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7. Synchronization in coupled systems

7.1. Graph representation

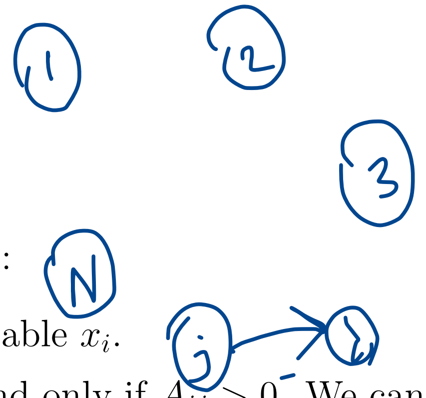
- Recall that a mathematical model to study synchronization in networks is given by

$$\dot{x}_i = f_i(x_i) + \alpha \sum_{j=1}^{N} A_{ij} H_i(x_j - x_i), \qquad \forall i \in \{1, \dots, N\}, \tag{7.1}$$

N nodes

where $x_i \in \mathbb{R}^n$ $(n \geq 1)$, $A_{ij} \geq 0$ are real constants, and $f_i, H_i \in \mathcal{C}^2(\mathbb{R}^n)$.

$A_{ij} > 0$



7.1.1 Adjacency matrix

- We can associate a (directed weighted) graph to system (7.1) as follows:
  - The graph has N nodes, and node number i corresponds to the variable x_i.
  - There exists an edge starting from node j and ending at node i if and only if $A_{ij} > 0$. We can also think of A_{ij} as the weight of this edge.
- The matrix A defined by $A := (A_{ij})$ is called the adjacency matrix of the graph.

$$\begin{pmatrix} A_{11} & A_{12} & A_{13} & - & A_{1N} \\ \vdots & & & & \\ \vdots & & & & \\ A_{N1} & - & & -- & A_{NN} \end{pmatrix} N \times N$$

Example 7.1. *For given functions f_i and H_i $(i = 1, \ldots, 5)$, consider the system*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1) + \alpha\left[H_1(x_2 - x_1) + 2H_1(x_4 - x_1)\right], \\ \dot{x}_2 &= f_2(x_2) + \alpha\left[H_2(x_3 - x_2)\right], \\ \dot{x}_3 &= f_3(x_3) + \alpha\left[7H_3(x_4 - x_3)\right], \\ \dot{x}_4 &= f_4(x_4) + \alpha\left[3H_4(x_5 - x_4)\right], \\ \dot{x}_5 &= f_5(x_5) + \alpha\left[H_5(x_1 - x_5) + 3H_5(x_2 - x_5)\right]. \end{aligned} \tag{7.2}$$

The corresponding graph is shown in Figure 45. The adjacency matrix is

$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 & 0 \end{pmatrix}. \tag{7.3}$$

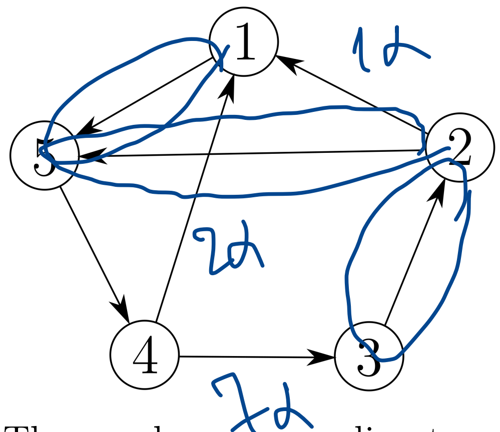


Figure 45: The graph corresponding to system (7.2).

7.1.2 Laplacian matrix

- Consider again the system

$$\dot{x}_i = f_i(x_i) + \alpha \sum_{j=1}^{N} A_{ij} H_i(x_j - x_i), \qquad \forall i \in \{1, \ldots, N\}. \tag{7.4}$$

- Suppose that the couplings are identical, i.e. $H_i = H_j$ for all i and j, and there exists H such that $H_i(x_j - x_i) = H(x_j) - H(x_i)$[20]. Then, (7.4) is written as

$$\dot{x}_i = f_i(x_i) + \alpha \sum_{j=1}^{N} A_{ij} [H(x_j) - H(x_i)]. \tag{7.5}$$

- Note that

$$\sum_{j=1}^{N} A_{ij} [H(x_j) - H(x_i)] = \left[\sum_{j=1, j\neq i}^{N} A_{ij} H(x_j) \right] - H(x_i) \sum_{j=1, j\neq i}^{N} A_{ij}. \tag{7.6}$$

