5.4. Periodic orbits: stability, limit cycles and Poincaré maps

Definition 5.31. *Consider a system $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is smooth. Let $\overline{x} \in \mathbb{R}^n$ and assume it is not an equilibrium point. Then, $\Gamma = \{\phi(t, \overline{x}), t \in \mathbb{R}\}$ is said to be a periodic orbit or a cycle if there exists $T > 0$, such that $\phi(T, \overline{x}) = \overline{x}$.*

- Geometrically, a periodic orbit is a closed curve.
- Assume Γ is a periodic orbit and consider $r > 0$. Let $U_r(\Gamma)$ be the set of all points $x \in \mathbb{R}^n$ whose distance from Γ is less than r.

$U_r(\Gamma)$

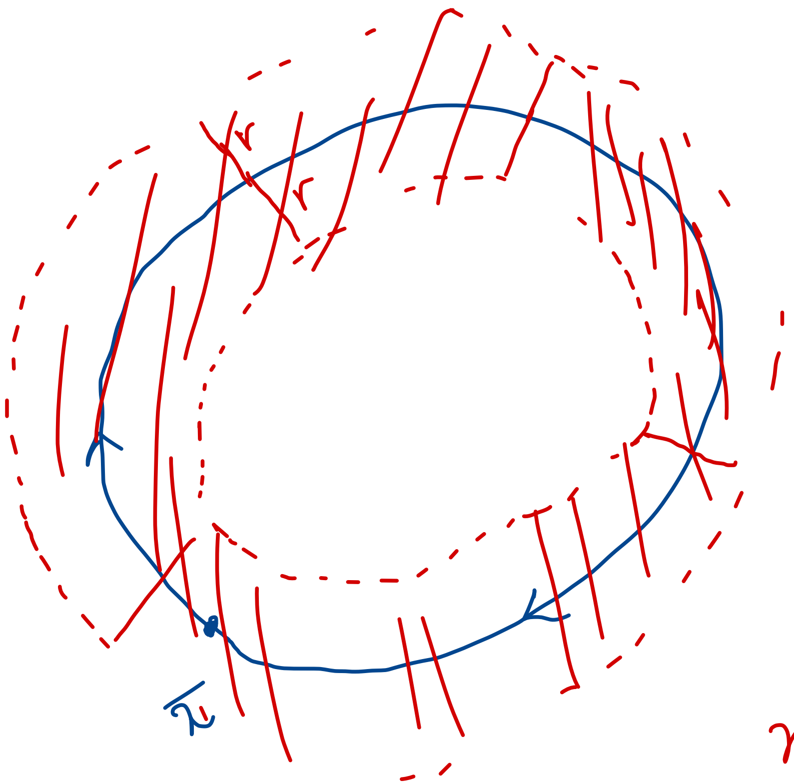
$T > 0$

