A\cancel{P}\cancel{P^{-1}}A\cancel{P}\cdots\cancel{P^{-1}}AP = P^{-1}A^kP \tag{4.26}$$

which implies $A^k = PB^kP^{-1}$. Thus,

$$e^A = \sum_{k=0}^{\infty}\frac{A^k}{k!} = \sum_{k=0}^{\infty}\frac{PB^kP^{-1}}{k!} = P\left(\sum_{k=0}^{\infty}\frac{B^k}{k!}\right)P^{-1} = Pe^BP^{-1}. \tag{4.27}$$

What relation (4.27) suggests is that if, for a given A, we can find B such that $B = P^{-1}AP$, for some invertible matrix P, and computing e^B be easy, then we can find e^A using relation (4.27), i.e. $e^A = Pe^BP^{-1}$. For example, if A is diagonalizable, we can choose B to be a diagonal matrix and then use Example 4.12.

Remark 4.16. *Most of the matrices are diagonalizable. For non-diagonalizable matrices, the matrix B can be chosen to be the Jordan form of A. In this course, we deal with non-diagonalizable case for 2×2 matrices and refer the reader to [VS18] for higher dimensional case.*

4.4. Planar linear systems

In this section, we study the dynamics of

$$\dot{x} = Ax, \tag{4.28}$$

where A is a 2×2 real matrix. Our approach is based on the following lemma

Lemma 4.17. *For a given $A \in \mathbb{R}^{2\times 2}$, there exists an invertible $P \in \mathbb{R}^{2\times 2}$ such that $B = P^{-1}AP$ has one of the following forms*

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \qquad B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad \text{or} \qquad B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \tag{4.29}$$

where λ, μ, a and $b \neq 0$ are real.

Proof. This lemma is the Jordan form theorem for the particular case of 2-dimensional matrices. See [Per01], Jordan canonical form theorem (Section 1.8). □

Let B and P be as in Lemma 4.17, and define the change of variables $y = P^{-1}x$. Thus, $y \in \mathbb{R}^2$ and $x = Py$. Then

$$\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APy = By. \tag{4.30}$$

This relation together with Lemma 4.17 suggests that by a linear change of variables, any given linear planar system $\dot{x} = Ax$ can be reduced to a system $\dot{y} = By$, where B is one of the three matrices given by Lemma 4.17.

4.4.1 Case I: $B = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$

In this section, we study the system $\dot{y} = By$, where $B = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Consider an initial point $(y_{10}, y_{20}) \in \mathbb{R}^2$. The solution of $\dot{y} = By$ passing through this initial point at $t = 0$ is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{Bt} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} y_{10} \\ e^{\mu t} y_{20} \end{pmatrix}. \tag{4.31}$$

Assume λ, y_{10} and y_{20} are non-zero. Then,

$$y_2(t) = e^{\mu t} y_{20} = \left(e^{\lambda t}\right)^{\frac{\mu}{\lambda}} y_{20} = \left(e^{\lambda t} y_{10}\right)^{\frac{\mu}{\lambda}} y_{10}^{-\frac{\mu}{\lambda}} y_{20} = y_{10}^{-\frac{\mu}{\lambda}} y_{20} \left[y_1(t)\right]^{\frac{\mu}{\lambda}}. \tag{4.32}$$

This means that for the case that λ, y_{10} and y_{20} are non-zero, the orbit of (y_{10}, y_{20}) lies in the set

$$\{(y_1, y_2) : \quad y_2 = y_{10}^{-\frac{\mu}{\lambda}} y_{20} y_1^{\frac{\mu}{\lambda}}\}. \tag{4.33}$$

When $\lambda \neq 0$ but $y_{10} = 0$, it follows from (4.31) that $y_1(t) = 0$ for all $t \in \mathbb{R}$. This implies that the orbit of $(0, y_{20})$ is the positive side of y_2-axis if $y_{20} > 0$, the negative side of y_2-axis if $y_{20} < 0$, and the origin if $y_{20} = 0$. Similarly, when $\lambda \neq 0$ but $y_{20} = 0$, it follows from (4.31) that $y_2(t) = 0$ for all $t \in \mathbb{R}$. Thus, the orbit of $(y_{10}, 0)$ is the positive side of y_1-axis if $y_{10} > 0$, and the negative side of y_1-axis if $y_{10} < 0$. This analysis also implies that the vertical and horizontal axes are invariant with respect to the dynamics.

In order to figure out the phase portrait of the system $\dot{y} = By$, where $B = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$, we consider the following scenarios:

(i) $\lambda < 0 < \mu$ or $\mu < 0 < \lambda$.

(ii) $\lambda = \mu > 0$ or $\lambda = \mu < 0$.

(iii) $\mu > \lambda > 0$ or $\mu < \lambda < 0$.

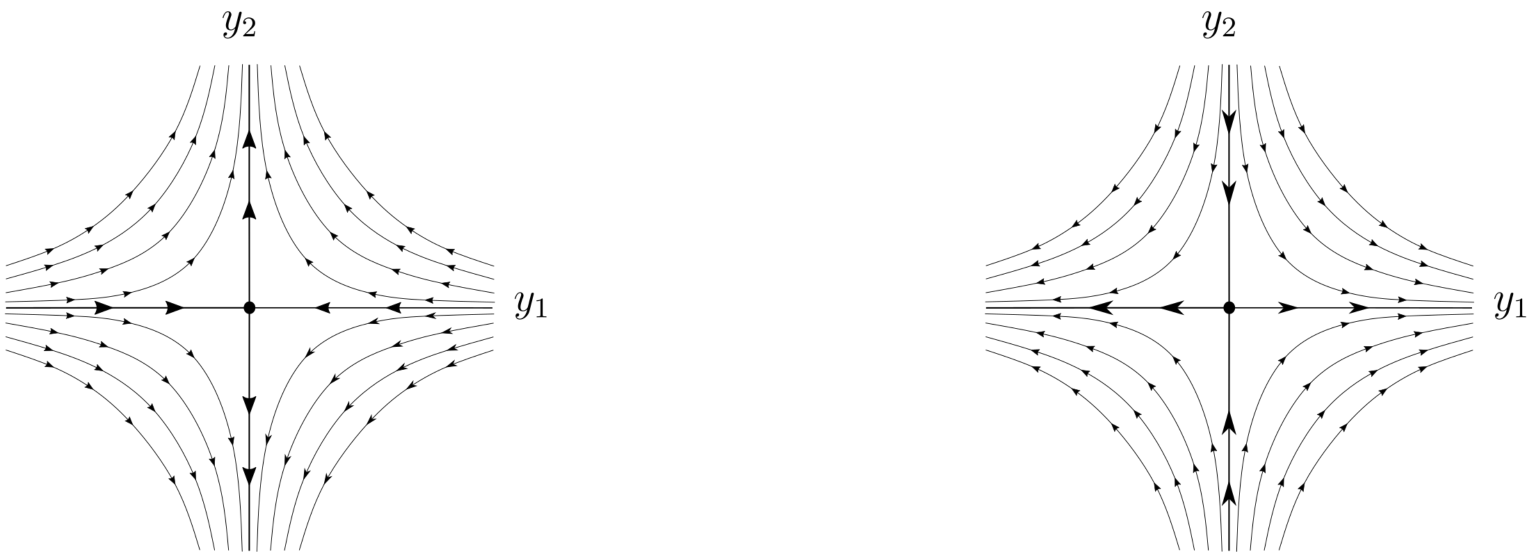
(iv) $\lambda > \mu > 0$ or $\lambda < \mu < 0$.

(v) $\lambda = 0$ or $\mu = 0$.

- **Scenario (i): $\lambda < 0 < \mu$ or $\mu < 0 < \lambda$.**

In this scenario, $\frac{\mu}{\lambda} < 0$. Define $\beta = \frac{\mu}{\lambda}$. According to (4.33), we need to plot the curves of the form $y_2 = \text{constant} \cdot y_1^{\beta}$, where $\beta < 0$. Taking into account that the horizontal and vertical axes are invariant, we can plot the phase portrait of the system for this scenario (see Figure 12).

- In this scenario, two orbits approach the origin as $t \to \infty$ and two other orbits approach the origin as $t \to -\infty$.
- The equilibrium point at the origin in such scenarios is called a *saddle* point.



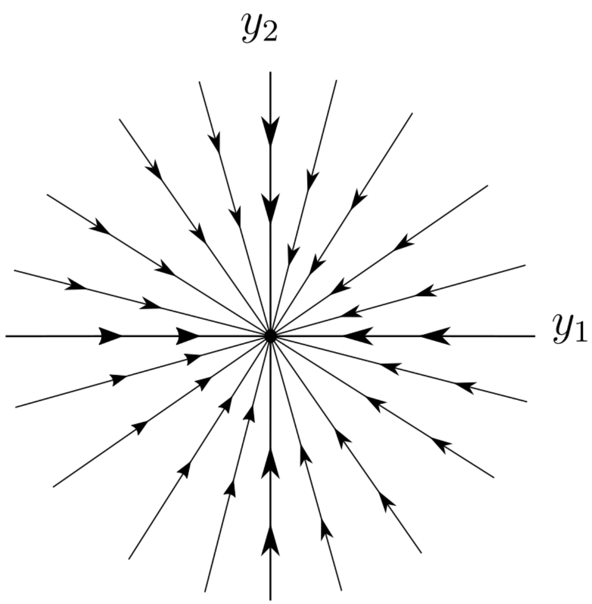
(a) $\lambda < 0 < \mu$.

(b) $\mu < 0 < \lambda$.

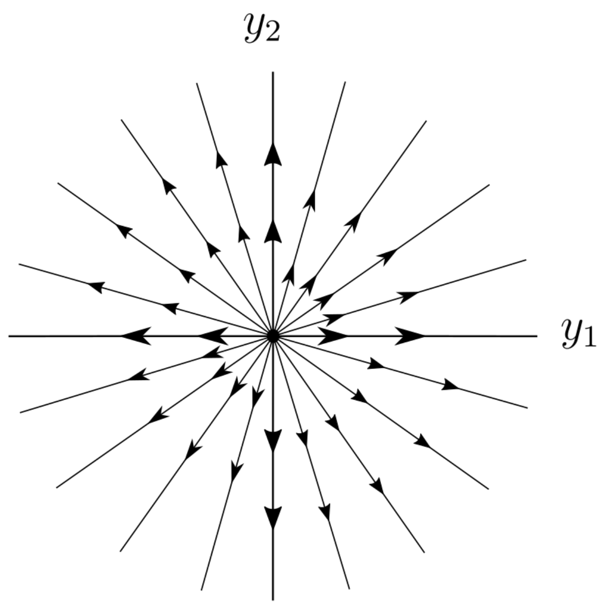
Figure 12: Phase portrait of scenario (i).

- **Scenario (ii):** $\lambda = \mu > 0$ **or** $\lambda = \mu < 0$.

In this scenario, $\frac{\mu}{\lambda} = 1$. According to (4.33), we need to plot the straight lines $y_2 = \text{constant} \cdot y_1$. Taking into account that the horizontal and vertical axes are invariant, we can plot the phase portrait of the system for this scenario (see Figure 13).



(a) $\lambda = \mu < 0$.

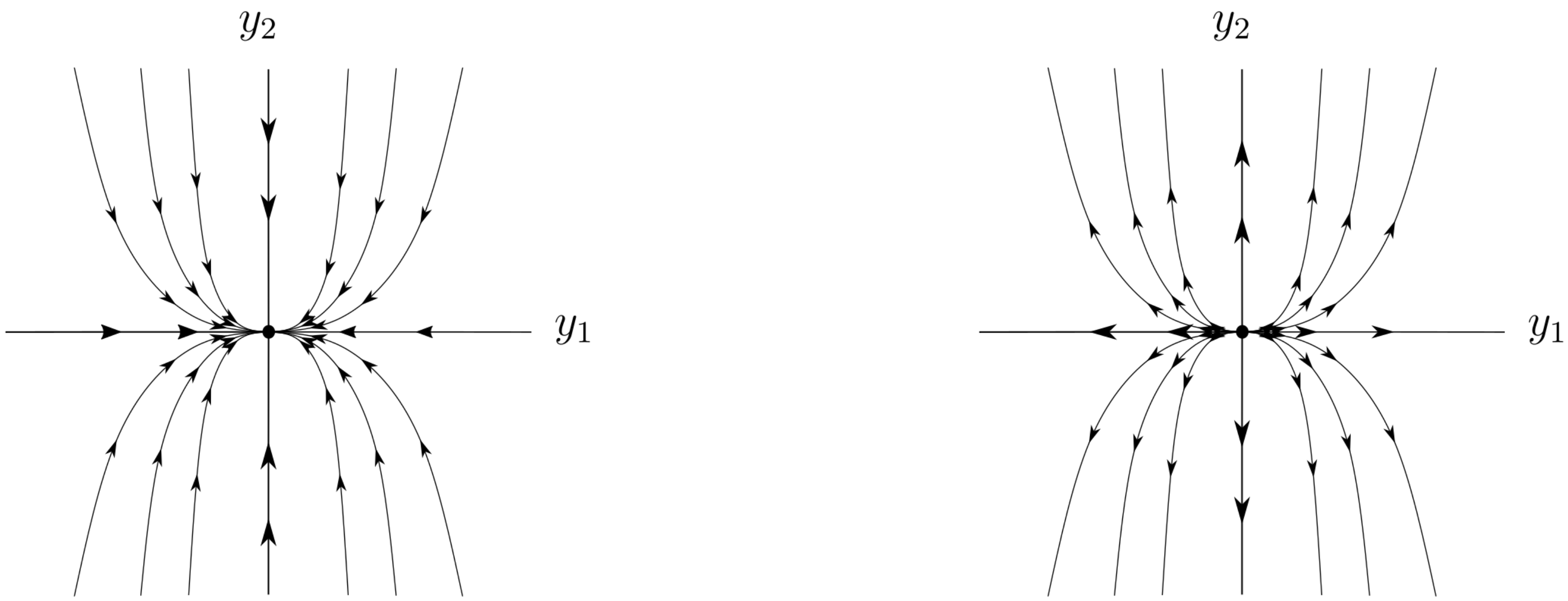


(b) $\lambda = \mu > 0$.

Figure 13: Phase portrait of scenario (ii).

- **Scenario (iii): $\mu > \lambda > 0$ or $\mu < \lambda < 0$.**

In this scenario, $\frac{\mu}{\lambda} > 1$. Define $\beta = \frac{\mu}{\lambda}$. According to (4.33), we need to plot the curves of the form $y_2 = \text{constant} \cdot y_1^{\beta}$, where $\beta > 1$. Taking into account that the horizontal and vertical axes are invariant, we can plot the phase portrait of the system for this scenario (see Figure 14).

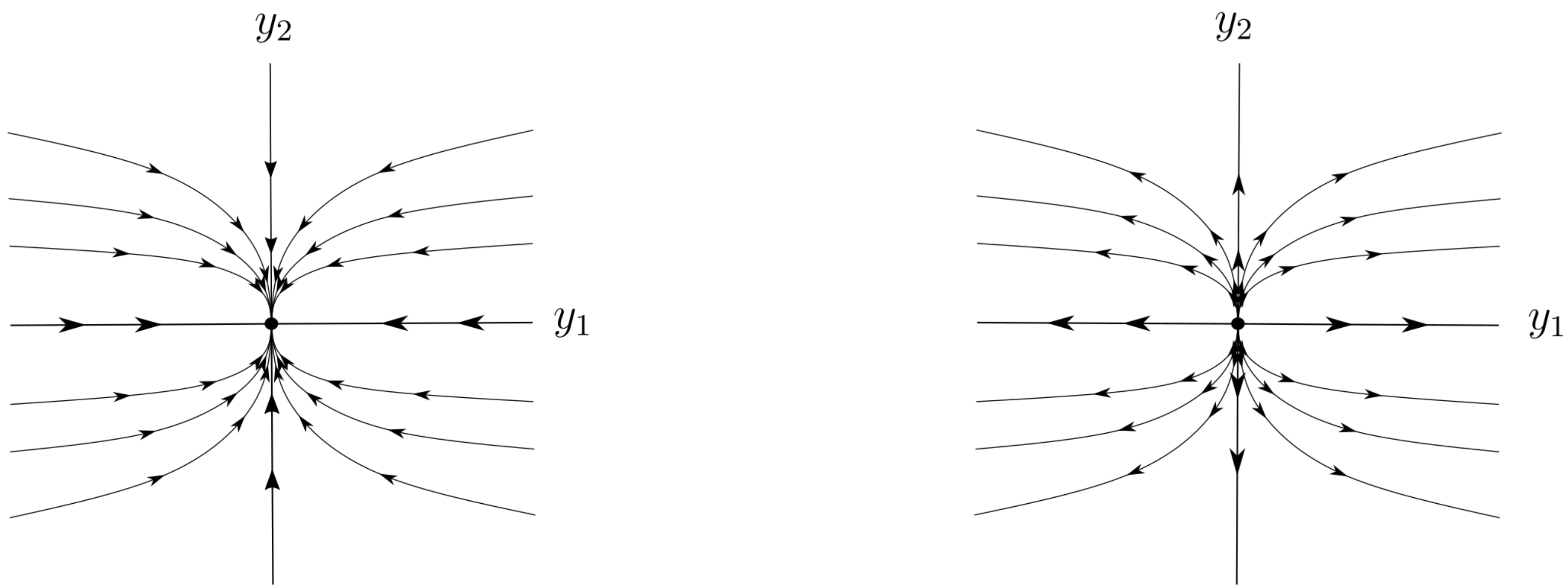


(a) Case $\mu < \lambda < 0$. (b) Case $\mu > \lambda > 0$.

Figure 14: Phase portrait of scenario (iii).

- **<u>Scenario (iv): $\lambda > \mu > 0$ or $\lambda < \mu < 0$.</u>**

In this scenario, $0 < \frac{\mu}{\lambda} < 1$. Define $\beta = \frac{\mu}{\lambda}$. According to (4.33), we need to plot the curves of the form $y_2 = \text{constant} \cdot y_1^{\beta}$, where $0 < \beta < 1$. Taking into account that the horizontal and vertical axes are invariant, we can plot the phase portrait of the system for this scenario (see Figure 15).



(a) Case $\lambda < \mu < 0$.

(b) Case $\lambda > \mu > 0$.

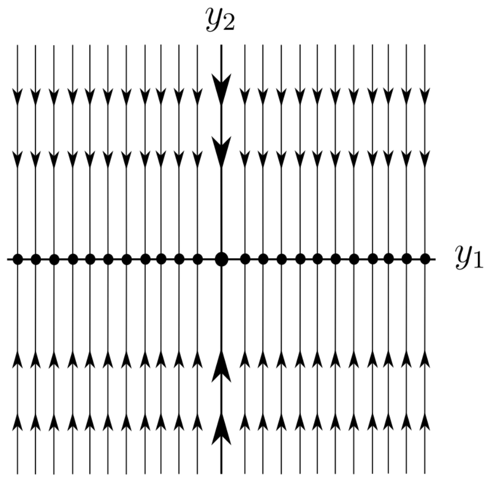
Figure 15: Phase portrait of scenario (iv).

