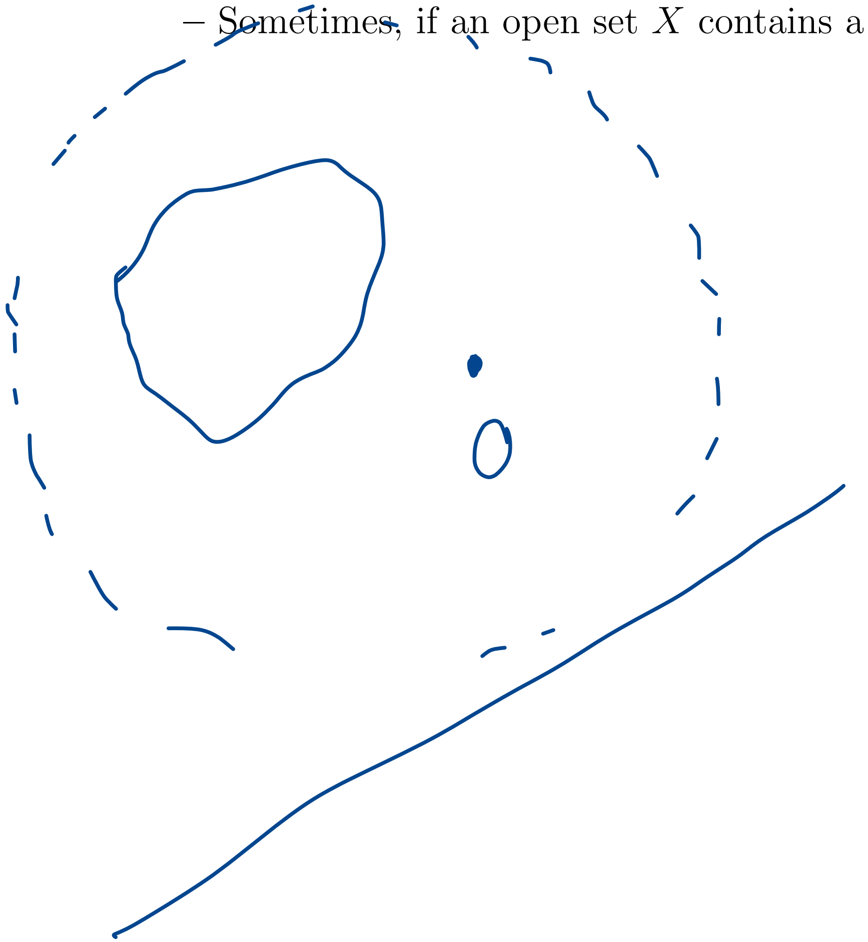
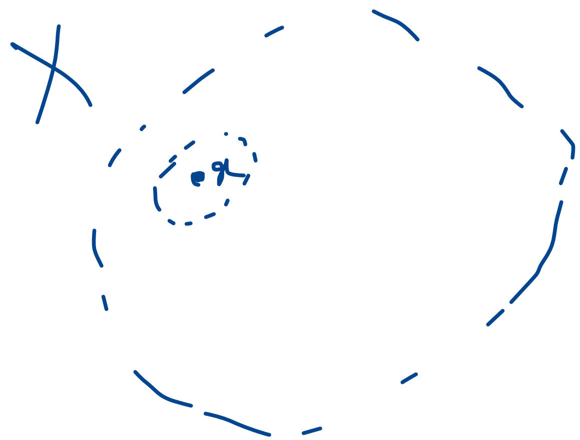
6. Chaos theory