$\varphi(T, \bar{x}) = \bar{x}$

$\varphi(T+t, \bar{x})$

$= \varphi(t, \bar{x})$

$\forall\, t \in \mathbb{R}$



$x \in \mathbb{R}^n$

$\| x - \Gamma$

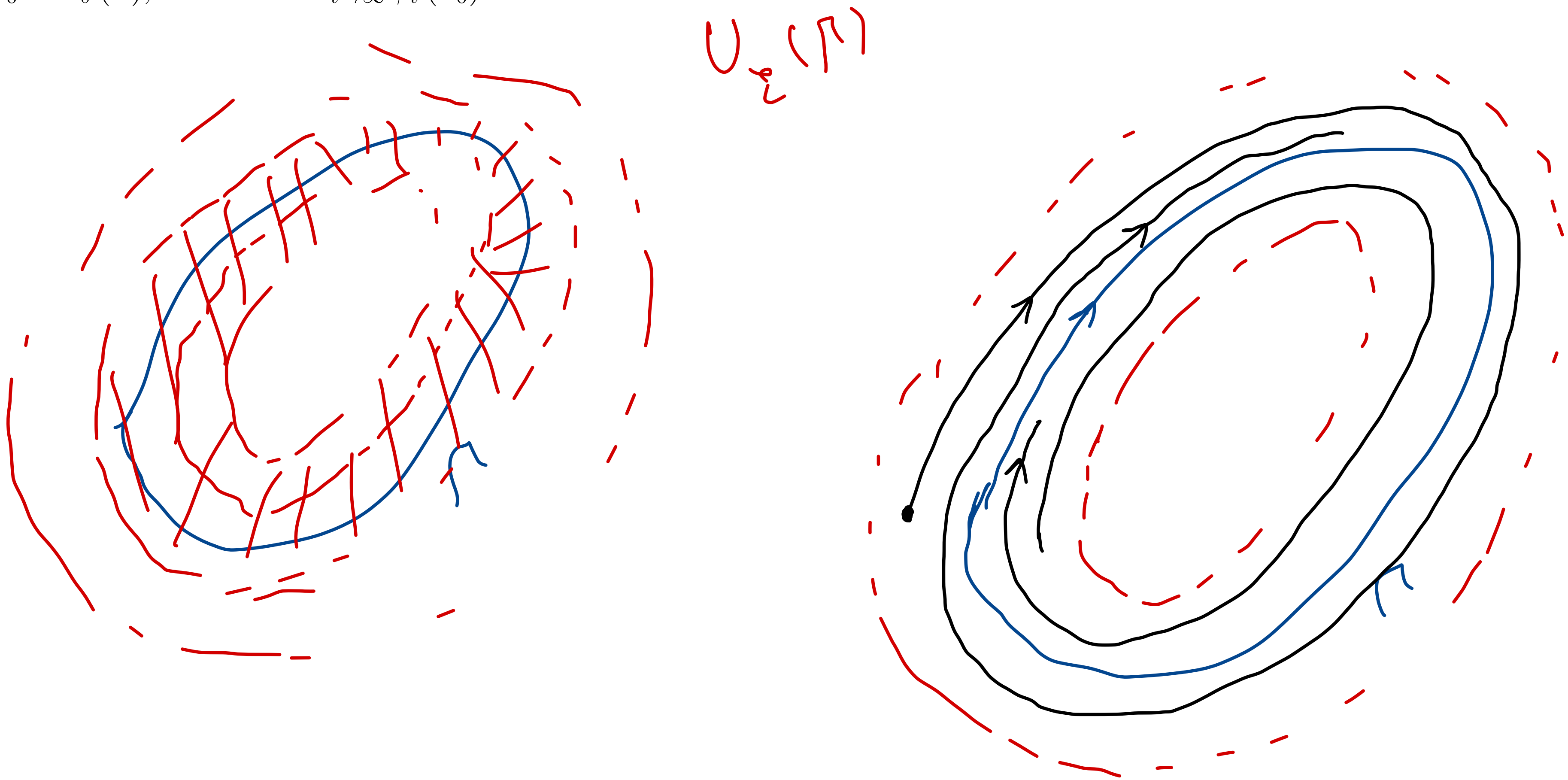
dista$(x, \Gamma) < r$

Analogous to equilibria, we can define the concept of stability for periodic orbits too. For arbitrary $x_0 \in \mathbb{R}^n$, let $\phi_t(x_0) = \phi(t, x_0)$ be the solution of the system with the initial condition $\phi(0, x_0) = x_0$.

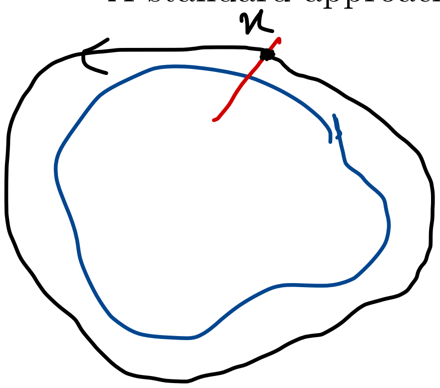
Definition 5.32. *The cycle* Γ *is called stable if for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that for all* $x_0 \in U_\delta(\Gamma)$ *and for all* $t \geq 0$*, we have* $\phi_t(x_0) \in U_\epsilon(\Gamma)$*.*