- **Scenario (v): $\lambda = 0$ or $\mu = 0$.**

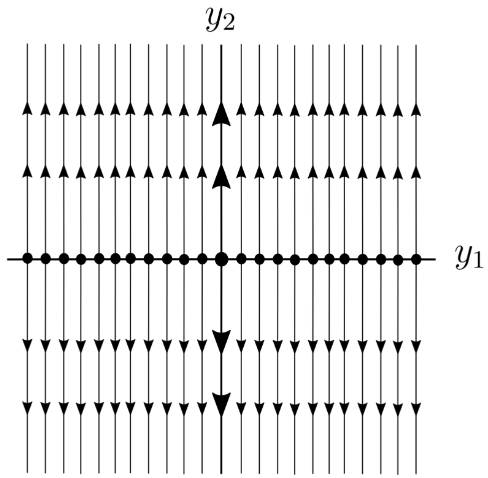
Assume $\lambda = 0$ and $\mu \neq 0$. Recall from (4.31) that

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} y_{10} \\ e^{\mu t} y_{20} \end{pmatrix}. \tag{4.34}$$

Suppose $\lambda = 0$ and observe that any point on the y_1-axis is an equilibrium. Moreover, by (4.34), we have $(y_1(t), y_2(t)) = (y_{10}, e^{\mu t} y_{20})$. This suggests that when $\lambda = 0$, the orbit of (y_{10}, y_{20}) is the positive side of the vertical line $y_1 = y_{10}$ if $y_{20} > 0$, the negative side of the vertical line $y_1 = y_{10}$ if $y_{20} < 0$, and the point $(y_{10}, 0)$ if $y_{20} = 0$. By this analysis, we have the phase portrait for the case $\lambda = 0$ and $\mu \neq 0$ as in Figure 16. By an analogous analysis, we obtain the phase portrait of the case $\lambda \neq 0$ and $\mu = 0$ as in Figure 17.

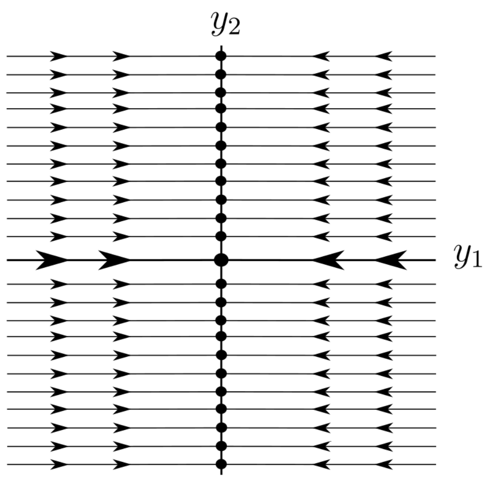


(a) Case $\lambda = 0$ and $\mu < 0$.

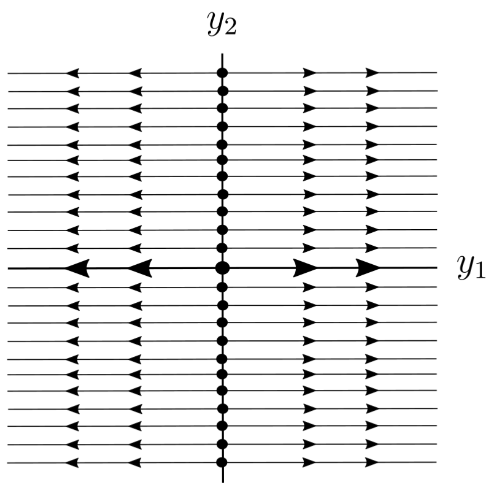


(b) Case $\lambda = 0$ and $\mu > 0$.

Figure 16: Phase portrait of scenario (v).



(a) Case $\mu = 0$ and $\lambda < 0$.



(b) Case $\mu = 0$ and $\lambda > 0$.

Figure 17: Phase portrait of scenario (v).

4.4.2 Case II: $B = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$

In this section, we study the system $\dot{y} = By$, where $B = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Consider an initial point $(y_{10}, y_{20}) \in \mathbb{R}^2$. The solution of $\dot{y} = By$ passing through this initial point at $t = 0$ is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{Bt} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} y_{10} + e^{\lambda t} t y_{20} \\ e^{\lambda t} y_{20} \end{pmatrix}. \tag{4.35}$$

In order to figure out the phase portrait of the system $\dot{y} = By$, where $B = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \mu \end{smallmatrix}\right)$, we consider the following scenarios:

(i) $\lambda \neq 0$.

(ii) $\lambda = 0$.

- **Scenario (i):** $\lambda \neq 0$.

Assume λ and y_{20} are non-zero. From the equation $y_2(t) = e^{\lambda t} y_{20}$, we obtain

$$t = \frac{1}{\lambda} \ln \frac{y_2(t)}{y_{20}}. \tag{4.36}$$

On the other hand, $\frac{y_1(t)}{y_2(t)} = \frac{y_{10}}{y_{20}} + t$. Thus, by (4.36), we have

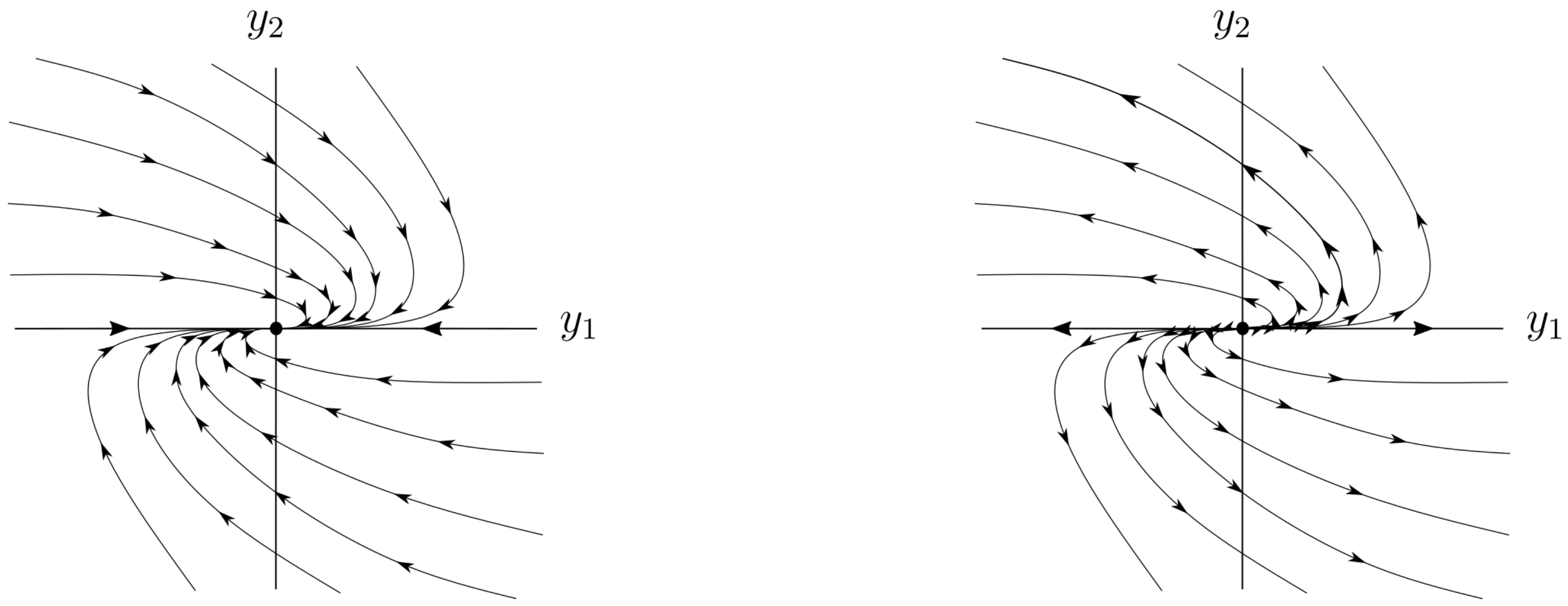
$$\frac{y_1(t)}{y_2(t)} = \frac{y_{10}}{y_{20}} + \frac{1}{\lambda} \ln \frac{y_2(t)}{y_{20}} = \left[\frac{y_{10}}{y_{20}} - \frac{1}{\lambda} \ln y_{20}\right] + \frac{1}{\lambda} \ln y_2(t), \tag{4.37}$$

which gives

$$y_1(t) = \frac{y_{10}}{y_{20}} + \frac{1}{\lambda} \ln \frac{y_2(t)}{y_{20}} = \left[\frac{y_{10}}{y_{20}} - \frac{1}{\lambda} \ln y_{20}\right] y_2(t) + \frac{1}{\lambda} y_2(t) \ln y_2(t). \tag{4.38}$$

Thus, to plot the phase portrait of this scenario, we need to consider the curves of the form $y_1 = \alpha y_2 + \frac{1}{\lambda} y_2 \ln y_2$, where α is some constant. This is also easily seen that the horizontal axis $y_2 = 0$ is invariant, and these curves are tangent to the horizontal axis at the origin. This analysis gives Figure 18.

Remark 4.18. *Note that the y_1-axis is invariant while the y_2-axis is not.*



(a) Case $\lambda < 0$.

(b) Case $\lambda > 0$.

Figure 18: Scenario (i): $\lambda \neq 0$

- **Scenario (ii):** $\lambda = 0$.

By (4.35), when $\lambda = 0$, we have

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y_{10} + t y_{20} \\ y_{20} \end{pmatrix}. \tag{4.39}$$

This implies that the horizontal lines $y_2 =$ constant are invariant. the phase portrait for this scenario is given in Figure 19.

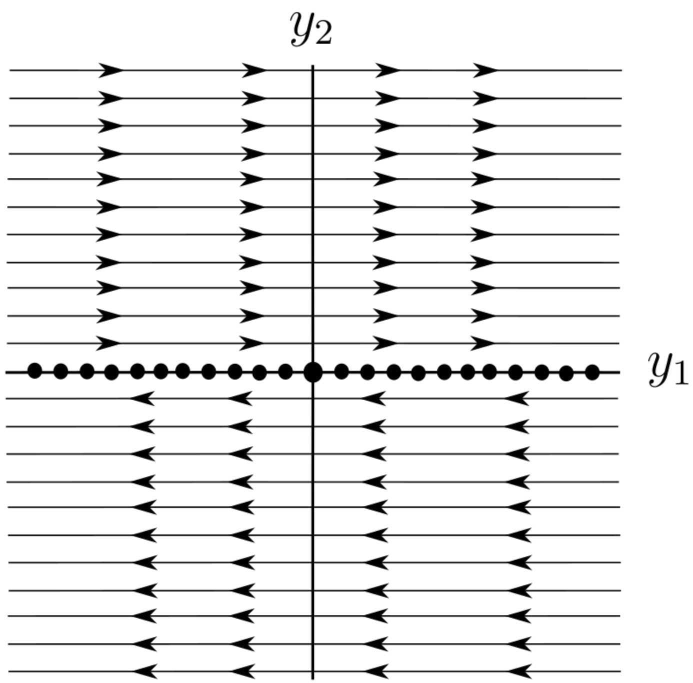


Figure 19: Phase portrait of scenario (ii): $\lambda = 0$.

4.4.3 Case III: $B = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$

In this section, we study the system $\dot{y} = By$, where $B = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Consider an initial point $(y_{10}, y_{20}) \in \mathbb{R}^2$. The solution of $\dot{y} = By$ passing through this initial point at $t = 0$ is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}. \tag{4.40}$$

In order to figure out the phase portrait of the system $B = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, we consider the following scenarios:

(i) $a = 0$.

(ii) $a \neq 0$.

Note that when $b = 0$, we have $B = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ which is the case that was studied earlier (see Figure 13).

Before we proceed to study the above scenarios, let us first see what the geometrical meaning of relation (4.40) is. Let $\theta \in \mathbb{R}$, and consider the matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4.41}$$

- The matrix R_θ is called a rotation matrix. This matrix rotates the points in the plane about the origin by the angle θ (see e.g. [Mey00]). The rotation is counter-clockwise when $\theta > 0$, and clockwise when $\theta < 0$.

- In (4.40), the matrix $\left(\begin{smallmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{smallmatrix}\right)$ rotates $\left(\begin{smallmatrix} y_{10} \\ y_{20} \end{smallmatrix}\right)$ by the angle bt. Thus, as t increases, this rotation is counter-clockwise if $b > 0$, and clockwise if $b < 0$. Then, after this rotation, the coefficient e^{at} in (4.40) controls the size of $\left(\begin{smallmatrix} y_1(t) \\ y_2(t) \end{smallmatrix}\right)$. In other words, b controls the angle (rotation) and a controls the size of $\left(\begin{smallmatrix} y_1(t) \\ y_2(t) \end{smallmatrix}\right)$.

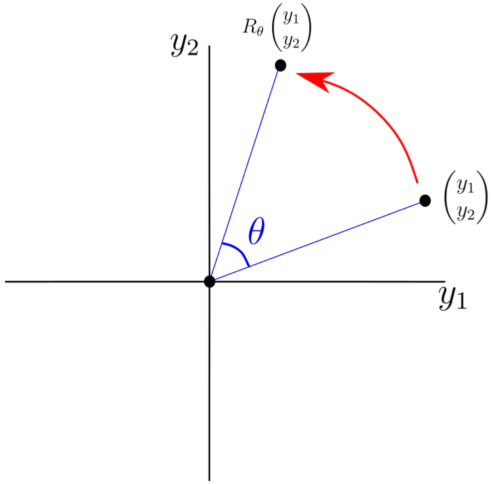
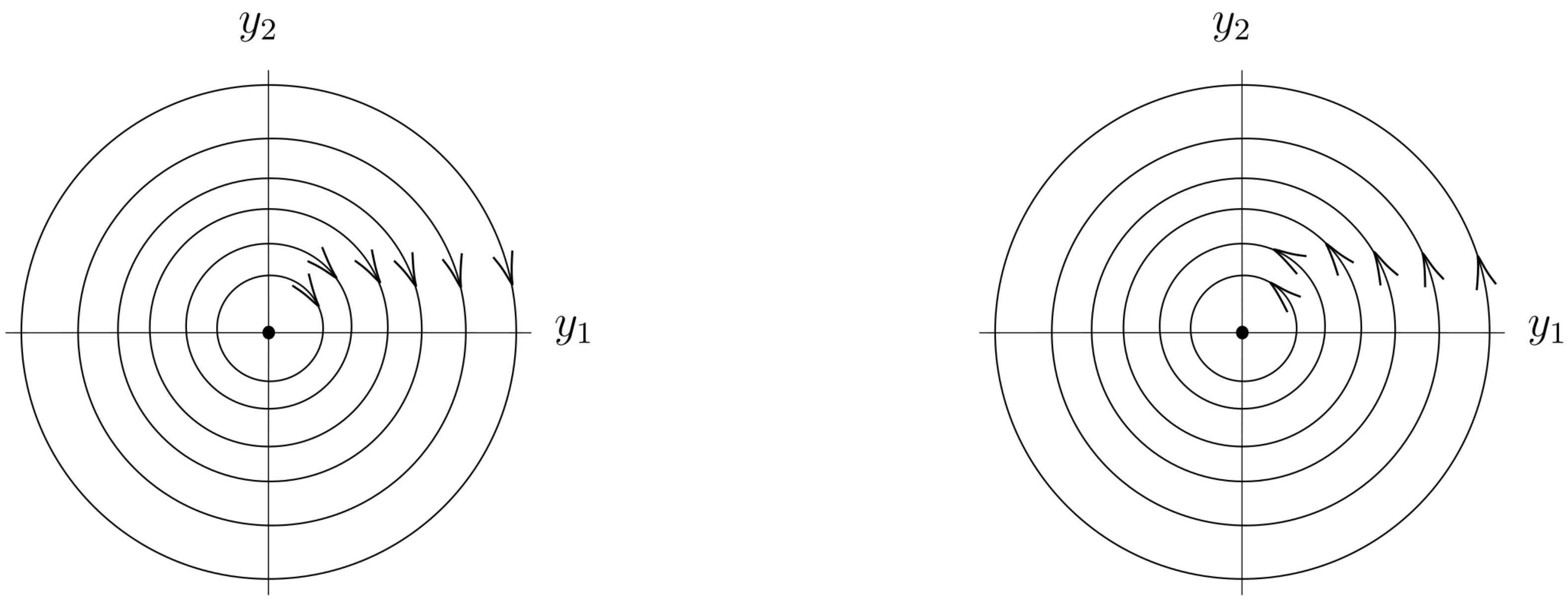


Figure 20: R_θ rotates the points in the plane by the angle θ about the origin.



(a) $b < 0$. (b) $b > 0$.

Figure 21: Scenario (i): $a = 0$.

(a) $b < 0$.　　(b) $b > 0$.

Figure 22: Scenario (ii): $a > 0$

(a) $b < 0$.　　(b) $b > 0$.

Figure 23: Scenario (ii): $a < 0$

4.5. **Stability of equilibria in linear systems**

4.5.1 Preliminaries from Linear Algebra

In this section, we briefly review some concepts from Linear Algebra. For a more detailed review on this topic, we recommed [HSD12] (Sections 2.3 and 5.1).

Vector norms

Let $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ be a vector in $\mathbb{R}^n$. In this course, we define the norm of x, denoted by $\|x\|$, by

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}. \tag{4.42}$$

This norm is called the standard norm of the Euclidean space $\mathbb{R}^n$.

Remark 4.19. *Norm is a function which assigns a non-negative real number to every vector of* $\mathbb{R}^n$*.*

Example 4.20. *(i) Consider* $v = (-3, 0, 3, 2) \in \mathbb{R}^4$*. Then*

$$\|v\| = \sqrt{(-3)^2 + 0^2 + 3^2 + 2^2} = \sqrt{9 + 0 + 9 + 4} = \sqrt{22}. \tag{4.43}$$

(ii) Let O *be the origin of* $\mathbb{R}^n$*. Then,* $\|O\| = 0$*.*

(iii) let $-1 = (-1) \in \mathbb{R}$*. Then* $\|(-1)\| = 1$*.*

Exercise 4.21. *Prove that the norm defined by (4.42) satisfies the following properties.*

(i) Let O *be the origin of* $\mathbb{R}^n$*, and* $x \in \mathbb{R}^n$ *be an arbitrary vector. Then,* $\|x\| = 0$ *if and only if* $x = O$*.*

(ii) Let r *be an arbitrary real number, and* v *be an arbitrary vector in* $\mathbb{R}^n$*. Then,* $\|rv\| = |r|\|v\|$*.*

(iii) (Triangular inequality) Let $x, y \in \mathbb{R}^n$*. Then,* $\|x + y\| \leq \|x\| + \|y\|$*.*

Linear independence

Definition 4.22. *Consider m vectors $v_1, v_2, \dots, v_m$ in $\mathbb{R}^n$. A linear combination of these m vectors is any vector of the form*

$$r_1 v_1 + r_2 v_2 + \cdots + r_m v_m, \tag{4.44}$$

where r_i are arbitrary real numbers.