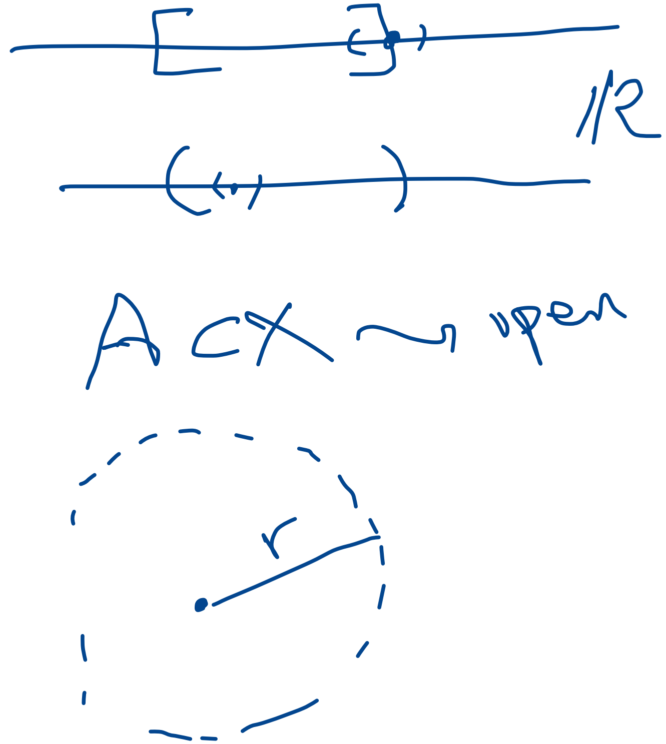
6.1. Preliminaries

Assume X is a subset of $\mathbb{R}^n$, where $n \geq 1$.

- We say $X \subset \mathbb{R}^n$ is bounded if there exists $r > 0$ such that $X \subset B_r(O)$, where $B_r(O)$ is the ball with radius r and centered at the origin.
- We say $X \subset \mathbb{R}^n$ is open if for every $x \in X$, there exists a ball $B_r(x)$, where $r > 0$, and $B_r(x) \subset X$.
  - Sometimes, if an open set X contains a set A, we say X is a neighborhood of A.





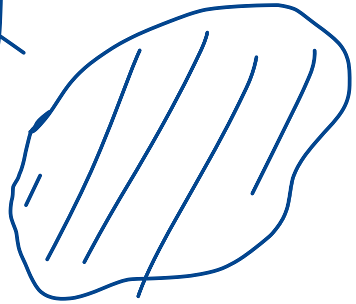


- We say $X \subset \mathbb{R}^n$ is closed if it contains all of its limit points ($x \in \mathbb{R}^n$ is a limit point of X if there exists a sequence $\{x_n\}$ in X which converges to x).

- We say $X \subset \mathbb{R}^n$ is compact if it is closed and bounded.

$\{x_n\} \to x \in \mathbb{R}^n$

X



6.2. **Attracting sets and attractors**

Consider a system

$$\dot{x} = f(x), \tag{6.1}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, for $n \geq 1$, is smooth. Suppose that $\phi_t = \phi(t, x)$ is the flow of the system.

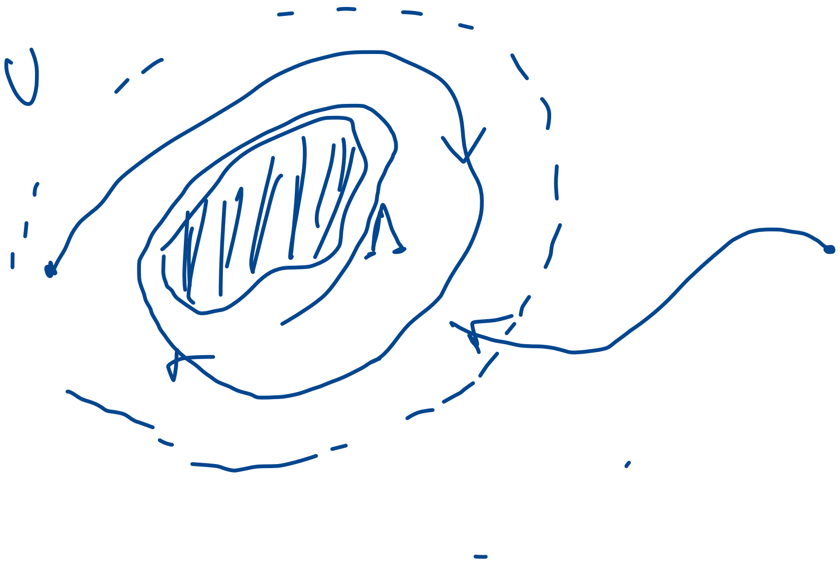
- Throughout, let $\Lambda \subset \mathbb{R}^n$ be closed and invariant.