- Define the $N \times N$ matrix $L := (L_{ij})$ by

$$L_{ij} := \begin{cases} -A_{ij} & \text{if } i \neq j, \\ \sum_{j=1, j\neq i}^{N} A_{ij} & \text{if } i = j. \end{cases} \tag{7.7}$$

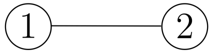
$$L = \begin{pmatrix} L_{11} & \cdots & L_{1N} \\ L_{N1} & & L_{NN} \end{pmatrix}$$

- The matrix L is called the Laplacian matrix.
- Using the Laplacian, we can write (7.5) as

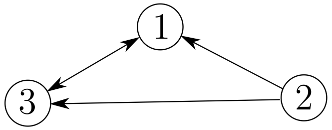
$$\dot{x}_i = f_i(x_i) - \alpha \sum_{j=1}^{N} L_{ij} H(x_j). \tag{7.8}$$

[20] Such a function H naturally appears when we linearize the system at the synchronization subspace $M := \{(x_1, \ldots, x_N) \in \mathbb{R}^{nN} : x_1 = x_2 = \cdots = x_N\}$.

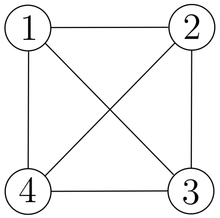
Remark 7.2. *It follows from (7.7) that the sum of all the entries of each arbitrary row of the Laplacian matrix is zero.*



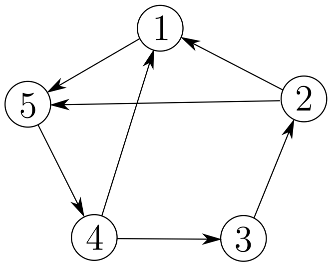
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$



$$A = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 2 & -1 & -1 \\ 0 & 0 & 0 \\ -1 & -1 & 2 \end{pmatrix}$$



$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix}$$



$$A = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 3 & 0 & 0 & 0 \end{pmatrix} \qquad L = \begin{pmatrix} 3 & -1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 7 & -7 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & -3 & 0 & 0 & 4 \end{pmatrix}$$

Figure 46: A few examples of (directed and undirected) graphs and their associated adjacency and Laplacian matrices.

7.1.3 Spectral properties of the Laplacian matrix

Theorem 7.3. *For a given arbitrary graph, the laplacian matrix L has a zero eigenvalue.*

Proof. Let

$$L = \begin{pmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{7.9}$$

be the Laplacian matrix, and consider the vector $\mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ (the n-dimensional vector whose entries are all one). Then,

$$L\mathbf{1} = \begin{pmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} l_{11} + l_{12} + \cdots + l_{1n} \\ \vdots \\ l_{n1} + l_{n2} + \cdots + l_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{7.10}$$

This means that the i-th entry of the vector $L1$ is the row-sum of the i-th row of the Laplacian L. However, the row-sum of each row of L is zero. Thus, $L\mathbf{1} = 0$. This implies that 0 is an eigenvalue of L, and $\mathbf{1}$ is an associated eigenvector. □

$$\underbrace{L\mathbf{1} = 0 \times \mathbf{1}}_{}$$

Theorem 7.4. *All the eigenvalues of the Laplacian matrix L of a given graph have non-negative real parts.*

Proof. Let

$0 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n$

$$L = \begin{pmatrix} l_{11} & \cdots & l_{1n} \\ \vdots & \ddots & \vdots \\ l_{n1} & \cdots & l_{nn} \end{pmatrix} \tag{7.11}$$

λ_1

be the Laplacian matrix. Since the row-sums of L are zero, for each i, we have $l_{ii} = \sum_{j \neq i} |l_{ij}|$. Thus, all the Gershgorin disks lie in the right side of the imaginary axis in the complex plane, as desired. □