Definition 4.23. *Consider the vectors $v_1, v_2, \dots, v_m$, where $m \geq 2$, in $\mathbb{R}^n$. We say that these m vectors are linearly independent if and only if none of these vectors can be written as a linear combination of the other $m-1$ vectors. An equivalent version of this definition is as follows: if $r_1 v_1 + r_2 v_2 + \cdots + r_m v_m = 0$, for some real r_i, then $r_1 = \cdots = r_m = 0$.*

Example 4.24. *The vectors $\left(\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 2 \\ 0 \end{smallmatrix}\right)$ are linearly independent. Here is why: let $r_1, r_2 \in \mathbb{R}$. Then,*

$$r_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + r_2 \begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} 2r_2 - r_1 \\ r_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow r_1 = 0 \Longrightarrow r_2 = 0. \tag{4.45}$$

Example 4.25. *The vectors $\left(\begin{smallmatrix} 3 \\ -1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 5 \\ 2 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} -7 \\ -5 \end{smallmatrix}\right)$ are not linearly independent. Here is why: let $r_1 = -2$, $r_2 = 4$ and $r_3 = 2$. Then,*

$$-2 \begin{pmatrix} 3 \\ -1 \end{pmatrix} + 4 \begin{pmatrix} 5 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} -7 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{4.46}$$

Generated vector subspaces

Suppose that a family of vectors $\{v_\alpha\}$ in $\mathbb{R}^n$ is given. Then, the set

$$V = \{r_1 v_1 + r_2 v_2 + \cdots + r_m v_m : m \geq 1 \text{ is an arbitrary integer, } r_i \text{ are arbitrary real numbers, and } v_i \in \{v_\alpha\}\} \quad (4.47)$$

is a vector subspace of $\mathbb{R}^n$.

Definition 4.26. *The set V is called the vector (sub)space generated by $\{v_\alpha\}$. We denote it by $\langle\{v_\alpha\}\rangle$.*

Let V_1, V_2, ..., and V_k be subspaces of $\mathbb{R}^n$. Assume that the intersection of any two of these subspaces is only the origin of $\mathbb{R}^n$, i.e. $V_i \cap V_j = \{0\}$, for all $1 \leq i, j \leq k$. We write

$$\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k \tag{4.48}$$

if $\mathbb{R}^n$ can be generated by V_1, V_2, ..., V_k, i.e. $\mathbb{R}^n = \langle V_1, \ldots, V_k \rangle$.

Remark 4.27. Assume $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$. Then, for any arbitrary $v \in \mathbb{R}^n$, there exists a unique vector $v_i \in V_i$, for every $i = 1, \ldots, k$, such that $v = v_1 + v_2 + \cdots + v_k$. In other words, an arbitrary vector $v \in \mathbb{R}^n$ can be uniquely decomposed to components in each of the subspaces V_i.

Eigenvalues, eigenvectors and (generalized) eigenspaces

Definition 4.28. *For a given $n \times n$ matrix A, define $p(x) := \det(A - xI)$. The expression $p(x)$ is a polynomial of degree n and is called the characteristic polynomial of A.*

Example 4.29. *Consider the matrix $Q = \left(\begin{smallmatrix} -1 & 4 \\ 5 & 2 \end{smallmatrix}\right)$. The characteristic polynomial of Q is*

$$p(x) = \det(Q - xI) = \det \begin{pmatrix} -1 - x & 4 \\ 5 & 2 - x \end{pmatrix} = (-1 - x) \times (2 - x) - 4 \times 5 = x^2 - x - 22. \tag{4.49}$$

Example 4.30. *Consider the matrix $Q = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, where a and b are real numbers. The characteristic polynomial of Q is*

$$p(x) = \det(Q - xI) = \det \begin{pmatrix} a - x & -b \\ b & a - x \end{pmatrix} = (a - x)^2 + b^2 = x^2 - 2ax + a^2 + b^2. \tag{4.50}$$

Definition 4.31. *The roots of the characteristic polynomial of the matrix A are called the eigenvalues of A.*

Remark 4.32. *Considering the multiple (repeated) roots of $p(A)$, the $n \times n$ matrix A has exactly n eigenvalues. The number of times that the eigenvalue λ appears as the root of the characteristic polynomial $p(x)$ is called the algebraic multiplicity of λ. An eigenvalue λ is said to be simple if its algebraic multiplicity is 1.*

Example 4.33. *The matrix given in Example 4.29 has two eigenvalues given by $\lambda_{1,2} = \frac{1}{2}\left(1 \pm \sqrt{89}\right)$.*

Example 4.34. *The matrix given in Example 4.30 has two eigenvalues given by $\lambda_{1,2} = a \pm bi$, where $i = \sqrt{-1}$.*

Remark 4.35. *As Example 4.30 suggests, the eigenvalues of a real matrix A can be non-real too. In this case, non-real eigenvalues appear as pairs. More precisely, if $\lambda = a + bi$ ($i = \sqrt{-1}$) is an eigenvalue of A then the complex conjugate of λ, i.e. $\overline{\lambda} = a - bi$, is an eigenvalue of A too.*

Example 4.36. *It is easily seen that the eigenvalues of diagonal matrices (or more generally, upper or lower triangular matrices) are exactly the elements on the diagonal. For example, each of the matrices*

$$\begin{pmatrix} -7 & & & & & \\ & 5 & & & \mathbf{0} & \\ & & 0 & & & \\ & & & -7 & & \\ & \mathbf{0} & & & 0 & \\ & & & & & -7 \end{pmatrix}, \qquad \begin{pmatrix} -7 & -2 & 21 & 0 & -1 & 0 \\ 0 & 5 & 4 & -4 & 1 & 0 \\ 0 & 0 & 0 & 11 & \sqrt{6} & 8 \\ 0 & 0 & 0 & -7 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3.23 \\ 0 & 0 & 0 & 0 & 0 & -7 \end{pmatrix} \tag{4.51}$$

has six eigenvalues: eigenvalue -7 with algebraic multiplicity 3, eigenvalue 0 with algebraic multiplicity 2, and a simple eigenvalue 5.

Suppose that λ is an eigenvalue of A. Thus, $\det(A-\lambda I)=0$. Equivalently, $A-\lambda I$ is a singular (noninvertible) matrix. Therefore, $\mathrm{Null}(A-\lambda I)$ is non-trivial (i.e. its dimension is at least 1), where $\mathrm{Null}(A-\lambda I)$ is the null space of $A-\lambda I$. Assume λ is real. Thus, there exists $0\neq v\in\mathbb{R}^n$ such that $(A-\lambda I)\,v=0$. Equivalently, $Av=\lambda v$. Analogously, for the case that λ is non-real, such a vector $0\neq v\in\mathbb{C}^n$ that satisfies $Av=\lambda v$ can be found.

Definition 4.37. *Let λ be an eigenvalue of A. Any vector $v\neq 0$ that satisfies $Av=\lambda v$ is called an eigenvector of A associated with λ.*

Example 4.38. *Consider the matrix $A=\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$. The characteristic polynomial of A is*

$$p(x)=\det(A-xI)=\det\begin{pmatrix}2-x&1\\1&2-x\end{pmatrix}=x^2-4x+3. \tag{4.52}$$

The eigenvalues of A are the roots of $p(x)=x^2-4x+3$ which are $\lambda=3$ and $\lambda=1$. For $\lambda=3$, any vector of the form $v=\left(\begin{smallmatrix}r\\r\end{smallmatrix}\right)$, where $r\in\mathbb{R}$ is an eigenvector. For instance $v=\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$. For $\lambda=1$, any vector of the form $v=\left(\begin{smallmatrix}r\\-r\end{smallmatrix}\right)$, where $r\in\mathbb{R}$ is an eigenvector. For instance $v=\left(\begin{smallmatrix}1\\-1\end{smallmatrix}\right)$.

Proposition 4.39. *Consider a real $n \times n$ matrix A and suppose that λ is an eigenvalue of it with a corresponding eigenvector v. Then, e^λ is an eigenvalue of e^A and v is an eigenvector of e^A associated with the eigenvalue e^λ.*

Proof. Since v is an eigenvector associated to λ, we have

$$e^A v = \left(I + \frac{A}{1!} + \frac{A^2}{2!} + \frac{A^3}{3!} + \frac{A^4}{4!} + \cdots \right) v = v + \frac{Av}{1!} + \frac{A^2 v}{2!} + \frac{A^3 v}{3!} + \frac{A^4 v}{4!} + \cdots . \tag{4.53}$$

Notice that, for a positive integer k, we have

$$A^k v = A^{k-1} A v = \lambda A^{k-1} v = \lambda A^{k-2} A v = \lambda^2 A^{k-2} v = \cdots = \lambda^{k-1} A v = \lambda^k v. \tag{4.54}$$

Thus, by (4.53), we have

$$\begin{aligned} e^A v &= v + \frac{Av}{1!} + \frac{A^2 v}{2!} + \frac{A^3 v}{3!} + \frac{A^4 v}{4!} + \cdots \\ &= v + \frac{\lambda v}{1!} + \frac{\lambda^2 v}{2!} + \frac{\lambda^3 v}{3!} + \frac{\lambda^4 v}{4!} + \cdots \\ &= \left(1 + \frac{\lambda}{1!} + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \frac{\lambda^4}{4!} + \cdots \right) v \\ &= e^\lambda v. \end{aligned} \tag{4.55}$$

$\square$

Generalized eigenvectors and eigenspaces

Having $Av = \lambda v$ is equivalent to $(A - \lambda I)\, v = 0$. The set of all such vectors v is called the eigenspace associated with the eigenvalue λ. We can generalize this concept as follows:

Definition 4.40. *Let λ be an eigenvalue of A. A vector v is said to be a generalized eigenvector if there exists an integer $k \geq 1$ such that $(A - \lambda I)^k\, v = 0$. The set of all such vectors is a vector subspace of $\mathbb{R}^n$ and called the generalized eigenspace associated with the eigenvalue λ.*

Example 4.41. *Consider the matrix $A = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$, where $\lambda \in \mathbb{R}$. This matrix has the eigenvalue λ with multiplicity* 2. *To find the associated eigenvectors $v = \left(\begin{smallmatrix} v_1 \\ v_2 \end{smallmatrix}\right)$, we have*

$$(A - \lambda I)\, v = 0 \Longrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow \begin{pmatrix} v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Longrightarrow v_2 = 0. \tag{4.56}$$

This suggests that $v = \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ is an eigenvector associated with λ. Moreover, we cannot find any other independent eigenvector for λ. Let us know try to find generalized eigenvectors. To this end, we need to solve the equation $(A - \lambda I)^2\, v = 0$ for $v \in \mathbb{R}^2$. We have

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \Longrightarrow (A - \lambda I)^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.57}$$

Since $(A - \lambda I)^2 = 0$, any arbitrary vector v satisfies $(A - \lambda I)^2\, v = 0$. Thus, any arbitrary $v \neq 0$ can be considered as a generalized eigenvector. If we look for vectors independent of from the eigenvector $\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$ founded earlier, we can consider, for instance $v = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$.

4.5.2 Examples of invariant sets for linear flows

LEMMA 4.42. *Let A be a real $n \times n$ matrix and $V \subseteq \mathbb{R}^n$ be a subspace such that $AV \subseteq V$. Then, $e^A V \subseteq V$.*

Proof. Let $x \in V$. Then

$$e^A x = x + \frac{Ax}{1!} + \frac{A^2x}{2!} + \frac{A^3x}{3!} + \frac{A^4x}{4!} + \cdots \tag{4.58}$$

Define $M_k := x + \frac{Ax}{1!} + \frac{A^2x}{2!} + \cdots + \frac{A^kx}{k!}$. Thus, $e^A x = \lim_{k\to\infty} M_k$. However, since for each positive integer j, we have $A^j x \in V$, we have $M_k \in V$ for all k. However, since $e^A x$ is well-defined, i.e. $\lim_{k\to\infty} M_k$ exists, and V is closed[6], we have that $e^A x \in V$. Thus, $e^A V \subseteq V$. □

The following proposition is an easy consequence of Lemma 4.42.

PROPOSITION 4.43. *Consider a system*

$$\dot{x} = Ax, \tag{4.59}$$

where A is an $n \times n$ real matrix. Let $V \subseteq \mathbb{R}^n$ be a subspace such that $AV \subseteq V$. If x_0 be an arbitrary point in V, we have $e^{tA} x_0 \subseteq V$ for all $t \in \mathbb{R}$. In other words, V is invariant with respect to the flow of system (4.59).

Example 4.44. *Consider system (4.59) and suppose λ is an eigenvalue of A. Let E_λ be the generalized eigenspace associated with λ. It can be shown that $AE_\lambda \subset E_\lambda$ (see [Per01], Section 1.9). Then, it follows from Proposition 4.43 that E_λ is invariant with respect to the flow of (4.59), i.e. $e^{At} E_\lambda \subseteq E_\lambda$ for all $t \in \mathbb{R}$.*

[6] Here, closedness is a topological property. A set X in $\mathbb{R}^n$ is said to be closed in $\mathbb{R}^n$, or simply closed, if for any sequence $\{x_k\}$ such that $x_k \in X$ for all k, and $\{x_k\}$ is convergent to some point $x^* \in \mathbb{R}^n$, we have $x^* \in X$. For example, the interval $(0, 1]$ is not closed in $\mathbb{R}$ because the sequence $\{\frac{1}{k}\}$ is in $(0, 1]$, however, this sequence is convergent to $0 \in \mathbb{R}$ but $0 \notin (0, 1]$.

4.5.3 Stable, unstable and center subspaces

Consider a system

$$\dot{x} = Ax, \tag{4.60}$$

where A is an $n \times n$ real matrix. The matrix A has n eigenvalues. Consider the real parts[7] of these eigenvalues. Some of these real parts are positive, some are negative and some equal to zero. Consider all the generalized eigenvectors of these eigenvalues[8].

Definition 4.45. *We define*

(i) $E^s := \left\langle \{v: \text{ the vector } v \text{ is a generalized eigenvector of some eigenvalue } \lambda, \text{ where } \operatorname{Re}(\lambda) < 0\} \right\rangle.$

(ii) $E^c := \left\langle \{v: \text{ the vector } v \text{ is a generalized eigenvector of some eigenvalue } \lambda, \text{ where } \operatorname{Re}(\lambda) = 0\} \right\rangle.$

(iii) $E^u := \left\langle \{v: \text{ the vector } v \text{ is a generalized eigenvector of some eigenvalue } \lambda, \text{ where } \operatorname{Re}(\lambda) > 0\} \right\rangle.$

We call E^s, E^c *and* E^u, *the stable, center and unstable subspaces of system (4.60), respectively.*

---

[7]Let $\lambda = a + bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$, be a complex number. By the real part and imaginary part of λ, we mean the real numbers a and b, respectively. We write $\operatorname{Re}(\lambda) = a$ and $\operatorname{Im}(\lambda) = b$. For example, $\operatorname{Re}(3-5i) = 3$, $\operatorname{Im}(3-5i) = -5$, $\operatorname{Re}(4i) = 0$, $\operatorname{Im}(4i) = 4$, $\operatorname{Re}(10) = 10$ and $\operatorname{Im}(10) = 0$

[8]If the eigenvalue λ is non-real, its generalized eigenvector can be written as $v = u + iw$, where $u, w \in \mathbb{R}^n$ and $i = \sqrt{-1}$. In such scenario, instead of $v = u + iw \in \mathbb{C}^n$, we consider the vectors u and w individually, and the vector subspace spanned by these two vectors.

Example 4.46. [9] *Consider system (4.60), where A is given by*

$$A = \begin{pmatrix} -2 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix}. \tag{4.61}$$

The matrix A has the eigenvalues $\lambda_{1,2} = -2 \pm i$ and $\lambda_3 = 3$. This matrix has the eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.62}$$

corresponding to $\lambda_{1,2}$, and

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.63}$$

corresponding to λ_3. Then

$$E^s = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \tag{4.64}$$

and

$$E^u = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle. \tag{4.65}$$

Thus, the stable subspace E^s is the (x_1, x_2)-plane, and the unstable subspace E^u is the x_3 axis (see Figure 24).

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[9] This example together with its figure is taken from [Per01]

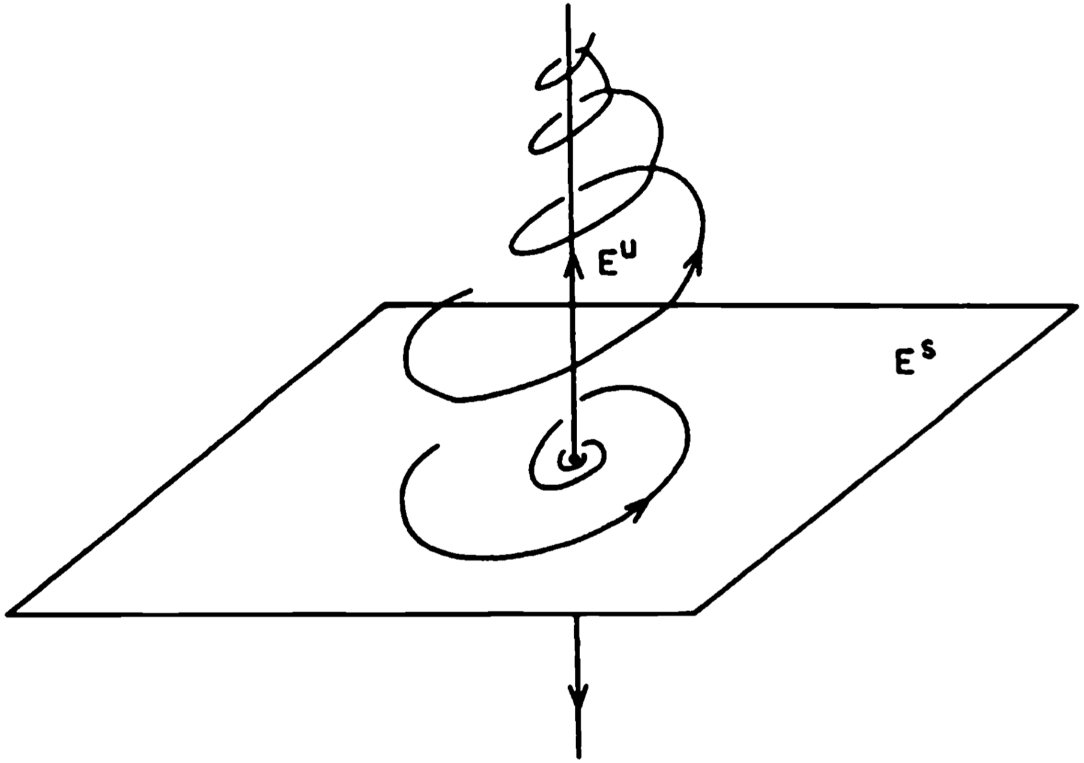


Figure 24: The stable subspace E^s is the (x_1, x_2)-plane, and the unstable subspace E^u is the x_3 axis.