Definition 6.1 (Attracting set). *A set Λ is an attracting set if there exists a neighborhood U of Λ such that U is positively invariant, i.e. $\phi(t, U) \subset U$ for all $t \geq 0$, and we have*

$$\bigcap_{t>0} \phi(t, U) = \Lambda. \tag{6.2}$$

Moreover, the set U is called a trapping region.

Definition 6.2 (Basin of attraction). *The basin of attraction of the attracting set Λ is the set of all initial conditions $x_0 \in \mathbb{R}^n$ whose forward orbits lies or enter the trapping region U. In other words, the basin of attraction is $\cup_{t\leq 0} \phi(t, U)$.*



Example 6.3. [18] *Consider the planar system*

$$\dot{x} = x - x^3, \\ \dot{y} = -y. \tag{6.3}$$

The phase portrait of this system is shown in Figure 43.

- *This system has three equilibria* $(-1, 0)$, $(0, 0)$ *and* $(1, 0)$. *An open set* U *containing* $A := [-1, 1] \times \{0\}$ *can be found such that* U *is positively invariant and every initial condition in it approaches* A *(see [GH13]).*
- *The set* A *is an attracting set. The open set* U *is a trapping region, and the basin of attraction is the whole* $\mathbb{R}^2$.

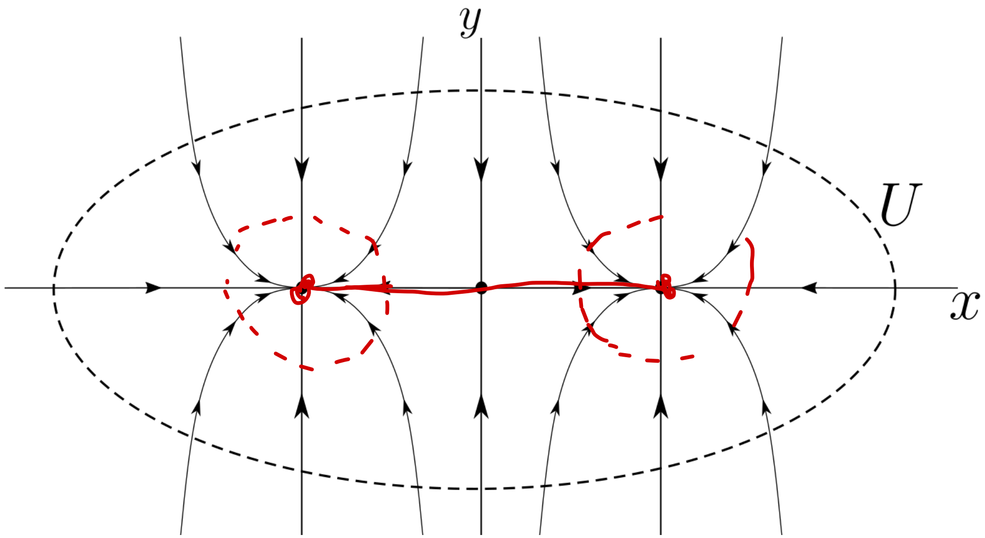
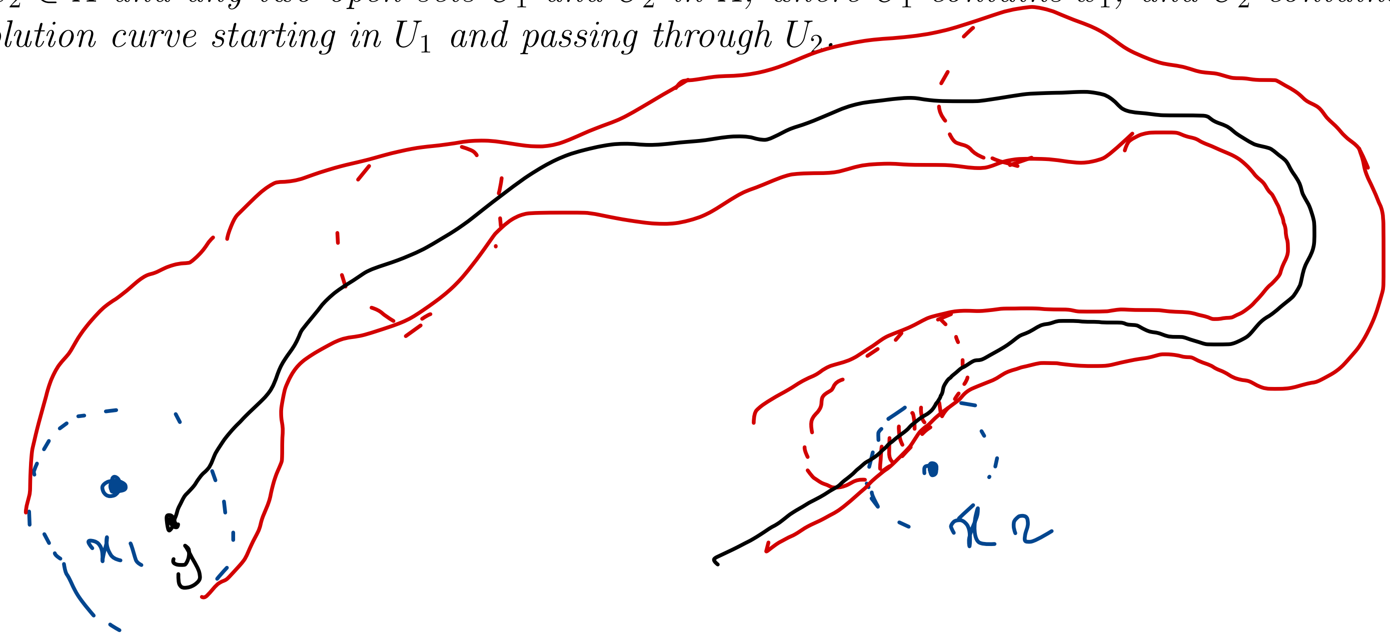