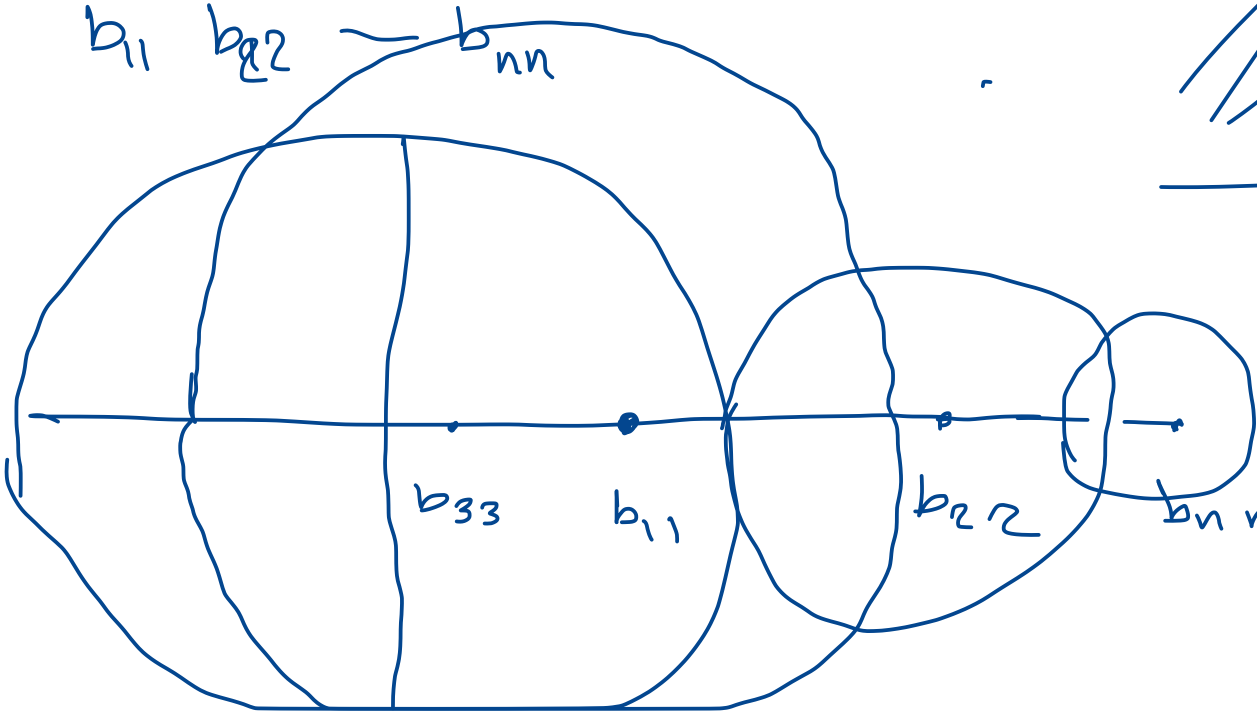
$B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{n1} & b_{n2} & \dots & b_{nn} \end{pmatrix}$