Example 4.47. [10] *Consider system (4.60), where A is given by*

$$A = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}. \tag{4.66}$$

The matrix A has the pure imaginary eigenvalue $\lambda_{1,2} = i$ (with multiplicity 2) and $\lambda_3 = 2$. This matrix has the eigenvectors

$$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \pm i \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{4.67}$$

corresponding to $\lambda_{1,2}$, and

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{4.68}$$

corresponding to λ_3. Then

$$E^s = \left\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \right\rangle, \tag{4.69}$$

and

$$E^u = \left\langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\rangle. \tag{4.70}$$

Thus, the center subspace E^c is the (x_1, x_2)-plane, and the unstable subspace E^u is the x_3 axis (See Figure 25).

[10] This example together with its figure is taken from [Per01]

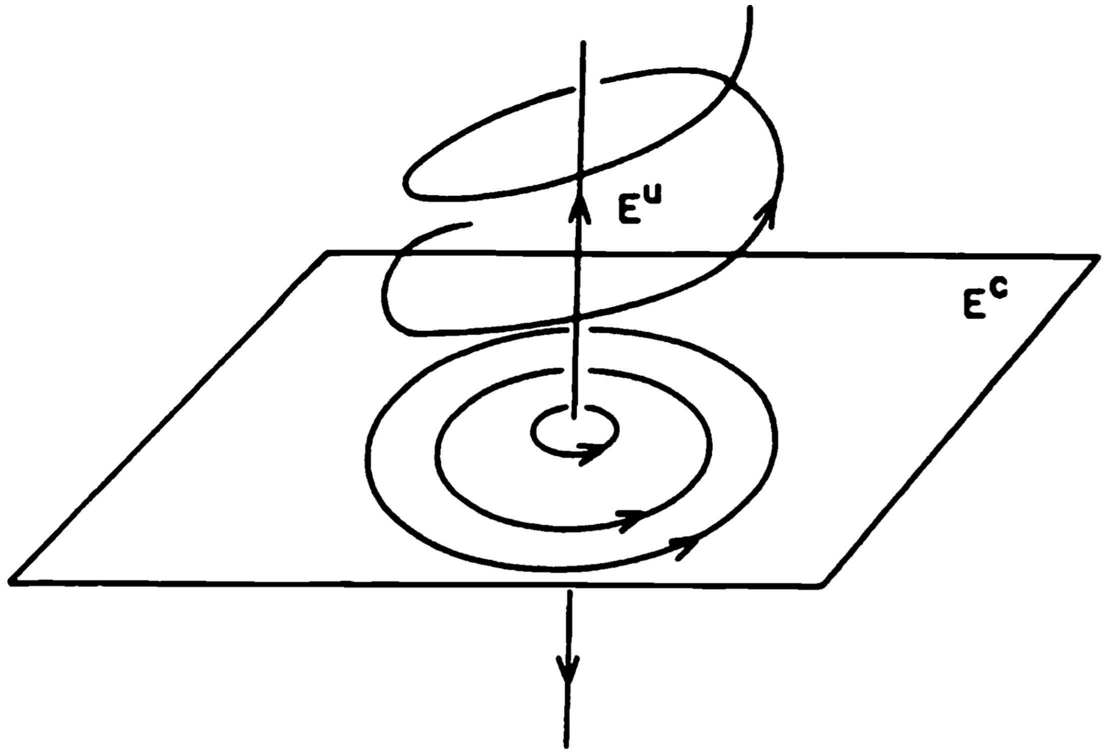


Figure 25: The center subspace E^c is the (x_1, x_2)-plane, and the unstable subspace E^u is the x_3 axis.

Example 4.48. *Consider system (4.60), where*

$$A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}, \tag{4.71}$$

and $\lambda \in \mathbb{R}$. *In Example 4.41, we discussed that the vectors*

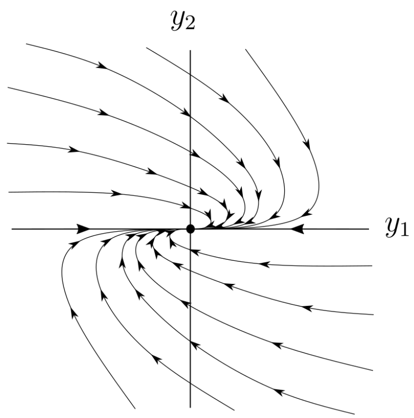
$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.72}$$

are the generalized eigenvectors associated with the eigenvalue λ. *On the other hand, we have*

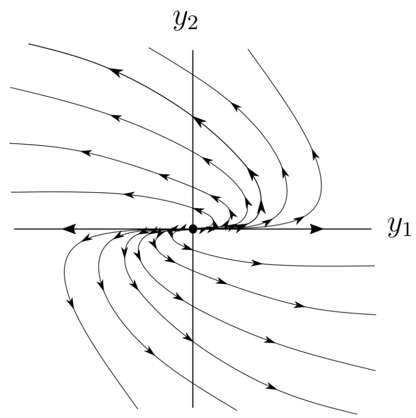
$$\mathbb{R}^2 = \left\langle \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\rangle. \tag{4.73}$$

Therefore,

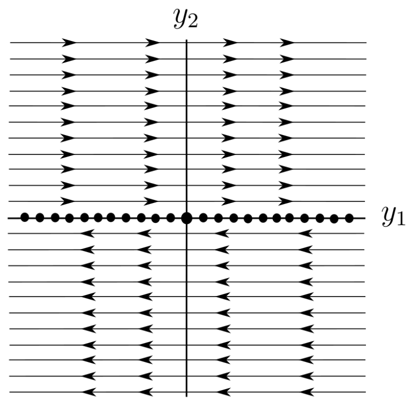
1. *if* $\lambda < 0$, *then* $E^s = \mathbb{R}^2$, $E^c = \{0\}$ *and* $E^u = \{0\}$,
2. *if* $\lambda = 0$, *then* $E^s = \{0\}$, $E^c = \mathbb{R}^2$ *and* $E^u = \{0\}$,
3. *if* $\lambda > 0$, *then* $E^s = \{0\}$, $E^c = \{0\}$ *and* $E^u = \mathbb{R}^2$.



(a) Case $\lambda < 0$: $E^s = \mathbb{R}^2$, $E^c = \{0\}$ and $E^u = \{0\}$.



(b) Case $\lambda > 0$: $E^s = \{0\}$, $E^c = \{0\}$ and $E^u = \mathbb{R}^2$.



(c) Case $\lambda = 0$: $E^s = \{0\}$, $E^c = \mathbb{R}^2$ and $E^u = \{0\}$

Figure 26: Stable, unstable and center spaces of the system $\dot{x} = Ax$, where A is given by (4.71).

THEOREM 4.49. *Consider a system*

$$\dot{x} = Ax, \tag{4.74}$$

where A is a real $n \times n$ matrix. Let E^s, E^u and E^c be the stable, unstable and center subspaces of the system. Then each of these three spaces are invariant with respect to the flow of system (4.74). Moreover, we have

$$\mathbb{R}^n = E^s \oplus E^u \oplus E^c. \tag{4.75}$$

Proof. See [Per01]. □

PROPOSITION 4.50. [11] *Let O be the origin of $\mathbb{R}^n$. Consider a point $x_0 \in \mathbb{R}^n$, and let E^s, E^u and E^c be the stable, unstable and center subspaces of system (4.74), respectively. Then, the following hold.*

(i) If $x_0 \in E^s$, then $e^{tA}x_0 \to O$ as $t \to \infty$.

(i) If $x_0 \in E^u$, then $e^{tA}x_0 \to O$ as $t \to -\infty$.

(i) If $e^{tA}x_0 \to O$ as $t \to \infty$, then $x_0 \in E^s \oplus E^c$.

(i) If $e^{tA}x_0 \to O$ as $t \to -\infty$, then $x_0 \in E^u \oplus E^c$.

Proof. See [Per01]. □

[11] A more comprehensive version of this result is valid: E^s (resp. E^u) is indeed the set of all points x_0 in $\mathbb{R}^n$ that converge to O exponentially fast as $t \to \infty$ (resp. $t \to -\infty$), i.e. $\exists\, a > 0$, $M > 0$ such that $\|e^{tA}x_0\| \leq Me^{-a|t|}$ for $t \geq 0$ (resp. $t \leq 0$). On the other hand, E^c is the set of the points $x_0 \in \mathbb{R}^n$ whose orbits grow at most sub-exponentially fast as $t \to \pm\infty$, i.e. $\forall a > 0$, $\frac{\|e^{tA}x_0\|}{e^{a|t|}} \to 0$ as $t \to \pm\infty$ (see [Rob98], Theorem 6.1).

Definition 4.51. *Let O be the origin of $\mathbb{R}^n$ and consider system (4.74).*

(i) We say the equilibrium O is hyperbolic if A has no eigenvalue with zero real part, i.e. $E^c = \{O\}$, or equivalently $\mathbb{R}^n = E^s \oplus E^u$. Otherwise, we say O is nonhyperbolic, i.e. A has at least one eigenvalue with zero real part.

(ii) We say the equilibrium O is a sink (resp. source) if all the eigenvalues of A have negative (resp. positive) real parts, i.e. $E^s = \mathbb{R}^n$ (resp. $E^u = \mathbb{R}^n$).

(iii) We say the equilibrium O is a saddle if it is hyperbolic, and the matrix A has at least one eigenvalue with negative real part and at least one eigenvalue with positive real part, i.e. $\mathbb{R}^n = E^s \oplus E^u$, $\dim(E^s) \geq 1$ and $\dim(E^u) \geq 1$.

4.6. **An example of synchronization in linear systems**

Recall from the first lecture that a mathematical model for understanding synchronization in networks is given by

$$\dot{x}_i = f_i\left(x_i\right) + \alpha \sum_{j=1}^{N} A_{ij} H_i\left(x_j - x_i\right), \qquad \forall i \in \{1, \ldots, N\}, \tag{4.76}$$

where $x_i \in \mathbb{R}^n$ $(n \geq 1)$, $A = (A_{ij})$ is the adjacency matrix of the network, and $f_i, H_i \in \mathcal{C}^2\left(\mathbb{R}^n\right)$.

We now discuss a simple case of this model: two identical linear systems defined on $\mathbb{R}$ that are linearly coupled together. First, consider two identical linear systems

$$\dot{x}_1 = ax_1 \tag{4.77}$$

and

$$\dot{x}_2 = ax_2, \tag{4.78}$$

where $x_i \in \mathbb{R}$ $(i = 1, 2)$ and a is a non-zero real constant. The dynamics of these systems are simple: for any initial point $x_i(0)$, we have $x_i\left(t\right) = e^{at}x_i(0)$, $i = 1, 2$. Thus, $x_i\left(t\right) \to 0$ if $a < 0$, and $\left|x_i\left(t\right)\right| \to \infty$ if $a > 0$.

(a) Case $a < 0$.

(b) Case $a > 0$.

Figure 27: Phase portrait of $\dot{x}_i = ax_i$, where $i = 1, 2$.

We can couple these two systems as in (4.76). We have

$$\begin{aligned}\dot{x}_1 &= ax_1 + \alpha\left(x_2 - x_1\right)\\ \dot{x}_2 &= ax_2 + \alpha\left(x_1 - x_2\right)\end{aligned} \tag{4.79}$$

where α, the coupling strength, is a real constant. In terms of the notations in (4.76), we are considering $n = 1$, $N = 2$, $f_1(x) = f_2(x) = ax$, $A = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $H_i =$ identity.

We say system (4.79) gets into (complete) synchrony if for any initial condition $\left(x_1\left(0\right), x_2\left(0\right)\right) \in \mathbb{R}^2$, we have

$$\lim_{t\to\infty} \left|x_1\left(t\right) - x_2\left(t\right)\right| = 0. \tag{4.80}$$

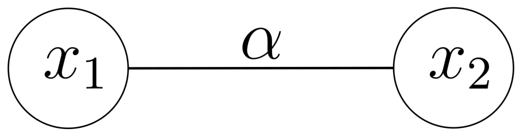


Figure 28: Two coupled systems.

- Define $z(t) := x_1(t) - x_2(t)$. To detect the synchrony in system (4.79), we need to see when $z(t) \to 0$ as $t \to \infty$.
- Recall that

$$\begin{aligned} \dot{x}_1 &= (a-\alpha)\, x_1 + \alpha x_2 \\ \dot{x}_2 &= \alpha x_1 + (a-\alpha)\, x_2 \end{aligned} \tag{4.81}$$

- We have

$$\dot{z} = \dot{x}_1 - \dot{x}_2 = (a - 2\alpha)\, z. \tag{4.82}$$

  This yields $z\,(t) = e^{(a-2\alpha)t} z\,(0)$. Thus, $\lim_{t\to\infty} z(t) = 0$ for arbitrary $z(0)$ if and only if $a - 2\alpha < 0$.

- Define $\alpha_c := \frac{a}{2}$. This constant is called the critical coupling value. We have that system (4.79) gets into (complete) synchrony if and only if $\alpha > \alpha_c$.

5. Nonlinear systems

5.1. Stability of equilibria

Consider a system

$$\dot{x} = f(x), \tag{5.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Assume that $x^* \in \mathbb{R}^n$ is an equilibrium of this system, i.e. $f(x^*) = 0$. Let $r > 0$. An open ball in $\mathbb{R}^n$ with radius r, centered at x^* is defined by

$$B_r(x^*) = \{x \in \mathbb{R}^n : \|x - x^*\| < r\}. \tag{5.2}$$

For arbitrary $x_0 \in \mathbb{R}^n$, let $\phi_t(x_0) = \phi(t, x_0)$ be the solution of (5.1) with the initial condition $\phi(0, x_0) = x_0$.

Definition 5.1. *The equilibrium x^* is called stable if for every $\epsilon > 0$, there exists $\delta > 0$ such that for all $x_0 \in B_\delta(x^*)$ and for all $t \geq 0$, we have $\phi_t(x_0) \in B_\epsilon(x^*)$.*

Definition 5.2. *The equilibrium x^* is called unstable if it is not stable.*

Definition 5.3. *The equilibrium x^* is called asymptotically stable if it is stable, and there exists a $\delta > 0$ such that for all $x_0 \in B_\delta(x^*)$, we have $\lim_{t\to\infty} \phi_t(x_0) = x^*$.*

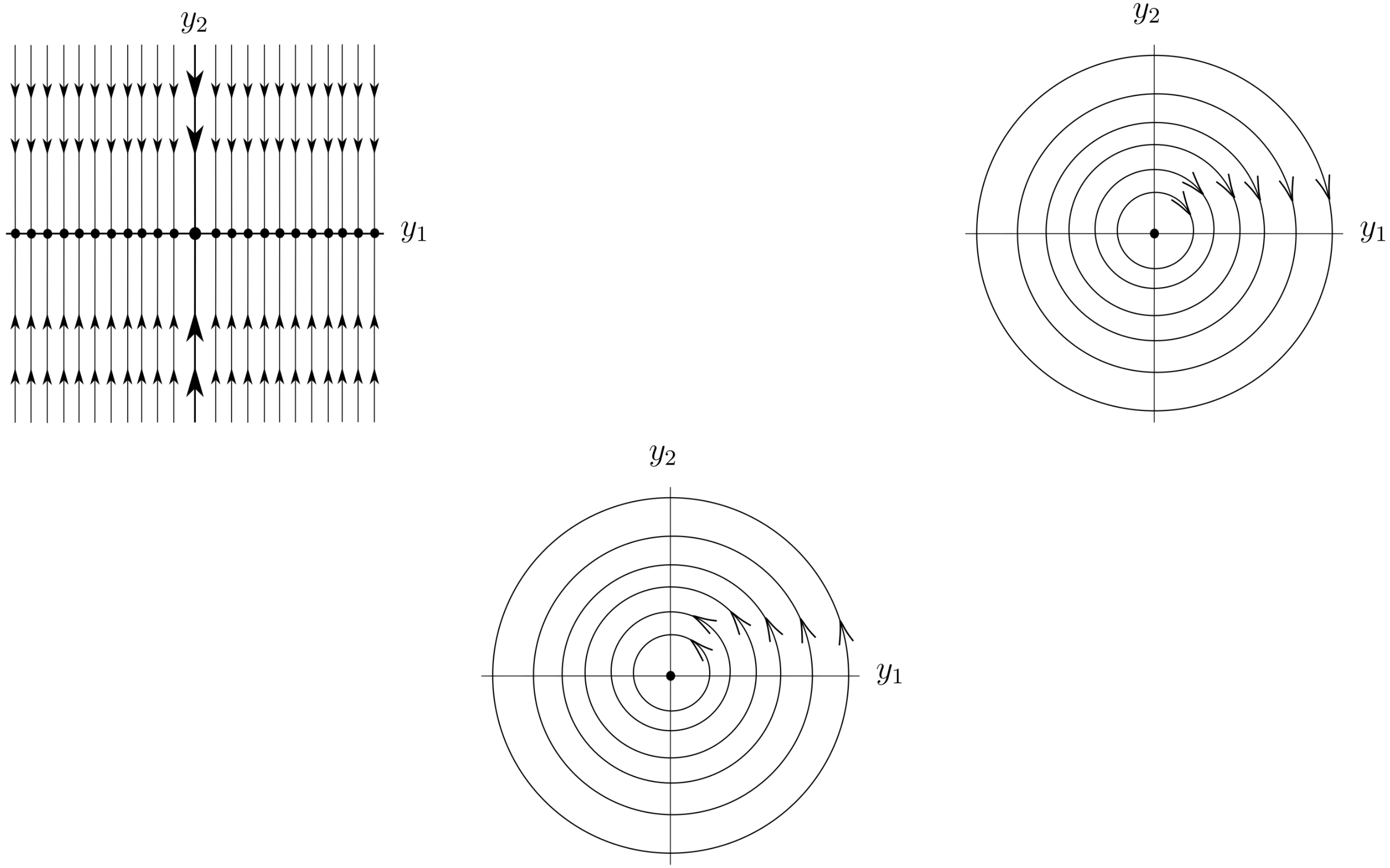


Figure 29: The origin is stable but not asymptotically stable.