Figure 43: Phase portrait of system (6.3).

[18]See also Example 8.2.2 in [Wig03] and the discussion about Figure 14.3 in [HSD12].

Definition 6.4. *We say Λ is an attractor if it is a topologically transitive attracting set.*

Definition 6.5 (Topological transitivity). *We say ϕ_t is topologically transitive on Λ if for any two arbitrary points $x_1, x_2 \in \Lambda$ and any two open sets U_1 and U_2 in Λ, where U_1 contains x_1, and U_2 contains x_2, we have that there exists a solution curve starting in U_1 and passing through U_2.*



6.3. Chaos and strange attractors

closed and bounded

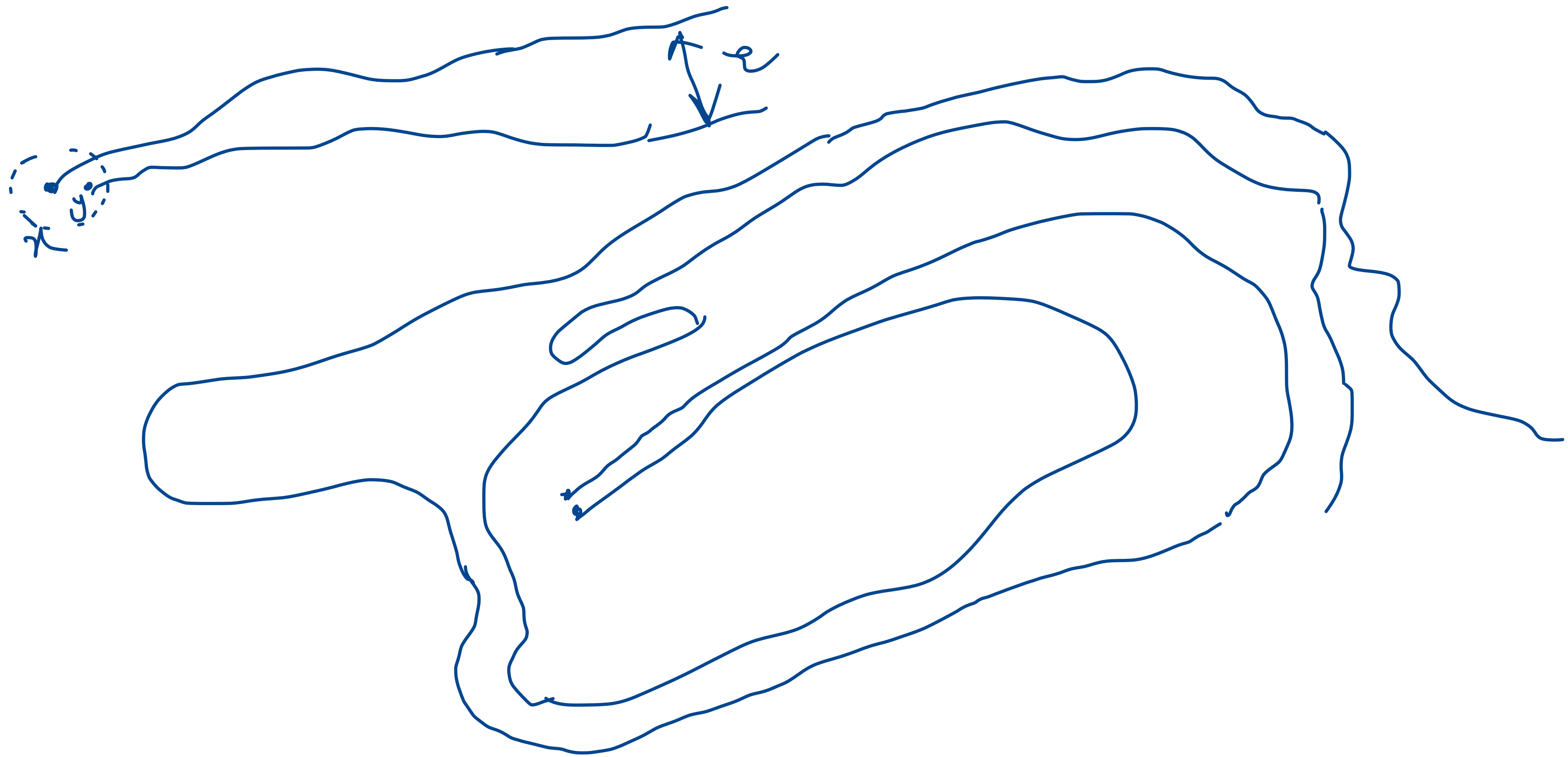
Consider again a smooth system

$$\dot{x} = f(x), \tag{6.4}$$