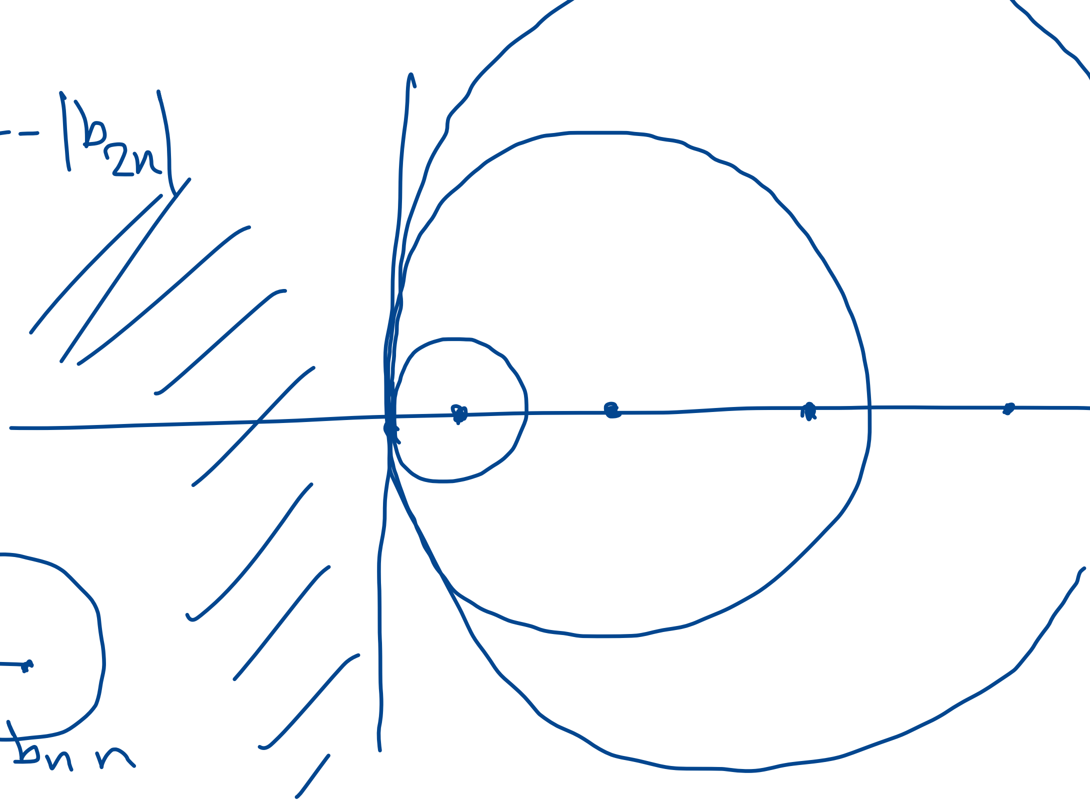
$r_1 = |b_{12}| + \dots + |b_{1n}|$

$r_2 = |b_{21}| + |b_{23}| + \dots |b_{2n}|$

$l_{ii} \to r_i = l_{ii}$

$b_{11} \quad b_{22} \quad \dots \quad b_{nn}$





7.2. Synchronization

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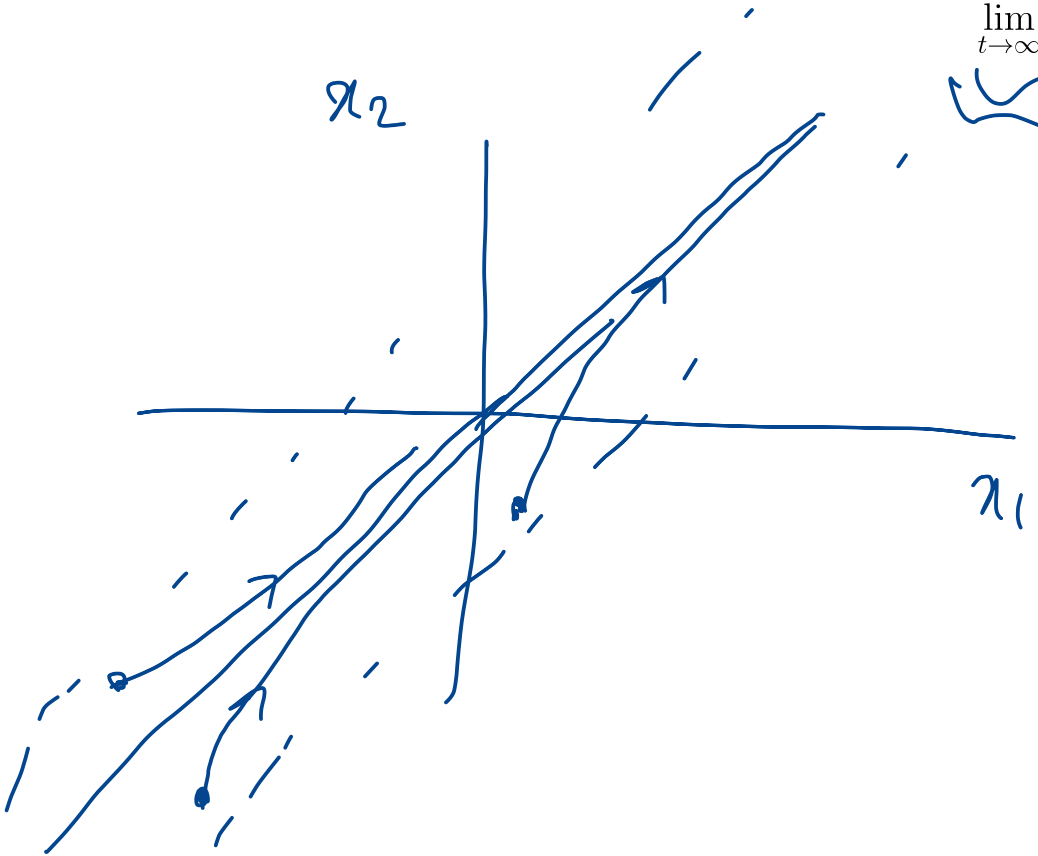
- Define $M := \{(x_1, \ldots, x_N) \in \mathbb{R}^{nN} : x_1 = x_2 = \cdots = x_N\}$.
  - M is a vector subspace of $\mathbb{R}^{nN}$.
  - We call M the synchronization subspace.
  - Suppose a solution $(x_1(t), \ldots, x_N(t))$ of system (7.1) entirely lies in M (for instance, this can happen when M is invariant). In this case, we have $x_1(t) = \cdots = x_N(t)$. Such a solution is called a synchronized solution.
- We say system (7.1) gets into (complete) synchrony if M attracts nearby orbits.
  - More precisely, if there exists an open neighborhood U of M such that for any initial condition $(x_1(0), \ldots, x_N(0)) \in U$, and any $1 \leq i, j \leq N$, we have