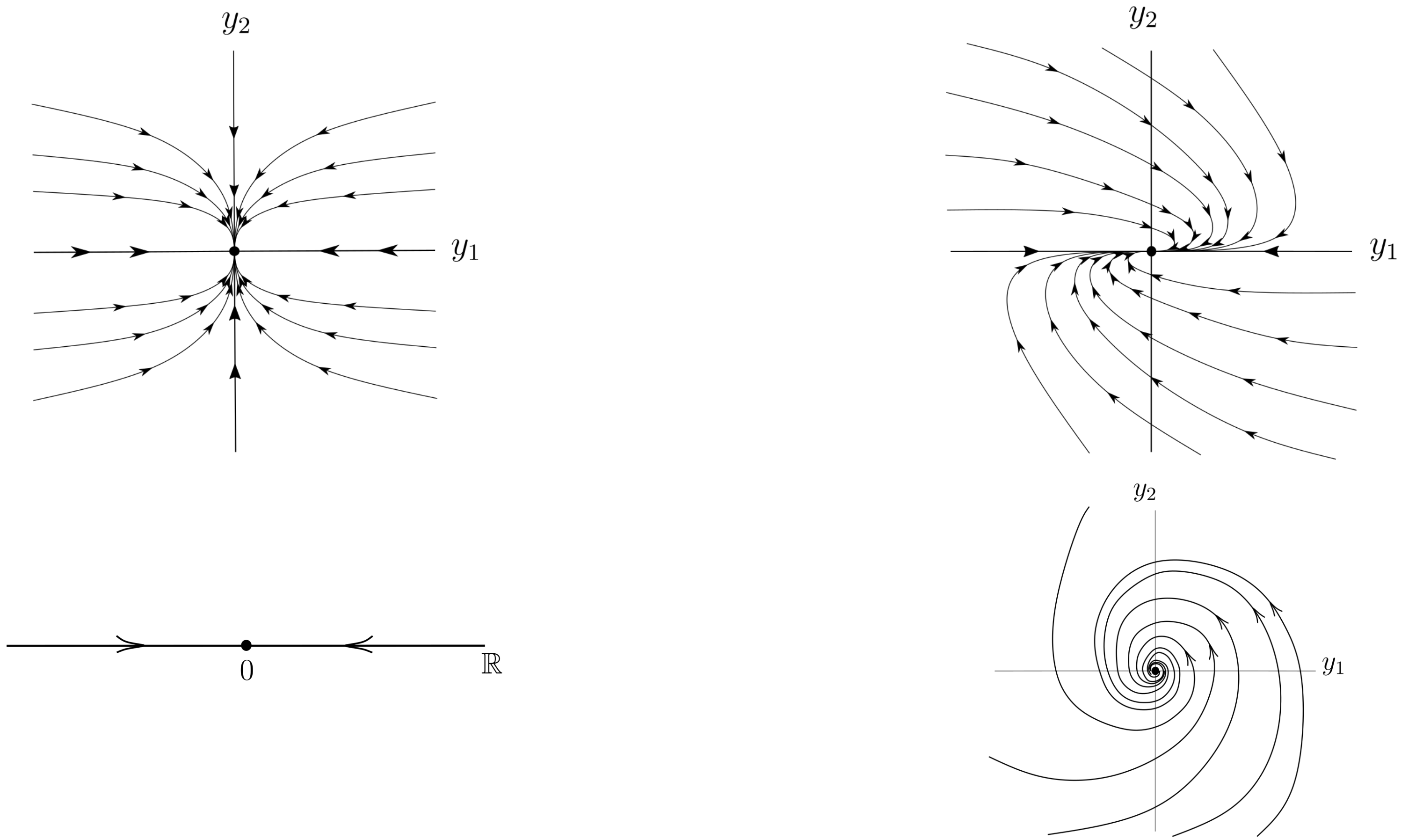


Figure 30: The origin is asymptotically stable.

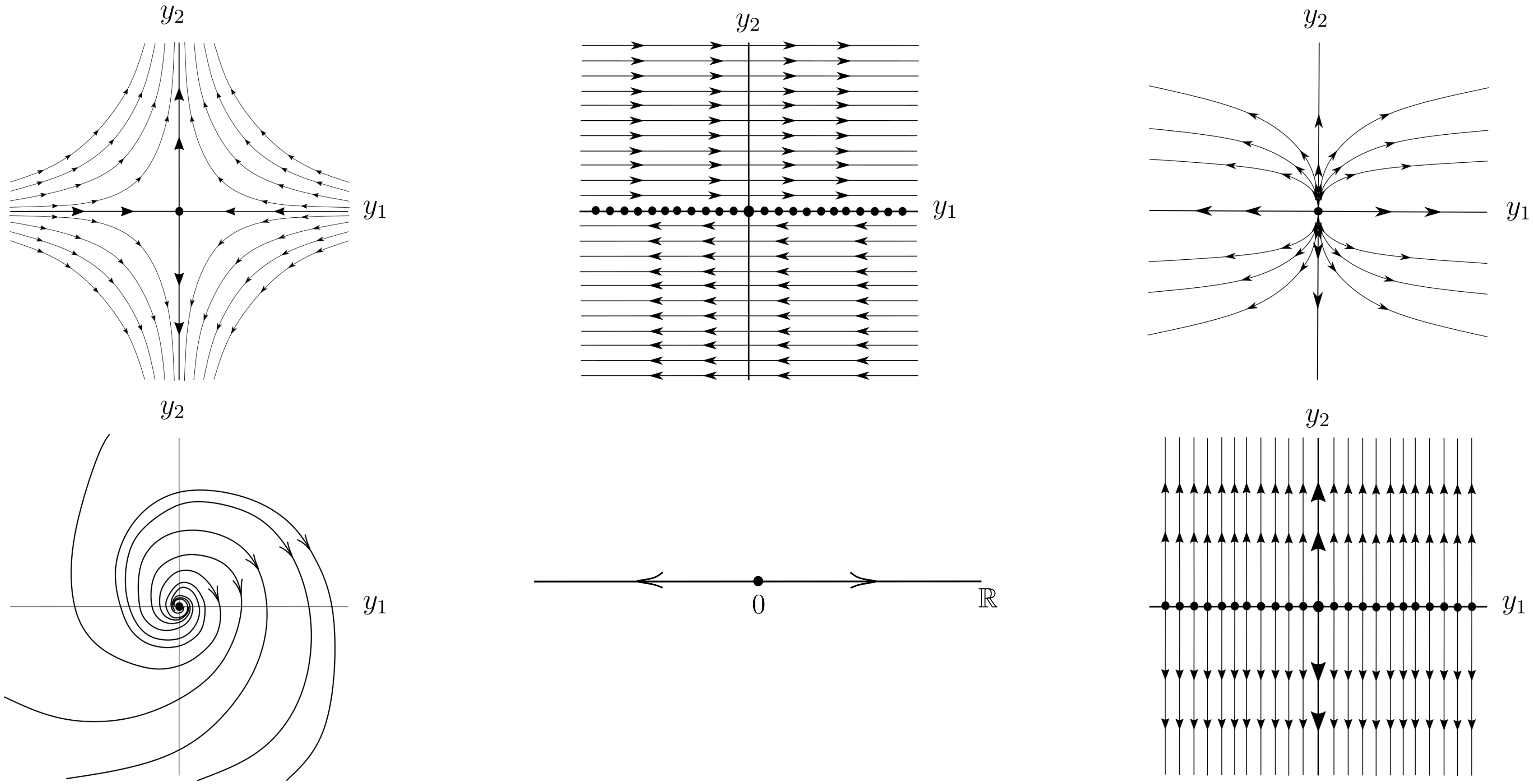


Figure 31: The origin is an unstable equilibrium.

Consider a system

$$\dot{x} = f(x), \tag{5.3}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Assume that the origin of $\in \mathbb{R}^n$ is an equilibrium of this system[12], i.e. $f(0) = 0$.

- **Question**: what is the stability of the equilibrium point at the origin? Stable? Asymptotically stable? Or unstable?

- **Answer to the question for the linear case:**
  Consider the case that f is linear, i.e. $f(x) = Ax$, for some $A \in \mathbb{R}^{n\times n}$. Indeed,

$$\dot{x} = Ax. \tag{5.4}$$

PROPOSITION 5.4. *Consider the decomposition $\mathbb{R}^n = E^s \oplus E^u \oplus E^c$ for system (5.4). Then*

(i) If the origin is stable, then $E^u = \{0\}$. In other words, if $E^u \neq \{0\}$, then the origin is unstable.

(ii) The origin is asymptotically stable if and only if $E^s = \mathbb{R}^n$, i.e. $E^u = E^c = \{0\}$.

Proof. See [Per01], Theorems 2 and 3 of Section 1.9 with their proofs. □

Remark 5.5. *In the case that $E^u = \{0\}$ and $E^c \neq \{0\}$, further investigation is needed to determine the stability of the equilibrium state at the origin (see e.g. [Per01], Problem 5 of the problem set of Section 1.9).*

[12] We do not lose generality by assuming that the equilibrium is located at the origin (see Remark 3.18).

5.2. Linearization

5.2.1 Preliminaries: Taylor's Theorem

Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^1$-smooth function. We write

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1\left(x_1, x_2, \ldots, x_n\right) \\ \vdots \\ f_n\left(x_1, x_2, \ldots, x_n\right) \end{pmatrix}, \tag{5.5}$$

where $f_i : \mathbb{R}^n \to \mathbb{R}$ $(i = 1, \ldots, n)$ are $\mathcal{C}^1$.

Theorem 5.6. *(Taylor's Theorem) Assume f is k-times continuously differentiable at the origin. Let α_i, $i = 1, \ldots, n$ be non-negative integers. Define $|\alpha| = \alpha_1 + \cdots + \alpha_n$, $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_n!$, and $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Then, for $|\alpha| = k$ and $i = 1, \ldots, n$, there exist functions $h_{i,\alpha}\left(x\right)$, such that $\lim_{x\to 0} h_{i,\alpha}\left(x\right) = 0$, and*

$$f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f_1\left(0\right) x^\alpha + \sum_{|\alpha|=k} h_{1,\alpha}\left(x\right) x^\alpha \\ \vdots \\ \sum_{|\alpha|\leq k} \frac{1}{\alpha!} D^\alpha f_n\left(0\right) x^\alpha + \sum_{|\alpha|=k} h_{n,\alpha}\left(x\right) x^\alpha \end{pmatrix}. \tag{5.6}$$

Remark 5.7. *Roughly speaking, Taylor's Theorem states that, close to the origin, we can write $f\left(x\right)$ as $P\left(x\right) +'$ remainder$'$, where P is a (non-zero) polynomial and the remainder is a function of x which often! can be neglected since, in comparison to $P(x)$, it is small.*

Example 5.8. *(Taylor's Theorem for $n = 1$) Suppose $f : \mathbb{R} \to \mathbb{R}$ is k-times continuously differentiable. Then, we can write $f(x) = P(x) + R(x)$, for*

$$P(x) = f(0) + \frac{1}{1!} f'(0)\, x + \frac{1}{2!} f''(0)\, x^2 + \frac{1}{3!} f'''(0)\, x^3 + \cdots + \frac{1}{k!} f^{(k)}(0)\, x^k, \tag{5.7}$$

and

$$R(x) = h(x)\, x^k \tag{5.8}$$

where $f^{(k)}$ stands for the k-th derivative, and $\lim_{x\to 0} h(x) = 0$.

Example 5.9. *Consider the function* $f(x) = \sin x$. *This function is $\mathcal{C}^\infty$ (we can differentiate it as many times as we want). Write* $\sin x = P(x) + R(x)$, *where P is a polynomial of degree k, and R satisfies* $\lim_{x\to 0} \frac{R(x)}{x^k} = 0$. *The following are some examples of P and R.*

(i) $P(x) = x$, *and* $R(x) = h(x)\, x$ *such that* $\lim_{x\to 0} h(x) = 0$.

(ii) $P(x) = x - \frac{x^3}{3!}$, *and* $R(x) = h(x)\, x^3$ *such that* $\lim_{x\to 0} h(x) = 0$.

(iii) $P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$, *and* $R(x) = h(x)\, x^{11}$ *such that* $\lim_{x\to 0} h(x) = 0$.

(iv) $P(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \frac{x^{11}}{11!}$, *and* $R(x) = h(x)\, x^{12}$ *such that* $\lim_{x\to 0} h(x) = 0$.

Example 5.10. *(Taylor's Theorem for $n = 2$) Suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$, given by $x \mapsto (f_1(x), f_2(x))$, where $x = (x_1, x_2)$, is 3-times continuously differentiable. We can write*

$$\begin{aligned} f_1(x_1, x_2) &= f_1(0,0) \\ &+ \frac{1}{1!} \cdot \frac{\partial f_1}{\partial x_1}(0,0)\, x_1 + \frac{1}{1!} \cdot \frac{\partial f_1}{\partial x_2}(0,0)\, x_2 \\ &+ \frac{1}{2!} \cdot \frac{\partial f_1}{\partial x_1^2}(0,0)\, x_1^2 + \frac{1}{1!1!} \cdot \frac{\partial f_1}{\partial x_1 \partial x_2}(0,0)\, x_1 x_2 + \frac{1}{2!} \cdot \frac{\partial f_1}{\partial x_2^2}(0,0)\, x_2^2 \\ &+ \frac{1}{3!} \cdot \frac{\partial f_1}{\partial x_1^3}(0,0)\, x_1^3 + \frac{1}{2!1!} \cdot \frac{\partial f_1}{\partial x_1^2 \partial x_2}(0,0)\, x_1^2 x_2 + \frac{1}{1!2!} \cdot \frac{\partial f_1}{\partial x_1 \partial x_2^2}(0,0)\, x_1 x_2^2 + \frac{1}{3!} \cdot \frac{\partial f_1}{\partial x_2^3}(0,0)\, x_2^3 \\ &+ h_{1,30}(x)\, x_1^3 + h_{1,21}(x)\, x_1^2 x_2 + h_{1,12}(x)\, x_1 x_2^2 + h_{1,03}(x)\, x_2^3 \end{aligned} \tag{5.9}$$

such that all the functions h converge to 0 as $x \to 0$. Regarding f_2, we have

$$\begin{aligned} f_2(x_1, x_2) &= f_2(0,0) \\ &+ \frac{1}{1!} \cdot \frac{\partial f_2}{\partial x_1}(0,0)\, x_1 + \frac{1}{1!} \cdot \frac{\partial f_2}{\partial x_2}(0,0)\, x_2 \\ &+ \frac{1}{2!} \cdot \frac{\partial f_2}{\partial x_1^2}(0,0)\, x_1^2 + \frac{1}{1!1!} \cdot \frac{\partial f_2}{\partial x_1 \partial x_2}(0,0)\, x_1 x_2 + \frac{1}{2!} \cdot \frac{\partial f_2}{\partial x_2^2}(0,0)\, x_2^2 \\ &+ \frac{1}{3!} \cdot \frac{\partial f_2}{\partial x_1^3}(0,0)\, x_1^3 + \frac{1}{2!1!} \cdot \frac{\partial f_2}{\partial x_1^2 \partial x_2}(0,0)\, x_1^2 x_2 + \frac{1}{1!2!} \cdot \frac{\partial f_2}{\partial x_1 \partial x_2^2}(0,0)\, x_1 x_2^2 + \frac{1}{3!} \cdot \frac{\partial f_2}{\partial x_2^3}(0,0)\, x_2^3 \\ &+ h_{2,30}(x)\, x_1^3 + h_{2,21}(x)\, x_1^2 x_2 + h_{2,12}(x)\, x_1 x_2^2 + h_{2,03}(x)\, x_2^3 \end{aligned} \tag{5.10}$$

such that all the functions h converge to 0 as $x \to 0$.

5.2.2 Linearization at equilibria: stable and unstable invariant manifolds theorem

Consider a system

$$\dot{x} = f(x), \tag{5.11}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Think of f as $f\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} f_1(x_1,x_2,\ldots,x_n) \\ \vdots \\ f_n(x_1,x_2,\ldots,x_n) \end{pmatrix}$. Assume that the origin is an equilibrium of this system, i.e. $f(0) = 0$. According to Taylor's theorem, we can write system (5.11) as

$$\dot{x} = Df(0)\,x + F(x), \tag{5.12}$$

where $F(x) = h(x)\,x$, for some h satisfying $\lim_{x\to 0} h(x) = 0$, and $Df(0)$ (called the derivative or differential of f at 0) is given by the $n \times n$ matrix

$$Df(0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(0) & \frac{\partial f_1}{\partial x_2}(0) & \cdots & \frac{\partial f_1}{\partial x_n}(0) \\ \frac{\partial f_2}{\partial x_1}(0) & \frac{\partial f_2}{\partial x_2}(0) & \cdots & \frac{\partial f_2}{\partial x_n}(0) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1}(0) & \frac{\partial f_n}{\partial x_2}(0) & \cdots & \frac{\partial f_n}{\partial x_n}(0) \end{pmatrix}. \tag{5.13}$$

Definition 5.11. *Consider system (5.11). The expressions $Df(0)\,x$ and $F(x)$ are called the linear and nonlinear parts of the system at the origin, respectively. Moreover, we call the linear system*

$$\dot{x} = Df(0)\,x, \tag{5.14}$$

the linearized system at the origin.

Definition 5.12. *Consider system (5.11) with its equilibrium state at the origin. We say that the origin is a hyperbolic equilibrium if it is a hyperbolic equilibrium for the linearized system (see Definition 4.51), i.e. all the eigenvalues of* $Df(0)$ *have non-zero real parts. An equilibrium which is not hyperbolic is called nonhyperbolic.*

Remark 5.13. *We can think of the linearized system as an approximation of system (5.11) near the origin. A natural question that arises here is that to what extent the stability of the linearized system gives information about the stability of the original (nonlinear) system.*

Theorem 5.14 (stable and unstable invariant manifolds theorem). *Consider a system* $\dot{x} = f(x)$, *where* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is* $\mathcal{C}^1$*-smooth. Assume that the origin* O *of* $\mathbb{R}^n$ *is a hyperbolic equilibrium. Let* E^s *and* E^u *be the stable and unstable subspaces of the linearized system at* O*, respectively, and suppose* $k := \dim(E^s)$*. Consider an arbitrary point* $x_0 \in \mathbb{R}^n$ *and let* $x(t)$*, where* $x(0) = x_0$*, be its orbit. Then, there exist manifolds* $W^s(O)$ *and* $W^u(O)$*, both containing the origin, such that*

(i) $x(t) \to O$ *as* $t \to \infty$ *if and only if* $x_0 \in W^s(O)$*.*

(ii) $x(t) \to O$ *as* $t \to -\infty$ *if and only if* $x_0 \in W^u(O)$*.*

(iii) $W^s(O)$ *and* $W^u(O)$ *are invariant with respect to the flow of the system and they are tangent to* E^s *and* E^u *at* O*, respectively.*

Proof. See [Per01]. □

Definition 5.15. *Consider a $\mathcal{C}^1$ system $\dot{x} = f(x)$ on $\mathbb{R}^n$ with an equilibrium at the origin O. We say O is a sink, source or saddle if it is a sink, source or saddle for the linearized system $\dot{x} = Df(0)x$, respectively.*

- *Recall that O is a sink (resp. source) of $\dot{x} = Df(O)x$ if all the eigenvalues of $Df(O)$ have negative (resp. positive) real parts. It is a saddle if it is hyperbolic, and $Df(O)$ has at least one eigenvalue with negative real part and at least one eigenvalue with positive real part.*

Remark 5.16. *By definition, sinks, sources and saddles are hyperbolic.*

Remark 5.17. *It follows from the stable and unstable invariant manifolds theorem that sinks are asymptotically stable, sources and saddles are unstable.*

Example 5.18. *Consider the nonlinear system*

$$\begin{aligned} \dot{x}_1 &= -x_1 - x_2^2, \\ \dot{x}_2 &= x_2 + x_1^2. \end{aligned} \tag{5.15}$$

The origin is an equilibrium of this system. The linearized system at the origin is given by

$$\begin{aligned} \dot{x}_1 &= -x_1, \\ \dot{x}_2 &= x_2. \end{aligned} \tag{5.16}$$

The corresponding stable subspace E^s and unstable subspace E^u are the horizontal and vertical axes, respectively. Figure 32 shows the stable and unstable manifolds of the origin of system (5.15) (see [Per01], Example 2 on page 111 for further details). The stable and unstable manifolds are one-dimensional and tangent to E^s and E^u, respectively.