on $\mathbb{R}^n$ with the flow $\phi_t = \phi(t, x)$. Throughout, let $\Lambda \subset \mathbb{R}^n$ be compact and invariant.

Definition 6.6 (Sensitivity to initial conditions). *We say ϕ_t has sensitivity to initial conditions on Λ if there exists $\epsilon > 0$, such that for any x in Λ and any $r > 0$, there exist $y \in B_r(x)$ and $t^* > 0$ such that $\|\phi(t^*, x) - \phi(t^*, y)\| > \epsilon$.*

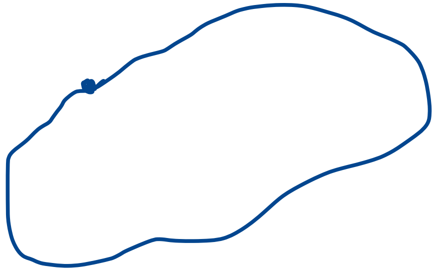
$\varepsilon > 0$ $\quad$ $r > 0$



Definition 6.7. *We say ϕ_t is chaotic on Λ if the following hold.*

(i) ϕ_t is topologically transitive on Λ.

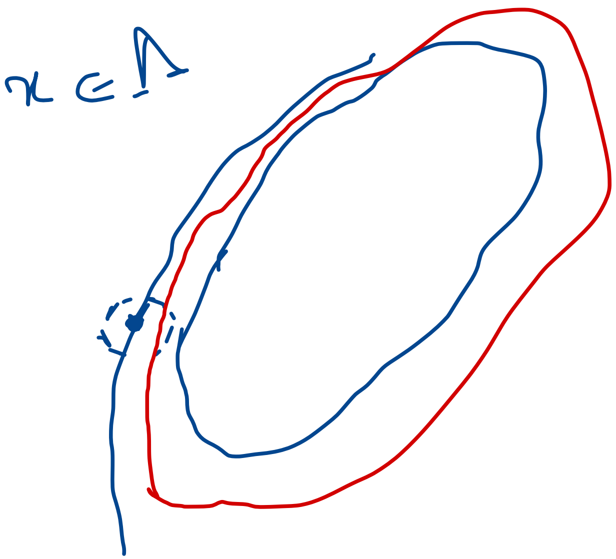
(ii) ϕ_t has sensitivity to initial conditions on Λ.