Definition 5.33. *The cycle* Γ *is called unstable if it is not stable.*

Definition 5.34. *The cycle* Γ *is called asymptotically stable if it is stable, and there exists a* $\delta > 0$ *such that for all* $x_0 \in U_\delta(\Gamma)$*, we have* $\lim_{t\to\infty} \phi_t(x_0) = \Gamma$*.*

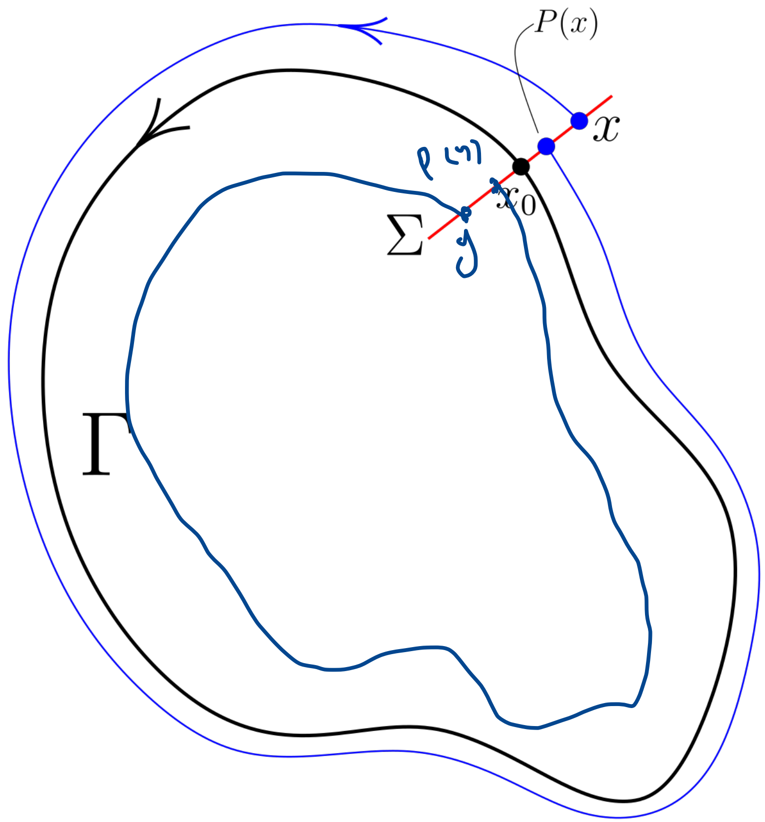


- A standard approach to investigate the dynamics near a periodic orbit is to study an associated Poincaré map.



$P(x) = x$

cross - section



$x \in \Sigma \longrightarrow P(x) \in \Sigma$

$y \longmapsto P(y)$

$x_0 \longmapsto x_0$

$P(x_0) = x_0$

$P : \Sigma \longrightarrow \Sigma$

$\subset \mathbb{R} \quad \subset \mathbb{R}$

Figure 34: Here, Γ is a periodic orbit in the plane. The Poincaré map P maps $x \in \Sigma$ to $P(x) \in \Sigma$.

$x \longrightarrow P(x) \longrightarrow P^2(x) = P(P(x)) \longmapsto \cdot \qquad x_0$