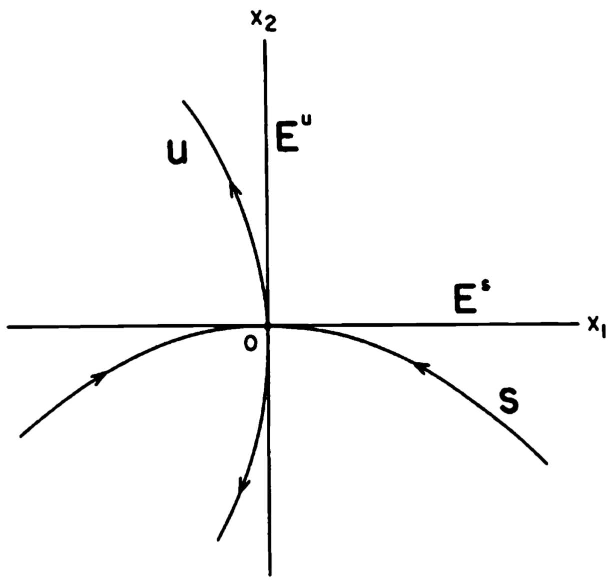


Figure 32: The stable and unstable invariant manifolds of the origin of system (5.15).

Example 5.19. *Consider the nonlinear system*

$$\begin{aligned}\dot{x}_1 &= -x_1,\\ \dot{x}_2 &= -x_2 + x_1^2,\\ \dot{x}_3 &= x_3 + x_1^2,\end{aligned} \tag{5.17}$$

The origin is the only equilibrium of this system. The linearization of this system at the origin is

$$\begin{pmatrix}-1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{pmatrix}. \tag{5.18}$$

The eigenvalues are -1 *and* 1. *The corresponding stable subspace* E^s *and unstable subspace* E^u *are the* (x_1, x_2)*-plane and* x_3 *axis, respectively. Figure 33 shows the stable and unstable manifolds of the origin of system (5.17) (see [Per01], Example 1 on page 105 for further details). The stable manifold is 2-dimensional and tangent to* E^s *at the origin. The unstable manifold is 1-dimensional and tangent to* E^u *at the origin.*

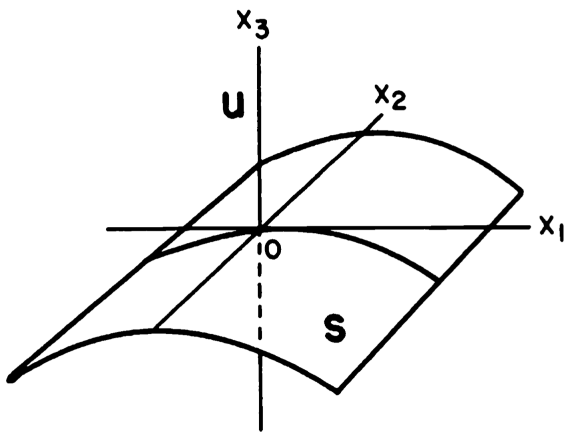


Figure 33: The stable and unstable invariant manifolds of the origin of system (5.17).

Example 5.20. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1 &= -3x_1 - 2x_2 - x_2^4 + x_1x_2,\\ \dot{x}_2 &= 2x_1 - 3x_2 + x_1^2 e^{x_1}.\end{aligned} \tag{5.19}$$

The origin is an equilibrium of this system. The linearization of this system at the origin is

$$\begin{pmatrix} -3 & -2 \\ 2 & -3 \end{pmatrix}. \tag{5.20}$$

The eigenvalues are $-3 \pm 2i$ *that have negative real parts. The stable manifold is a neighborhood of the origin (an open ball around the origin). The unstable manifold is the origin itself, i.e.* $W^u(0) = \{0\}$.

Example 5.21. Consider a nonlinear system $\dot{x} = f(x)$, *where* $f : \mathbb{R} \to \mathbb{R}$ *is smooth. Assume* 0 *is an equilibrium, i.e.* $f(0) = 0$. *We have*

(i) If $f'(0) < 0$, *then* $W^s(0)$ *is an open interval containing* 0, *and* $W^u(0) = \{0\}$.

(ii) If $f'(0) > 0$, *then* $W^s(0) = \{0\}$, *and* $W^u(0)$ *is an open interval containing* 0.

5.3. **Lyapunov functions**

- The method of linearization can be used to determine the stability of hyperbolic equilibria.
- We're going to introduce a method which can be used to investigate the stability of nonhyperbolic equilibria.

Definition 5.22. *Consider a system $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$-smooth, and assume that x^* is an equilibrium. Let U be an open subset of $\mathbb{R}^n$ containing x^*. A $\mathcal{C}^1$-smooth function $V : U \to \mathbb{R}$ is called a Lyapunov function if*

(i) $V(x^*) = 0$.

(ii) for all $x \in U \setminus \{x^*\}$, we have $V(x) > 0$.

(iii) for any arbitrary solution $x(t)$ of $\dot{x} = f(x)$, we have that $V(x(t))$ is a non-increasing function of t, i.e. if $t_1 < t_2$, then $V(x(t_2)) \leq V(x(t_1))$.

Remark 5.23. *An equivalent formulation of item (iii) of the definition of Lyapunov function is $\dot{V} \leq 0$, where*

$$\dot{V} := \frac{dV(x(t))}{dt}\bigg|_{t=0} = DV\bigg|_{x(0)} \frac{dx(t)}{dt}\bigg|_{t=0} = DV\bigg|_{x(0)} f(x(0)). \tag{5.21}$$

Example 5.24. [13] *Consider the system*

$$\begin{aligned} \dot{x}_1 &= -x_2^3, \\ \dot{x}_2 &= x_1^3. \end{aligned} \tag{5.22}$$

The origin is an (and the only) equilibrium of this system. Linearization of this system at the origin is the zero matrix, i.e. $\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. *Thus, the origin is a nonhyperbolic equilibrium point. Define* $V : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$V(x_1, x_2) = x_1^4 + x_2^4. \tag{5.23}$$

We now show that V *is a Lyapunov function.*

(i) V *vanishes at the origin, i.e.* $V(0,0) = 0$.

(ii) Consider an arbitrary point $(x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. *Since* $(x_1, x_2) \neq (0,0)$, *we have* $x_1 \neq 0$ *or* $x_2 \neq 0$. *Therefore,* $V(x_1, x_2) = x_1^4 + x_2^4 > 0$.

(iii) Let $(x_1(t), x_2(t))$ *be an arbitrary solution of system (5.22). We have*

$$\dot{V} = \frac{dV(x_1(t), x_2(t))}{dt} = \frac{d}{dt}\left([x_1(t)]^4 + [x_2(t)]^4\right) = -4[x_1(t)]^3 \cdot [x_2(t)]^3 + 4[x_2(t)]^3 \cdot [x_1(t)]^3 = 0. \tag{5.24}$$

Thus, the condition $\dot{V} \leq 0$ *is satisfied.*

As shown above, the function V *satisfies all the three conditions of Definition 5.22, and therefore, it is a Lyapunov function.*

[13] This example is taken from [Per01]

Example 5.25. [14] *Consider the system*

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= -x_1 - x_1^2 x_2.\end{aligned} \tag{5.25}$$

The origin is an (and the only) equilibrium of this system. Linearization of this system at the origin is $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, *with eigenvalues* $\pm i$. *Thus, the origin is a nonhyperbolic equilibrium point. Define* $V : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$V(x_1, x_2) = x_1^2 + x_2^2. \tag{5.26}$$

We now show that V *is a Lyapunov function.*

(i) V *vanishes at the origin, i.e.* $V(0,0) = 0$.

(ii) *Consider an arbitrary point* $(x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$. *Since* $(x_1, x_2) \neq (0,0)$, *we have* $x_1 \neq 0$ *or* $x_2 \neq 0$. *Therefore,* $V(x_1, x_2) = x_1^2 + x_2^2 > 0$.

(iii) *Let* $(x_1(t), x_2(t))$ *be an arbitrary solution of system (5.22). We have*

$$\dot{V} = \frac{\partial V}{\partial x_1} \cdot \dot{x}_1 + \frac{\partial V}{\partial x_2} \cdot \dot{x}_2 = 2x_1(x_2) + 2x_2\left(-x_1 - x_1^2 x_2\right) = -2x_1^2 x_2^2 \tag{5.27}$$

Thus, the condition $\dot{V} \leq 0$ *is satisfied.*

As shown above, the function V *satisfies all the three conditions of Definition 5.22, and therefore, it is a Lyapunov function.*

[14] This example is taken from [Wig03]

Example 5.26. [15] *Consider the system*

$$\begin{aligned}\dot{x} &= 2y\left(z-1\right),\\ \dot{y} &= -x\left(z-1\right),\\ \dot{z} &= xy.\end{aligned} \tag{5.28}$$

The origin is an equilibrium and the linearization of the system at the origin is $\begin{pmatrix} 0 & -2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. *The eigenvalues of this matrix are* 0 *and* $\pm\sqrt{2}i$. *This means that the origin is a nonhyperbolic equilibrium.*

Consider $V\left(x,y,z\right) = x^2 + 2y^2$. *Then,* $\dot{V} = 2x\left[2y\left(z-1\right)\right] + 4y\left[-x\left(z-1\right)\right] = 0$. *Moreover,* $V\left(0,0,0\right) = 0$. *However,* V *is not a Lyapunov function. This is because* V *vanishes at all the points on* z*-axis, i.e.* $V\left(0,0,z\right) = 0$ *for all* $z \in \mathbb{R}$.

[15] This example is taken from [VS18]

THEOREM 5.27. *Consider a system $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is $\mathcal{C}^1$-smooth, and assume that x^* is an equilibrium. Let U be an open*[16] *subset of $\mathbb{R}^n$ containing x^*. Let $V : U \to \mathbb{R}$ be a Lyapunov function. We have*

(i) If $\dot{V} \leq 0$, then x^* is stable.

(ii) If $\dot{V} < 0$ for all $x \in U \setminus \{x^*\}$, then x^* is asymptotically stable.

[16] A set $U \subset \mathbb{R}^n$ is open in $\mathbb{R}^n$ if for every point $x \in U$, there exists an open ball $B_r(x)$, for $r > 0$ (see (5.2)), such that $B_r(x) \subset U$.

Remark 5.28. *According to Theorem 5.27, we have that the origin is an stable equilibrium for systems (5.22) and (5.25). Note that, in both of these cases, the origin is a nonhyperbolic equilibrium and so the linearization at these equilibria does not directly give information about stability of them.*

Remark 5.29. *Assume that the stability of an equilibrium of a system is guaranteed due to the existence of a Lyapunov function* V *satisfying* $\dot{V} \leq 0$. *Sometimes, it is possible that there exists another Lyapunov function* L *such that* $\dot{L} < 0$ *which implies that the equilibrium point is in fact asymptotically stable.*

Example 5.30. [17] *Consider the system*

$$\begin{aligned}\dot{x}_1 &= -2x_2 + x_2x_3 - x_1^3,\\ \dot{x}_2 &= x_1 - x_1x_3 - x_2^3,\\ \dot{x}_3 &= x_1x_2 - x_3^3.\end{aligned} \tag{5.29}$$

The origin is an equilibrium of this system. Linearization of this system at the origin is $\left(\begin{smallmatrix} 0 & -2 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{smallmatrix}\right)$. *The eigenvalues of this matrix are* 0 *and* $\pm\sqrt{2}i$. *Define* $V : \mathbb{R}^3 \to \mathbb{R}$ *by*

$$V\left(x_1, x_2\right) = x_1^2 + 2x_2^2 + x_3^2. \tag{5.30}$$

We now show that V *is a Lyapunov function.*

(i) V *vanishes at the origin, i.e.* $V(0,0,0) = 0$.

(ii) Consider an arbitrary point $(x_1, x_2, x_3) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$. *Since* $(x_1, x_2, x_3) \neq (0,0,0)$, *we have* $x_1 \neq 0$ *or* $x_2 \neq 0$ *or* $x_3 \neq 0$. *Therefore,* $V(x_1, x_2, x_3) = x_1^2 + 2x_2^2 + x_3^2 > 0$.

(iii) Similar to Example 5.24, we show that $\dot{V} \leq 0$. *We have*

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial x_1}\cdot\dot{x}_1 + \frac{\partial V}{\partial x_2}\cdot\dot{x}_2 + \frac{\partial V}{\partial x_3}\cdot\dot{x}_3 = 2x_1\left(-2x_2 + x_2x_3 - x_1^3\right) + 4x_2\left(x_1 - x_1x_3 - x_2^3\right) + 2x_3\left(x_1x_2 - x_3^3\right)\\ &= -2\left(x_1^4 + 2x_2^4 + x_3^4\right)\end{aligned} \tag{5.31}$$

Thus, $\dot{V} < 0$ *if* $(x_1, x_2, x_3) \neq (0,0,0)$, *and* $\dot{V} = 0$ *if* $(x_1, x_2, x_3) = (0,0,0)$. *Therefore, the condition* $\dot{V} \leq 0$ *holds.*

The function V *satisfies all the three conditions of Definition 5.22, and therefore, it is a Lyapunov function.*

- According to Theorem 5.27, the origin is an asymptotically stable equilibrium for system (5.29).

[17] This example is taken from [Per01]

5.4. **Periodic orbits: stability, limit cycles and Poincaré maps**

Definition 5.31. *Consider a system* $\dot{x} = f(x)$*, where* $f : \mathbb{R}^n \to \mathbb{R}^n$ *is smooth. Let* $\overline{x} \in \mathbb{R}^n$ *and assume it is not an equilibrium point. Then,* $\Gamma = \{\phi(t, \overline{x}),\, t \in \mathbb{R}\}$ *is said to be a periodic orbit or a cycle if there exists* $T > 0$*, such that* $\phi(T, \overline{x}) = \overline{x}$*.*

- Geometrically, a periodic orbit is a closed curve.
- Assume Γ is a periodic orbit and consider $r > 0$. Let $U_r(\Gamma)$ be the set of all points $x \in \mathbb{R}^n$ whose distance from Γ is less than r.

Analogous to equilibria, we can define the concept of stability for periodic orbits too. For arbitrary $x_0 \in \mathbb{R}^n$, let $\phi_t(x_0) = \phi(t, x_0)$ be the solution of the system with the initial condition $\phi(0, x_0) = x_0$.

Definition 5.32. *The cycle* Γ *is called stable if for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x_0 \in U_\delta(\Gamma)$ *and for all* $t \geq 0$, *we have* $\phi_t(x_0) \in U_\epsilon(\Gamma)$.

Definition 5.33. *The cycle* Γ *is called unstable if it is not stable.*

Definition 5.34. *The cycle* Γ *is called asymptotically stable if it is stable, and there exists a* $\delta > 0$ *such that for all* $x_0 \in U_\delta(\Gamma)$, *we have* $\lim_{t\to\infty} \phi_t(x_0) = \Gamma$.

- A standard approach to investigate the dynamics near a periodic orbit is to study an associated Poincaré map.

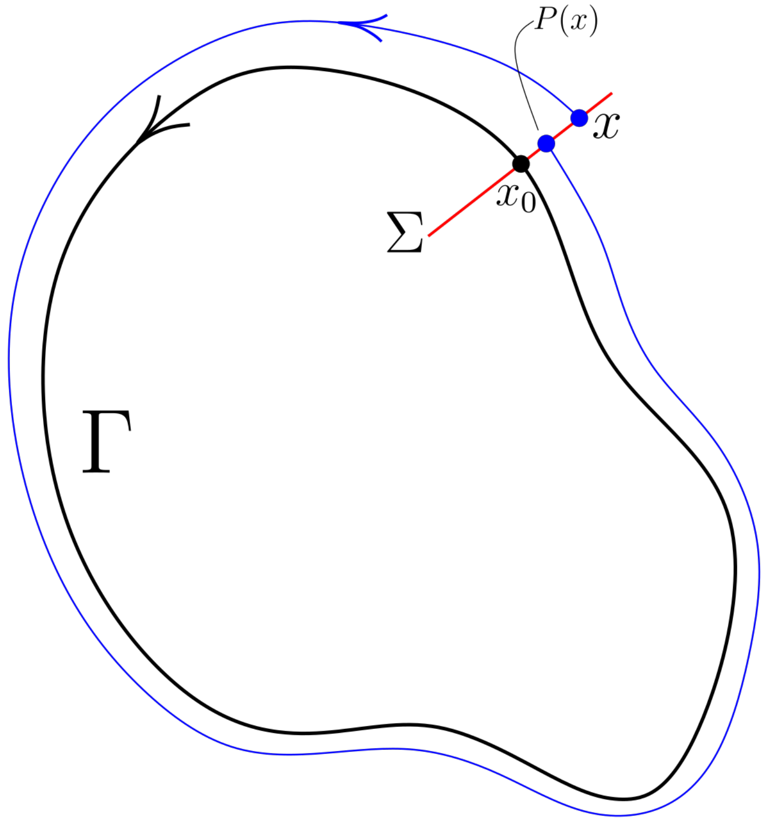


Figure 34: Here, Γ is a periodic orbit in the plane. The Poincaré map P maps $x \in \Sigma$ to $P(x) \in \Sigma$.