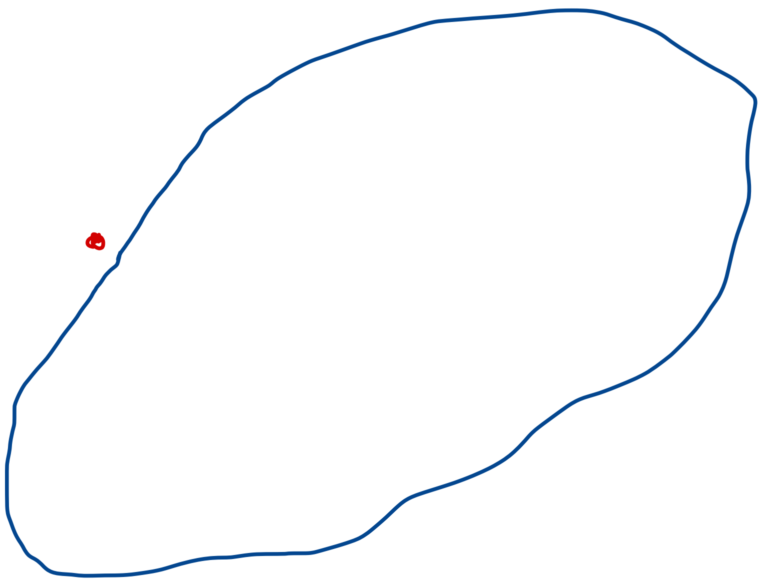


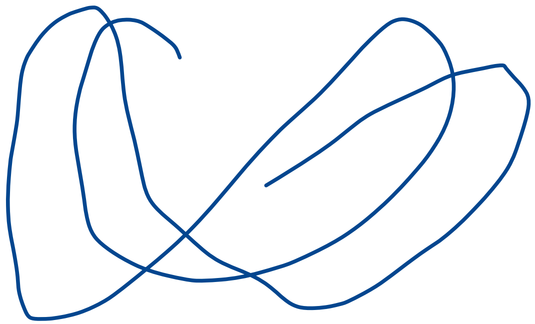
Remark 6.8. *Some literature requires the following extra condition in the previous definition: the periodic orbits of $\phi(t,x)$ are dense in Λ.*

Definition 6.9 (Strange attractor). *Let Λ be an attractor. We say Λ is a strange attractor if ϕ_t is chaotic on Λ.*

Example 6.10. *Lorenz attractor!*







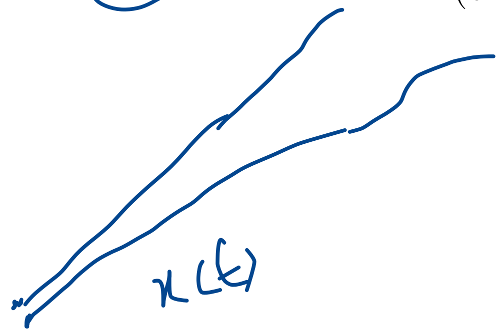
6.4. **Lyapunov exponents**

Consider a smooth system

C^1 C^0

$$\dot{x} = f(x), \qquad x \in \mathbb{R}^n. \tag{6.5}$$

For an initial state x_0, let $x(t)$ be the solution of the system satisfying $x(0) = x_0$.



- We are interested in how nearby orbits behave relative to each other as $t \to \infty$.
- We linearize system (6.5) about $x(t)$. This gives

$$\dot{v} = Df(x(t))\, v, \qquad v \in \mathbb{R}^n \tag{6.6}$$

  - Equation (6.6) is called the variation equation along the solution $x(t)$.
  - The variational equation is nonautonomous (directly depends on t).
  - There exists a matrix-valued function $X(t, x_0)$, called the fundamental solution matrix, such that, for any $v_0 \in \mathbb{R}^n$, the function $v(t) = X(t, x_0)\, v_0$ is the unique solution of the initial value problem $\dot{v} = Df(x(t))\, v$ and $v(0) = v_0$.