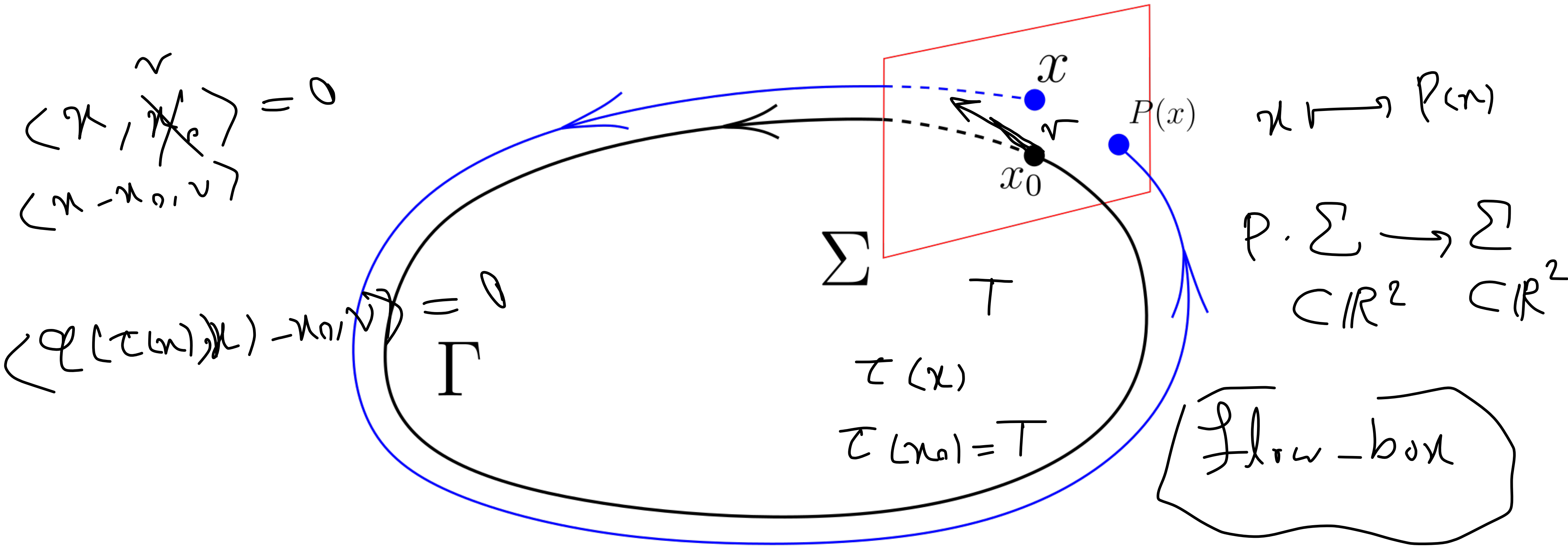


Figure 35: Here, Γ is a periodic orbit in the 3-dimensional space. The cross-section Σ is 2-dimensional. The Poincaré map P maps $x \in \Sigma$ to $P(x) \in \Sigma$.

5.5. Bifurcations

$\dot{x} = f(x)$

Consider a system

$\dot{x} = f(x, T)$

$$\dot{x} = f(x, \alpha), \tag{5.32}$$

where $x \in \mathbb{R}^n$ and $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{R}^k$. The variable x is the phase variable.

- We can think of system (5.32) as a model for a physical problem for which, α is a controlling parameter, such as temperature, pressure, etc.
- We can also think of system (5.32) as a family of systems. For each fixed α, we have a system of ODEs.

Example 5.35. *Consider the system*

$$\dot{x} = x^3 - \alpha x, \tag{5.33}$$

where $x \in \mathbb{R}$, and $\alpha \in \mathbb{R}$. For each fixed α, we have a system of ODEs. For instance,

(i) When $\alpha = 4$, we consider the system $\dot{x} = x^3 - 4x$.

(ii) When $\alpha = 0$, we consider the system $\dot{x} = x^3$.

(iii) When $\alpha = -10$, we consider the system $\dot{x} = x^3 + 10x$.

Example 5.36. *Consider the system*

$$\begin{aligned}\dot{x}_1 &= x_1 - \alpha_1 x_2^3 + \alpha_2 x_1 x_2,\\ \dot{x}_2 &= 4x_2 + \alpha_3 x_1 x_2^3 - \alpha_2 x_1^4,\end{aligned} \tag{5.34}$$

where $x = (x_1, x_2) \in \mathbb{R}^2$, *and* $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{R}^3$. *For each fixed* α, *we have a system of ODEs. For instance,*

(i) When $\alpha = (0, 0, 0)$, *we consider the system*

$$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= 4x_2.\end{aligned} \tag{5.35}$$

(ii) When $\alpha = (1, 2, 0)$, *we consider the system*

$$\begin{aligned}\dot{x}_1 &= x_1 - x_2^3 + 2x_1 x_2,\\ \dot{x}_2 &= 4x_2 - 2x_1^4,\end{aligned} \tag{5.36}$$