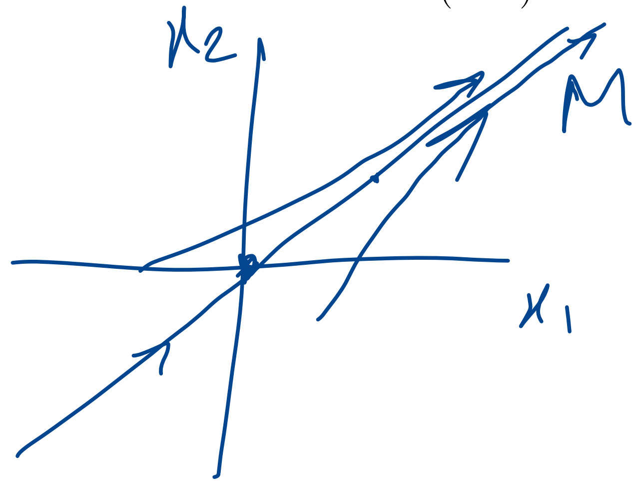
$$\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0. \tag{7.12}$$



7.3. An example of synchronization between two coupled nonlinear systems

We now discuss model (7.1) for two coupled identical systems with identity coupling. Consider the system

$$\begin{aligned} \dot{x}_1 &= f\left(x_1\right) + \alpha\left(x_2 - x_1\right), \\ \dot{x}_2 &= f\left(x_2\right) + \alpha\left(x_1 - x_2\right), \end{aligned} \tag{7.13}$$



where $f : \mathbb{R}^n \to \mathbb{R}^n$, and α is the coupling strength.

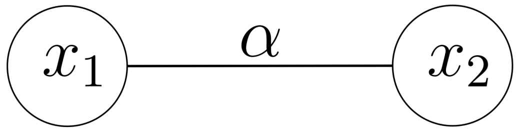


Figure 47: Two coupled systems.

- The synchronization subspace is $M = \{(x_1, x_2) \in \mathbb{R}^{2n} : x_1 = x_2\}$.
  - Observe that M is invariant with respect to the flow of system (7.13).
- System (7.13) gets into synchrony if for any initial condition $(x_1(0), x_2(0))$ close to M, we have

$$\lim_{t\to\infty} \|x_1(t) - x_2(t)\| = 0. \tag{7.14}$$

- We show that if the coupling is sufficiently strong, i.e. α is sufficiently large, then system (7.13) synchronizes.
- Define $z(t) := x_1(t) - x_2(t)$. To detect the synchrony, we need to see if $z(t) \to 0$ as $t \to \infty$ when $z(0)$ is small.
- Recall that

$$\begin{cases} \dot{x}_1 = f(x_1) + \alpha(x_2 - x_1), \\ \dot{x}_2 = f(x_2) + \alpha(x_1 - x_2). \end{cases}$$

Thus

$$\dot{z} = \dot{x}_1 - \dot{x}_2 = f(x_1) - f(x_2) - 2\alpha z. \tag{7.15}$$

- Taylor expanding $f(x_1 - z)$ at $z = 0$ gives

$$f(x_2) = f(x_1 - z) = f(x_1) - Df(x_1)\, z + O\left(\|z\|^2\right). \tag{7.16}$$

Thus, near $z = 0$, we have

$$\dot{z} = [Df(x_1(t)) - 2\alpha \mathrm{I}]\, z + O\left(\|z\|^2\right), \qquad (\mathrm{I} = \text{identity matrix}) \tag{7.17}$$

- To analyze the stability of the solution $z = 0$, we consider the linear part of the system, i.e.

$$\dot{z} = [Df(x_1(t)) - 2\alpha \mathrm{I}]\, z. \tag{7.18}$$

- Define a new variable $w(t) = e^{2\alpha t} z(t)$. Then,

$$\begin{aligned} \dot{w} &= 2\alpha e^{2\alpha t} z + e^{2\alpha t} \dot{z} \\ &= 2\alpha w + e^{2\alpha t} [Df(x_1(t)) - 2\alpha \mathrm{I}]\, z \\ &= [Df(x_1(t))]\, w. \end{aligned} \tag{7.19}$$

- Equation $\dot{w} = [Df(x_1(t))]\, w$ is the variational equation for the system $\dot{x}_1 = f(x_1)$ along the orbit $x_1(t)$.
- Let Λ be the maximal Lyapunov exponent of the orbit $\{x_1(t)\}$. Then,

$$\|w(t)\| \leq Ce^{\Lambda t}, \qquad \text{for some constant } C > 0. \tag{7.20}$$

- Thus, $\|z(t)\| \leq Ce^{(\Lambda - 2\alpha)t}$, and therefore $\alpha_c = \frac{\Lambda}{2}$.

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