$\dot{v} = A(t)\, v$

$X(t)\, v_0$

v_0

Let $X(t, x_0)$ be a fundamental matrix solution, and $e \neq 0$ be an arbitrary vector in $\mathbb{R}^n$. For the linearized system along $x(t)$, the expression

$$\frac{\|X(t, x_0)\, e\|}{\|e\|} \tag{6.7}$$

measures the expansion along $x(t)$ in the direction of e.

Definition 6.11. *The Lyapunov exponent along the orbit of x_0 and in the direction of $e \neq 0$* [19] *is given by*

lim

$$\lambda(x_0, e) = \limsup_{t \to \infty} \frac{1}{t} \log \frac{\|X(t, x_0)\, e\|}{\|e\|}. \tag{6.8}$$

$e^{\lambda t}$ $\|e\|$

$e \in \mathbb{R}^n$

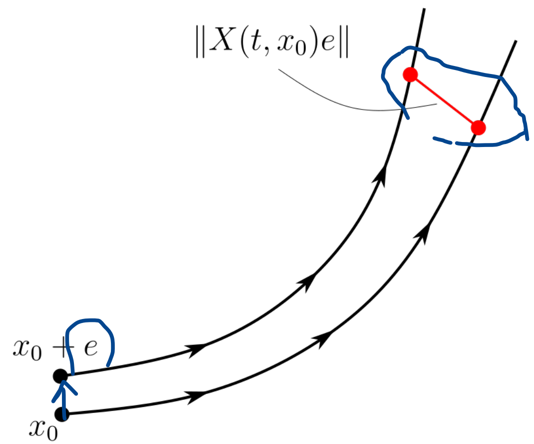


Figure 44: We can approximate the separation rate of the orbits infinitesimally close to $\{x(t)\}$ by linearizing the vector field along $x(t)$.

[19] We make the agreement $\lambda(x_0, 0) = -\infty$.

PROPOSITION 6.12. *Let* $r \in \mathbb{R}$. *Then, the set* $\{e \in \mathbb{R}^n : \lambda(x_0, e) \leq r\}$ *is a vector subspace of* $\mathbb{R}^n$.

Proof. See [Wig03], Lemma 29.1.1. □

COROLLARY 6.13. *It follows from Proposition 6.12 that there are at most* n *(the dimension of the phase space) distinct Lyapunov exponents associated with the orbit of* x_0.

Definition 6.14. *The set of all the Lyapunov exponents associated with an orbit is called the Lyapunov spectrum of that orbit.*

$\varepsilon > 0$ $e^{\lambda t}$

6.4.1 Maximum Lyapunov exponent

Let $\lambda_{\max} = \lambda_{\max}(x_0)$ be the maximum Lyapunov exponent associated with a given orbit. Then, $\boxed{\lambda_{\max}} > 0$ implies sensitivity to initial conditions.

Lemma 6.15 (A lemma from Linear Algebra). *Let A be an $n \times n$ real matrix. Consider the standard euclidean vector norm $\|\cdot\|$ (2-norm) and define $\|A\| := \max \frac{\|Ax\|}{\|x\|}$, for $0 \neq x \in \mathbb{R}^n$. Then, $\|A\|$ is equal to the largest eigenvalue of $A^\top A$ (also known as the largest singular value of A).*

$A^T A$ $\sqrt{}$

According to Lemma 6.15, if $\alpha(t)$ is the largest eigenvalue of $[X(t, x_0)]^\top X(t, x_0)$ (i.e. largest singular value of $X(t, x_0)$), then

square root

$$\lambda_{\max}(x_0) = \limsup_{t\to\infty} \frac{1}{t} \log \alpha(t). \tag{6.9}$$

$$\|(x_1, \ldots, x_n)\| = \sqrt{x_1^2 + \cdots + x_n^2}$$

$$\left\{ \lim_{t \to \infty} \right.$$

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