(iii) When $\alpha = (-1, 8, -2)$, *we consider the system*

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2^3 + 8x_1 x_2,\\ \dot{x}_2 &= 4x_2 - 2x_1 x_2^3 - 8x_1^4,\end{aligned} \tag{5.37}$$

- In this course, we focus on the case that the parameter space is one dimensional, i.e. $\alpha \in \mathbb{R}$.
- Consider a system

$\alpha \in \mathbb{R}$

$$\dot{x} = f(x, \alpha), \tag{5.38}$$

where $x \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. We say a bifurcation occurs at $\alpha = \alpha_c$ if the dynamics of the system changes suddenly at $\alpha = \alpha_c$.

- The parameter value α_c is called the bifurcation value.
- Recall from the first lecture that a mathematical model for understanding synchronization in networks is given by

$$\dot{x}_i = f_i(x_i) + \alpha \sum_{j=1}^{N} A_{ij} H_i(x_j - x_i), \qquad \forall i \in \{1, \ldots, N\}, \tag{5.39}$$

where $x_i \in \mathbb{R}^n$ $(n \geq 1)$, $A = (A_{ij})$ is the adjacency matrix of the network, and $f_i, H_i \in \mathcal{C}^2(\mathbb{R}^n)$. As it was mentioned in the first lecture, there exists a parameter value α_c (and we also discussed for the linear case in Section 4.6), called critical coupling strength, such that for $\alpha < \alpha_c$, system (5.39) is not in synchrony but it gets into synchrony for $\alpha \geq \alpha_c$.

$\alpha < \alpha_c$

Example 5.37. *Consider the system*

$x = x^2$

$$\dot{x} = x^2 - \alpha, \qquad x, \alpha \in \mathbb{R}. \tag{5.40}$$

- *For* $\alpha < 0$*, system (5.40) has no equilibria. The phase portrait of this system is shown in Figure 36.*

$x^2 - \alpha = 0$

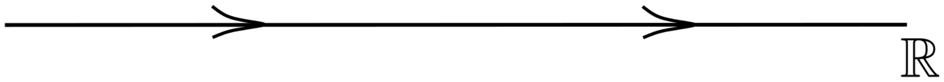


Figure 36: Case $\alpha < 0$.

$-\sqrt{\alpha}, \sqrt{\alpha}$

- *For* $\alpha = 0$*, system (5.40) has a nonhyperbolic equilibrium at* $x = 0$*. The phase portrait of this system is shown in Figure 37.*

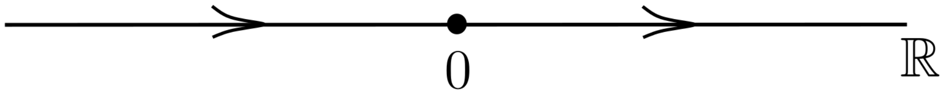


Figure 37: Case $\alpha = 0$.

- *For* $\alpha > 0$*, system (5.40) has two hyperbolic equilibria at* $x = -\sqrt{\alpha}$ *(sink) and* $x = \sqrt{\alpha}$ *(source). The phase portrait of this system is shown in Figure 38.*

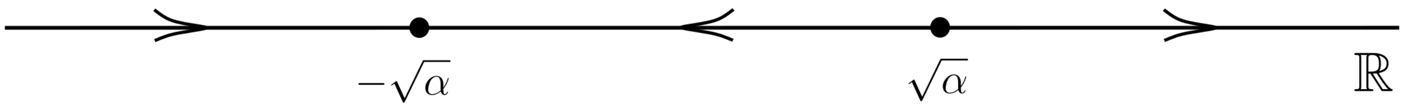


Figure 38: Case $\alpha > 0$.