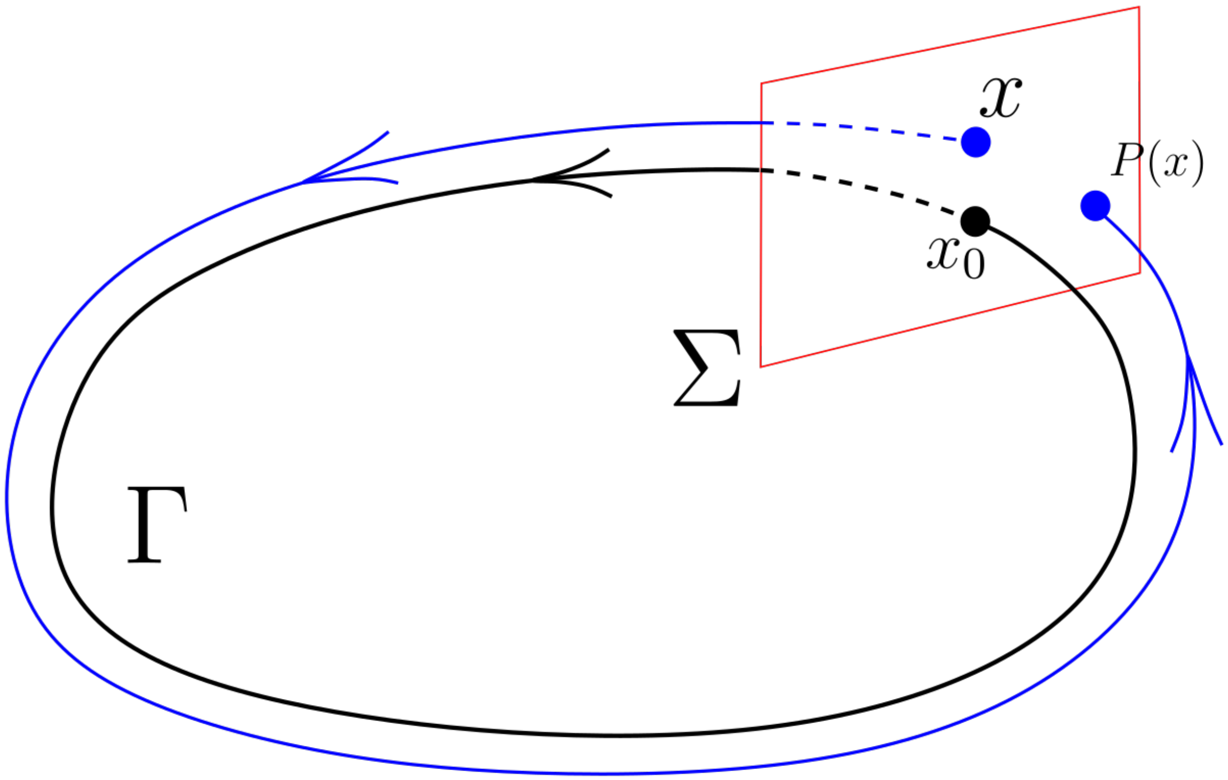


Figure 35: Here, Γ is a periodic orbit in the 3-dimensional space. The cross-section Σ is 2-dimensional. The Poincaré map P maps $x \in \Sigma$ to $P(x) \in \Sigma$.

5.5. **Bifurcations**

Consider a system

$$\dot{x} = f(x, \alpha), \tag{5.32}$$

where $x \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$. The variable x is the phase variable.

- We can think of system (5.32) as a model for a physical problem for which, α is a controlling parameter, such as temperature, pressure, etc.
- We can also think of system (5.32) as a family of systems. For each fixed α, we have a system of ODEs.

Example 5.35. *Consider the system*

$$\dot{x} = x^3 - \alpha x, \tag{5.33}$$

where $x \in \mathbb{R}$, and $\alpha \in \mathbb{R}$. For each fixed α, we have a system of ODEs. For instance,

(i) When $\alpha = 4$, we consider the system $\dot{x} = x^3 - 4x$.

(ii) When $\alpha = 0$, we consider the system $\dot{x} = x^3$.

(iii) When $\alpha = -10$, we consider the system $\dot{x} = x^3 + 10x$.

Example 5.36. *Consider the system*

$$\begin{aligned}\dot{x}_1 &= x_1 - \alpha_1 x_2^3 + \alpha_2 x_1 x_2,\\ \dot{x}_2 &= 4x_2 + \alpha_3 x_1 x_2^3 - \alpha_2 x_1^4,\end{aligned} \tag{5.34}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$*, and* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$*. For each fixed* α*, we have a system of ODEs. For instance,*

(i) When $\alpha = (0, 0, 0)$*, we consider the system*

$$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= 4x_2.\end{aligned} \tag{5.35}$$

(ii) When $\alpha = (1, 2, 0)$*, we consider the system*

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2^3 + 2x_1 x_2,\\ \dot{x}_2 &= 4x_2 - 2x_1^4,\end{aligned} \tag{5.36}$$

(iii) When $\alpha = (-1, 8, -2)$*, we consider the system*

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2^3 + 8x_1 x_2,\\ \dot{x}_2 &= 4x_2 - 2x_1 x_2^3 - 8x_1^4,\end{aligned} \tag{5.37}$$

- In this course, we focus on the case that the parameter space is one dimensional, i.e. $\alpha \in \mathbb{R}$.
- Consider a system
$$\dot{x} = f(x, \alpha), \tag{5.38}$$
where $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. We say a bifurcation occurs at $\alpha = \alpha_c$ if the dynamics of the system changes suddenly at $\alpha = \alpha_c$.
- The parameter value α_c is called the bifurcation value.
- Recall from the first lecture that a mathematical model for understanding synchronization in networks is given by
$$\dot{x}_i = f_i(x_i) + \alpha \sum_{j=1}^{N} A_{ij} H_i(x_j - x_i), \qquad \forall i \in \{1, \ldots, N\}, \tag{5.39}$$
where $x_i \in \mathbb{R}^n$ $(n \geq 1)$, $A = (A_{ij})$ is the adjacency matrix of the network, and $f_i, H_i \in \mathcal{C}^2(\mathbb{R}^n)$. As it was mentioned in the first lecture, there exists a parameter value α_c (and we also discussed for the linear case in Section 4.6), called critical coupling strength, such that for $\alpha < \alpha_c$, system (5.39) is not in synchrony but it gets into synchrony for $\alpha \geq \alpha_c$.

Example 5.37. *Consider the system*

$$\dot{x} = x^2 - \alpha, \qquad x, \alpha \in \mathbb{R}. \tag{5.40}$$

- *For $\alpha < 0$, system (5.40) has no equilibria. The phase portrait of this system is shown in Figure 36.*

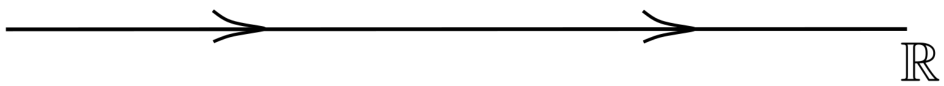


Figure 36: Case $\alpha < 0$.

- *For $\alpha = 0$, system (5.40) has a nonhyperbolic equilibrium at $x = 0$. The phase portrait of this system is shown in Figure 37.*

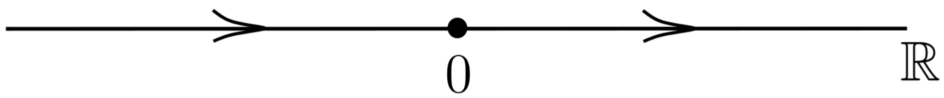


Figure 37: Case $\alpha = 0$.

- *For $\alpha > 0$, system (5.40) has two hyperbolic equilibria at $x = -\sqrt{\alpha}$ (sink) and $x = \sqrt{\alpha}$ (source). The phase portrait of this system is shown in Figure 38.*

Figure 38: Case $\alpha > 0$.

- *A bifurcation occurs in this system at $\alpha = 0$.*

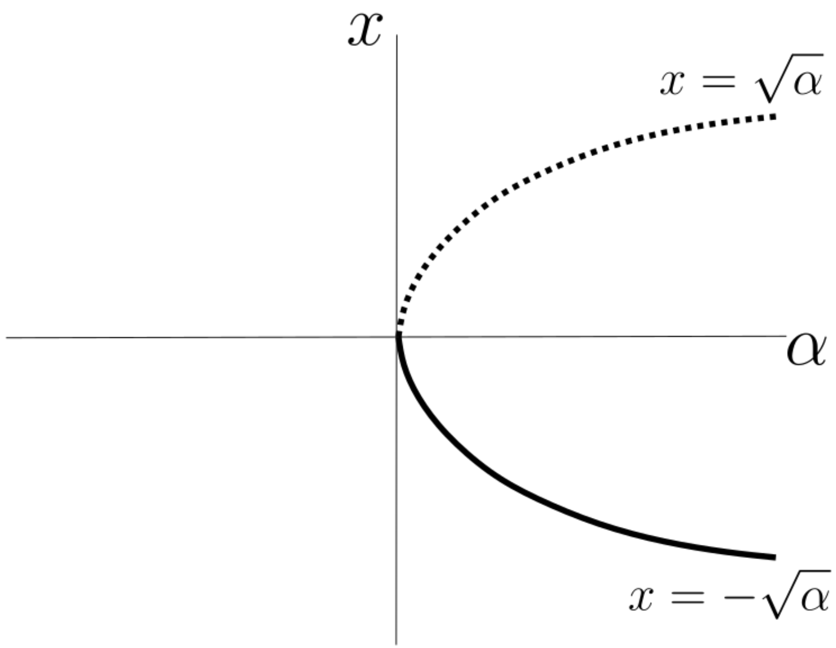


Figure 39: Bifurcation diagram for the system $\dot{x} = x^2 - \alpha$, where $x, \alpha \in \mathbb{R}$.

Hopf bifurcation

- Consider the nonlinear planar system
$$\begin{aligned}\dot{x} &= -y + x\left(\mu - x^2 - y^2\right),\\ \dot{y} &= x + y\left(\mu - x^2 - y^2\right),\end{aligned} \tag{5.41}$$
where $\mu \in \mathbb{R}$.

- The origin is an equilibrium. The linearization at the origin is $\left(\begin{smallmatrix}\mu & -1\\ 1 & \mu\end{smallmatrix}\right)$. The corresponding eigenvalues are $\mu \pm i$.

- We can write system (5.41) in polar coordinates. Define
$$r := \sqrt{x^2+y^2}, \qquad \text{and} \qquad \theta := \arctan\frac{y}{x}, \qquad \text{where } (x \neq 0). \tag{5.42}$$
Equivalently, we have $x = r\cos\theta$ and $y = r\sin\theta$.

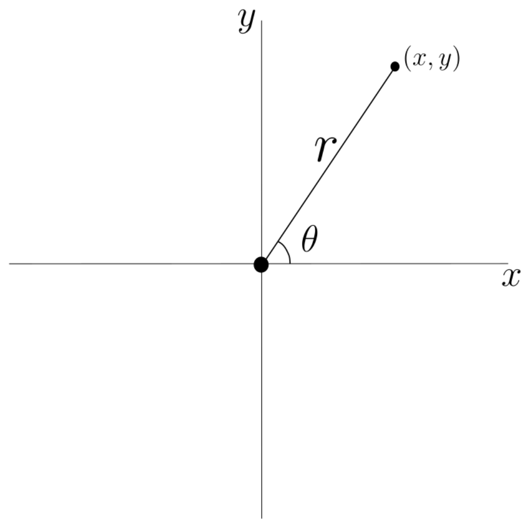


Figure 40: Polar coordinates.

- Writing system (5.41) in (r,θ)-coordinates, we have

$$\begin{aligned} \dot{r} &= r\left(\mu - r^2\right), \\ \dot{\theta} &= 1. \end{aligned} \tag{5.43}$$

- $\dot{r} = 0$ means that $r\,(t)$ is constant.
- Corresponding to the non-zero roots of $\dot{r}$, we have a periodic orbit for system (5.41), i.e. if $\dot{r}$ vanishes at $r = r_0$ and $r_0 \neq 0$, then $(x\,(t)\,, y\,(t))$, such that $[x\,(t)]^2 + [y\,(t)]^2 = r_0^2$, is a periodic orbit of system (5.41).
- Analyzing system (5.43), we can draw the phase portrait of system (5.41); see Figure 42.

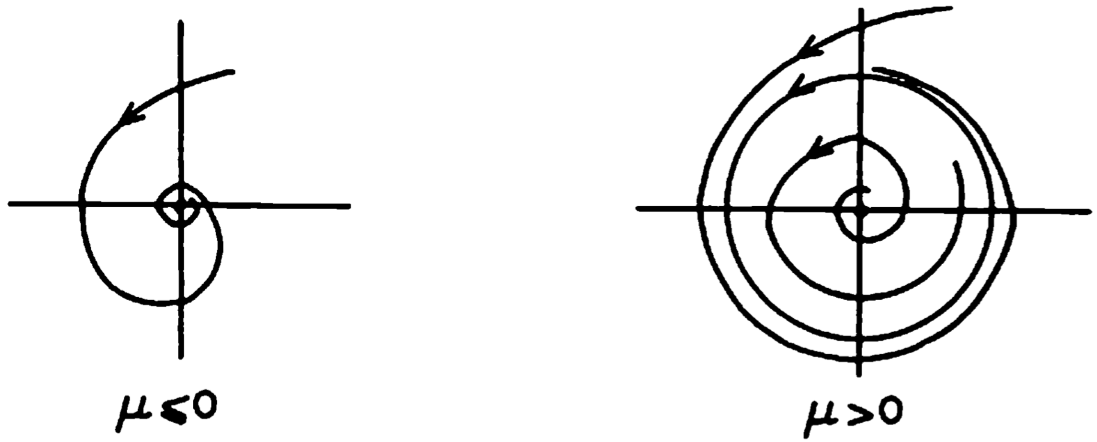


Figure 41: For $\mu \leq 0$, system (5.41) has an asymptotically stable equilibrium at the origin. Once μ becomes positive, the origin loses its stability and an asymptotically stable cycle gets born.

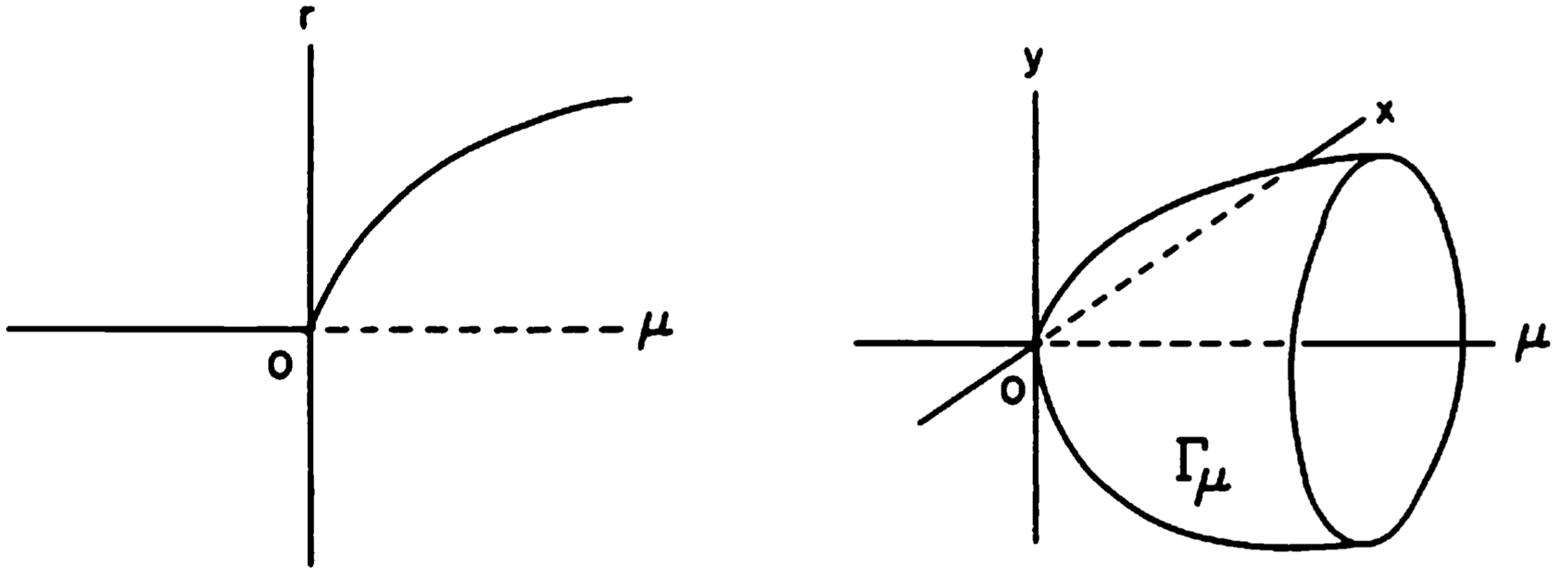


Figure 42: The periodic orbit associated with μ is shown by Γ_μ. This periodic orbit appears only when $\mu > 0$.

6. Chaos theory

6.1. Preliminaries

Assume X is a subset of $\mathbb{R}^n$, where $n \geq 1$.

- We say $X \subset \mathbb{R}^n$ is bounded if there exists $r > 0$ such that $X \subset B_r(O)$, where $B_r(O)$ is the ball with radius r and centered at the origin.

- We say $X \subset \mathbb{R}^n$ is open if for every $x \in X$, there exists a ball $B_r(x)$, where $r > 0$, and $B_r(x) \subset X$.
  - Sometimes, if an open set X contains a set A, we say X is a neighborhood of A.

- We say $X \subset \mathbb{R}^n$ is closed if it contains all of its limit points ($x \in \mathbb{R}^n$ is a limit point of X if there exists a sequence $\{x_n\}$ in X which converges to x).

- We say $X \subset \mathbb{R}^n$ is compact if it is closed and bounded.

6.2. **Attracting sets and attractors**

Consider a system

$$\dot{x} = f(x), \tag{6.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, for $n \geq 1$, is smooth. Suppose that $\phi_t = \phi(t, x)$ is the flow of the system.

- Throughout, let $\Lambda \subset \mathbb{R}^n$ be closed and invariant.

Definition 6.1 (Attracting set). *A set Λ is an attracting set if there exists a neighborhood U of Λ such that U is positively invariant, i.e. $\phi(t, U) \subset U$ for all $t \geq 0$, and we have*

$$\bigcap_{t>0} \phi(t, U) = \Lambda. \tag{6.2}$$

Moreover, the set U is called a trapping region.

Definition 6.2 (Basin of attraction). *The basin of attraction of the attracting set Λ is the set of all initial conditions $x_0 \in \mathbb{R}^n$ whose forward orbits lies or enter the trapping region U. In other words, the basin of attraction is $\cup_{t\leq 0} \phi(t, U)$.*

Example 6.3. [18] *Consider the planar system*

$$\begin{aligned} \dot{x} &= x - x^3, \\ \dot{y} &= -y. \end{aligned} \tag{6.3}$$

The phase portrait of this system is shown in Figure 43.

- *This system has three equilibria* $(-1, 0)$, $(0, 0)$ *and* $(1, 0)$. *An open set* U *containing* $A := [-1, 1] \times \{0\}$ *can be found such that* U *is positively invariant and every initial condition in it approaches* A *(see [GH13]).*
- *The set* A *is an attracting set. The open set* U *is a trapping region, and the basin of attraction is the whole* $\mathbb{R}^2$.

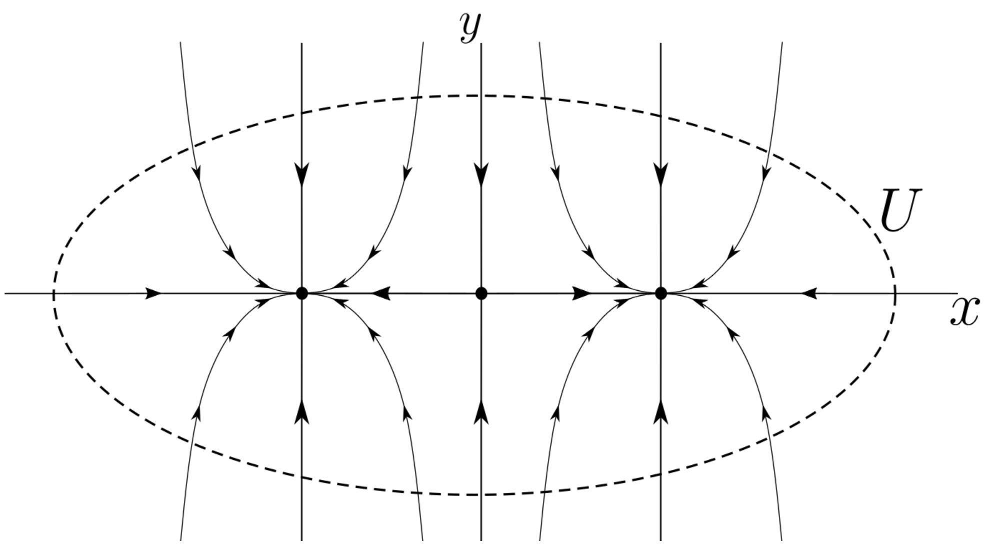


Figure 43: Phase portrait of system (6.3).

[18] See also Example 8.2.2 in [Wig03] and the discussion about Figure 14.3 in [HSD12].

Definition 6.4. *We say Λ is an attractor if it is a topologically transitive attracting set.*

Definition 6.5 (Topological transitivity). *We say ϕ_t is topologically transitive on Λ if for any two arbitrary points $x_1, x_2 \in \Lambda$ and any two open sets U_1 and U_2 in Λ, where U_1 contains x_1, and U_2 contains x_2, we have that there exists a solution curve starting in U_1 and passing through U_2.*

6.3. **Chaos and strange attractors**

Consider again a smooth system

$$\dot{x} = f(x), \tag{6.4}$$

on $\mathbb{R}^n$ with the flow $\phi_t = \phi(t, x)$. Throughout, let $\Lambda \subset \mathbb{R}^n$ be compact and invariant.

Definition 6.6 (Sensitivity to initial conditions). *We say ϕ_t has sensitivity to initial conditions on Λ if there exists $\epsilon > 0$, such that for any x in Λ and any $r > 0$, there exist $y \in B_r(x)$ and $t^* > 0$ such that $\|\phi(t^*, x) - \phi(t^*, y)\| > \epsilon$.*

Definition 6.7. *We say ϕ_t is chaotic on Λ if the following hold.*

(i) ϕ_t is topologically transitive on Λ.

(ii) ϕ_t has sensitivity to initial conditions on Λ.

Remark 6.8. *Some literature requires the following extra condition in the previous definition: the periodic orbits of $\phi(t,x)$ are dense in Λ.*

Definition 6.9 (Strange attractor). *Let Λ be an attractor. We say Λ is a strange attractor if ϕ_t is chaotic on Λ.*

Example 6.10. *Lorenz attractor!*

6.4. **Lyapunov exponents**

Consider a smooth system

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n. \tag{6.5}$$

For an initial state x_0, let $x(t)$ be the solution of the system satisfying $x(0) = x_0$.

- We are interested in how nearby orbits behave relative to each other as $t \to \infty$.
- We linearize system (6.5) about $x(t)$. This gives

$$\dot{v} = Df(x(t))\, v, \qquad v \in \mathbb{R}^n \tag{6.6}$$

  - Equation (6.6) is called the variation equation along the solution $x(t)$.
  - The variational equation is nonautonomous (directly depends on t).
  - There exists a matrix-valued function $X(t, x_0)$, called the fundamental solution matrix, such that, for any $v_0 \in \mathbb{R}^n$, the function $v(t) = X(t, x_0)\, v_0$ is the unique solution of the initial value problem $\dot{v} = Df(x(t))\, v$ and $v(0) = v_0$.

Let $X(t, x_0)$ be a fundamental matrix solution, and $e \neq 0$ be an arbitrary vector in $\mathbb{R}^n$. For the linearized system along $x(t)$, the expression

$$\frac{\|X(t, x_0) e\|}{\|e\|} \tag{6.7}$$

measures the expansion along $x(t)$ in the direction of e.

Definition 6.11. *The Lyapunov exponent along the orbit of x_0 and in the direction of $e \neq 0$* [19] *is given by*

$$\lambda(x_0, e) = \limsup_{t \to \infty} \frac{1}{t} \log \frac{\|X(t, x_0) e\|}{\|e\|}. \tag{6.8}$$

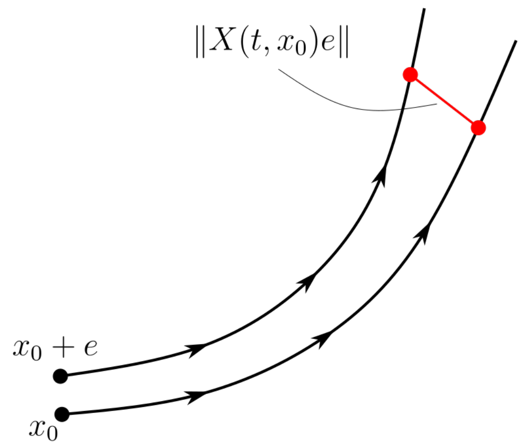


Figure 44: We can approximate the separation rate of the orbits infinitesimally close to $\{x(t)\}$ by linearizing the vector field along $x(t)$.

[19]We make the agreement $\lambda(x_0, 0) = -\infty$.

Proposition 6.12. *Let* $r \in \mathbb{R}$. *Then, the set* $\{e \in \mathbb{R}^n : \lambda(x_0, e) \leq r\}$ *is a vector subspace of* $\mathbb{R}^n$.

Proof. See [Wig03], Lemma 29.1.1. □

Corollary 6.13. *It follows from Proposition 6.12 that there are at most* n *(the dimension of the phase space) distinct Lyapunov exponents associated with the orbit of* x_0.

Definition 6.14. *The set of all the Lyapunov exponents associated with an orbit is called the Lyapunov spectrum of that orbit.*

6.4.1 Maximum Lyapunov exponent

Let $\lambda_{\max} = \lambda_{\max}(x_0)$ be the maximum Lyapunov exponent associated with a given orbit. Then, $\lambda_{\max} > 0$ implies sensitivity to initial conditions.

Lemma 6.15 (A lemma from Linear Algebra). *Let A be an $n \times n$ real matrix. Consider the standard euclidean vector norm $\|\cdot\|$ (2-norm) and define $\|A\| := \max \frac{\|Ax\|}{\|x\|}$, for $0 \neq x \in \mathbb{R}^n$. Then, $\|A\|$ is equal to the largest eigenvalue of $A^\top A$ (also known as the largest singular value of A).*

According to Lemma 6.15, if $\alpha(t)$ is the largest eigenvalue of $[X(t, x_0)]^\top X(t, x_0)$ (i.e. largest singular value of $X(t, x_0)$), then

$$\lambda_{\max}(x_0) = \limsup_{t \to \infty} \frac{1}{t} \log \alpha(t). \tag{6.9}$$

7. Synchronization in coupled systems

7.1. Graph representation

- Recall that a mathematical model to study synchronization in networks is given by

$$\dot{x}_i = f_i\left(x_i\right) + \alpha \sum_{j=1}^{N} A_{ij} H_i\left(x_j - x_i\right), \qquad \forall i \in \{1, \ldots, N\}, \tag{7.1}$$

where $x_i \in \mathbb{R}^n$ $(n \geq 1)$, $A_{ij} \geq 0$ are real constants, and $f_i, H_i \in \mathcal{C}^2\left(\mathbb{R}^n\right)$.

7.1.1 Adjacency matrix

- We can associate a (directed weighted) graph to system (7.1) as follows:
  - The graph has N nodes, and node number i corresponds to the variable x_i.
  - There exists an edge starting from node j and ending at node i if and only if $A_{ij} > 0$. We can also think of A_{ij} as the weight of this edge.
- The matrix A defined by $A := (A_{ij})$ is called the adjacency matrix of the graph.

Example 7.1. *For given functions f_i and H_i ($i = 1, \dots, 5$), consider the system*

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + \alpha\left[H_1(x_2 - x_1) + 2H_1(x_4 - x_1)\right], \\
\dot{x}_2 &= f_2(x_2) + \alpha\left[H_2(x_3 - x_2)\right], \\
\dot{x}_3 &= f_3(x_3) + \alpha\left[7H_3(x_4 - x_3)\right], \\
\dot{x}_4 &= f_4(x_4) + \alpha\left[3H_4(x_5 - x_4)\right], \\
\dot{x}_5 &= f_5(x_5) + \alpha\left[H_5(x_1 - x_5) + 3H_5(x_2 - x_5)\right].
\end{aligned} \tag{7.2}$$

The corresponding graph is shown in Figure 45. The adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 & 0 \end{pmatrix}. \tag{7.3}$$

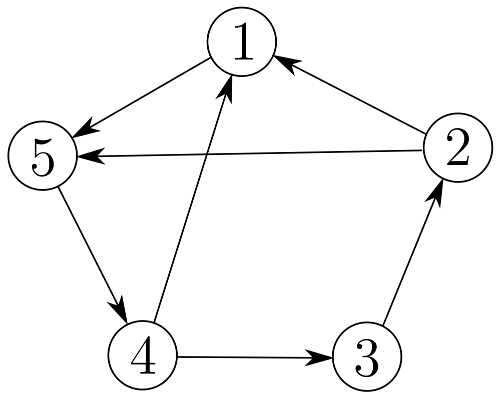


Figure 45: The graph corresponding to system (7.2).

7.1.2 Laplacian matrix

- Consider again the system

$$\dot{x}_i = f_i(x_i) + \alpha \sum_{j=1}^{N} A_{ij} H_i(x_j - x_i), \qquad \forall i \in \{1, \ldots, N\}. \tag{7.4}$$

- Suppose that the couplings are identical, i.e. $H_i = H_j$ for all i and j, and there exists H such that $H_i(x_j - x_i) = H(x_j) - H(x_i)$[20]. Then, (7.4) is written as

$$\dot{x}_i = f_i(x_i) + \alpha \sum_{j=1}^{N} A_{ij} \left[H(x_j) - H(x_i)\right]. \tag{7.5}$$

- Note that

$$\sum_{j=1}^{N} A_{ij} \left[H(x_j) - H(x_i)\right] = \left[\sum_{j=1,\, j\neq i}^{N} A_{ij} H(x_j)\right] - H(x_i) \sum_{j=1,\, j\neq i}^{N} A_{ij}. \tag{7.6}$$

- Define the $N \times N$ matrix $L := (L_{ij})$ by

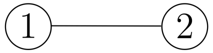
$$L_{ij} := \begin{cases} -A_{ij} & \text{if } i \neq j, \\ \sum_{j=1,\, j\neq i}^{N} A_{ij} & \text{if } i = j. \end{cases} \tag{7.7}$$

- The matrix L is called the Laplacian matrix.
- Using the Laplacian, we can write (7.5) as

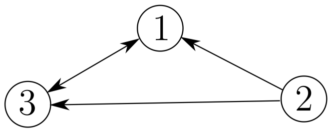
$$\dot{x}_i = f_i(x_i) - \alpha \sum_{j=1}^{N} L_{ij} H(x_j). \tag{7.8}$$

[20] Such a function H naturally appears when we linearize the system at the synchronization subspace $M := \{(x_1, \ldots, x_N) \in \mathbb{R}^{nN} : x_1 = x_2 = \cdots = x_N\}$.

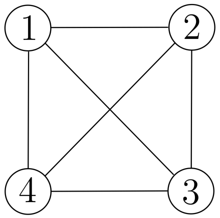
Remark 7.2. *It follows from (7.7) that the sum of all the entries of each arbitrary row of the Laplacian matrix is zero.*



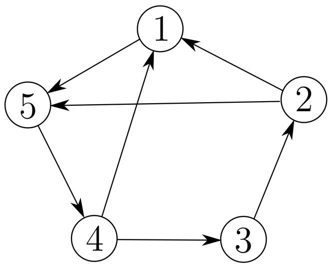
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$



$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$



$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 3 & -1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 7 & -7 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & -3 & 0 & 0 & 4 \end{pmatrix}$$

Figure 46: A few examples of (directed and undirected) graphs and their associated adjacency and Laplacian matrices.

7.1.3 Spectral properties of the Laplacian matrix

Theorem 7.3. *For a given arbitrary graph, the laplacian matrix L has a zero eigenvalue.*

Proof. Let

$$L = \begin{pmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{pmatrix} \tag{7.9}$$

be the Laplacian matrix, and consider the vector $\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ (the n-dimensional vector whose entries are all one). Then,

$$L\mathbf{1} = \begin{pmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} l_{11} + l_{12} + \cdots + l_{1n} \\ \vdots \\ l_{n1} + l_{n2} + \cdots + l_{nn} \end{pmatrix}. \tag{7.10}$$

This means that the i-th entry of the vector $L1$ is the row-sum of the i-th row of the Laplacian L. However, the row-sum of each row of L is zero. Thus, $L\mathbf{1} = 0$. This implies that 0 is an eigenvalue of L, and $\mathbf{1}$ is an associated eigenvector. $\square$

THEOREM 7.4. *All the eigenvalues of the Laplacian matrix L of a given graph have non-negative real parts.*

Proof. Let

$$L = \begin{pmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{pmatrix} \tag{7.11}$$

be the Laplacian matrix. Since the row-sums of L are zero, for each i, we have $l_{ii} = \sum_{j \neq i} |l_{ij}|$. Thus, all the Gershgorin disks lie in the right side of the imaginary axis in the complex plane, as desired. □

7.2. **Synchronization**

- Define $M := \{(x_1, \ldots, x_N) \in \mathbb{R}^{nN} : x_1 = x_2 = \cdots = x_N\}$.
  - M is a vector subspace of $\mathbb{R}^{nN}$.
  - We call M the synchronization subspace.
  - Suppose a solution $(x_1(t), \ldots, x_N(t))$ of system (7.1) entirely lies in M (for instance, this can happen when M is invariant). In this case, we have $x_1(t) = \cdots = x_N(t)$. Such a solution is called a synchronized solution.
- We say system (7.1) gets into (complete) synchrony if M attracts nearby orbits.
  - More precisely, if there exists an open neighborhood U of M such that for any initial condition $(x_1(0), \ldots, x_N(0)) \in U$, and any $1 \le i, j \le N$, we have
$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0. \tag{7.12}$$

7.3. An example of synchronization between two coupled nonlinear systems

We now discuss model (7.1) for two coupled identical systems with identity coupling. Consider the system

$$\begin{aligned} \dot{x}_1 &= f(x_1) + \alpha(x_2 - x_1), \\ \dot{x}_2 &= f(x_2) + \alpha(x_1 - x_2), \end{aligned} \tag{7.13}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, and α is the coupling strength.

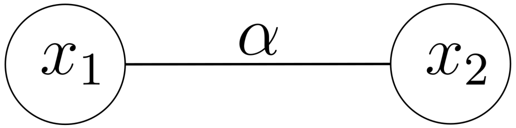


Figure 47: Two coupled systems.

- The synchronization subspace is $M = \{(x_1, x_2) \in \mathbb{R}^{2n} : x_1 = x_2\}$.
  - Observe that M is invariant with respect to the flow of system (7.13).
- System (7.13) gets into synchrony if for any initial condition $(x_1(0), x_2(0))$ close to M, we have

$$\lim_{t\to\infty} \|x_1(t) - x_2(t)\| = 0. \tag{7.14}$$

- We show that if the coupling is sufficiently strong, i.e. α is sufficiently large, then system (7.13) synchronizes.

- Define $z(t) := x_1(t) - x_2(t)$. To detect the synchrony, we need to see if $z(t) \to 0$ as $t \to \infty$ when $z(0)$ is small.

- Recall that

$$\begin{aligned} \dot{x}_1 &= f(x_1) + \alpha (x_2 - x_1), \\ \dot{x}_2 &= f(x_2) + \alpha (x_1 - x_2). \end{aligned}$$

  Thus

$$\dot{z} = \dot{x}_1 - \dot{x}_2 = f(x_1) - f(x_2) - 2\alpha z. \tag{7.15}$$

- Taylor expanding $f(x_1 - z)$ at $z = 0$ gives

$$f(x_2) = f(x_1 - z) = f(x_1) - Df(x_1) z + O\left(\|z\|^2\right). \tag{7.16}$$

  Thus, near $z = 0$, we have

$$\dot{z} = [Df(x_1(t)) - 2\alpha \mathrm{I}] z + O\left(\|z\|^2\right), \qquad (\mathrm{I} = \text{identity matrix}) \tag{7.17}$$

- To analyze the stability of the solution $z = 0$, we consider the linear part of the system, i.e.

$$\dot{z} = \left[Df\left(x_1\left(t\right)\right) - 2\alpha \mathrm{I}\right] z. \tag{7.18}$$

- Define a new variable $w\left(t\right) = e^{2\alpha t} z\left(t\right)$. Then,

$$\begin{aligned} \dot{w} &= 2\alpha e^{2\alpha t} z + e^{2\alpha t} \dot{z} \\ &= 2\alpha w + e^{2\alpha t}\left[Df\left(x_1\left(t\right)\right) - 2\alpha \mathrm{I}\right] z \\ &= \left[Df\left(x_1\left(t\right)\right)\right] w. \end{aligned} \tag{7.19}$$

- Equation $\dot{w} = \left[Df\left(x_1\left(t\right)\right)\right] w$ is the variational equation for the system $\dot{x}_1 = f\left(x_1\right)$ along the orbit $x_1\left(t\right)$.

- Let Λ be the maximal Lyapunov exponent of the orbit $\{x_1\left(t\right)\}$. Then,

$$\|w\left(t\right)\| \leq Ce^{\Lambda t}, \qquad \text{for some constant } C > 0. \tag{7.20}$$

- Thus, $\|z\left(t\right)\| \leq Ce^{\left(\Lambda - 2\alpha\right)t}$, and therefore $\alpha_c = \frac{\Lambda}{2}$.

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