- *A bifurcation occurs in this system at* $\alpha = 0$*.*

$x^2 - \alpha = 0$

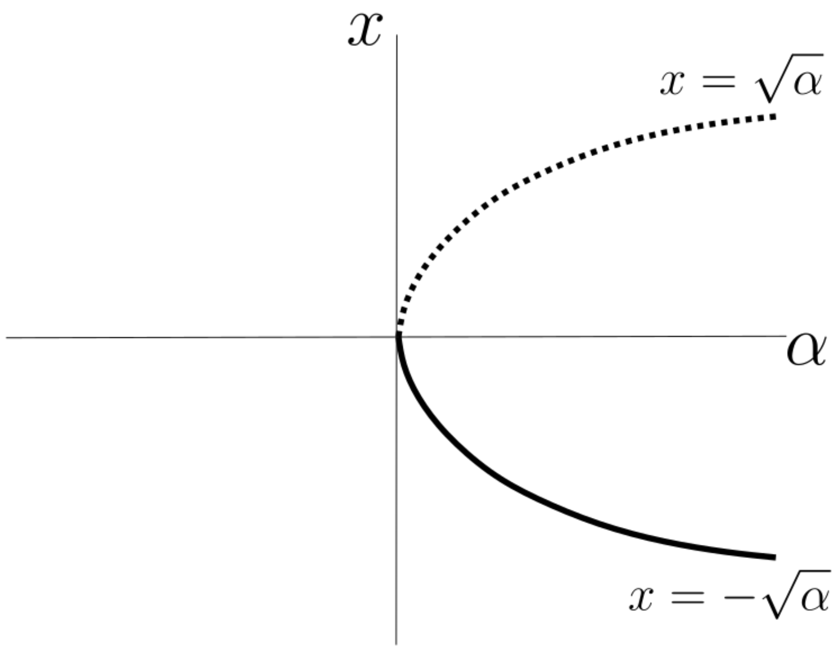


Figure 39: Bifurcation diagram for the system $\dot{x} = x^2 - \alpha$, where $x, \alpha \in \mathbb{R}$.

$\alpha_c = 0$

Hopf bifurcation

$\dot{x} = -y + \mu x$

- Consider the nonlinear planar system

$$\dot{x} = -y + x\left(\mu - x^2 - y^2\right),$$
$$\dot{y} = x + y\left(\mu - x^2 - y^2\right), \tag{5.41}$$

$\dot{y} = x + \mu y$

  where $\mu \in \mathbb{R}$.

- The origin is an equilibrium. The linearization at the origin is $\left(\begin{smallmatrix} \mu & -1 \\ 1 & \mu \end{smallmatrix}\right)$. The corresponding eigenvalues are $\mu \pm i$.

- We can write system (5.41) in polar coordinates. Define

$$r := \sqrt{x^2 + y^2}, \quad \text{and} \quad \theta := \arctan \frac{y}{x}, \quad \text{where } (x \neq 0). \tag{5.42}$$

  Equivalently, we have $x = r\cos\theta$ and $y = r\sin\theta$.

$\dot{r}$

$\dot{\theta}$

$r = (x^2 + y^2)^{\frac{1}{2}}$

$r = r_0$

$\dot{r} = \frac{1}{2}(x^2 + y^2)^{-\frac{1}{2}}\left[2x\dot{x} + 2y\dot{y}\right]$

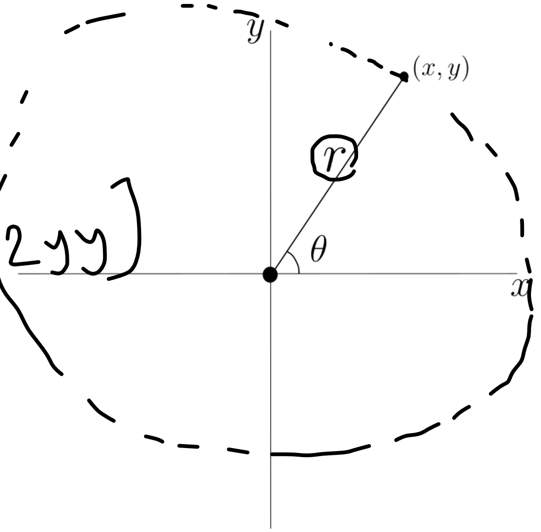


Figure 40: Polar coordinates.

- Writing system (5.41) in (r, θ)-coordinates, we have

$$\begin{cases} \dot{r} = r\left(\mu - r^2\right), \\ \dot{\theta} = 1. \end{cases} \tag{5.43}$$

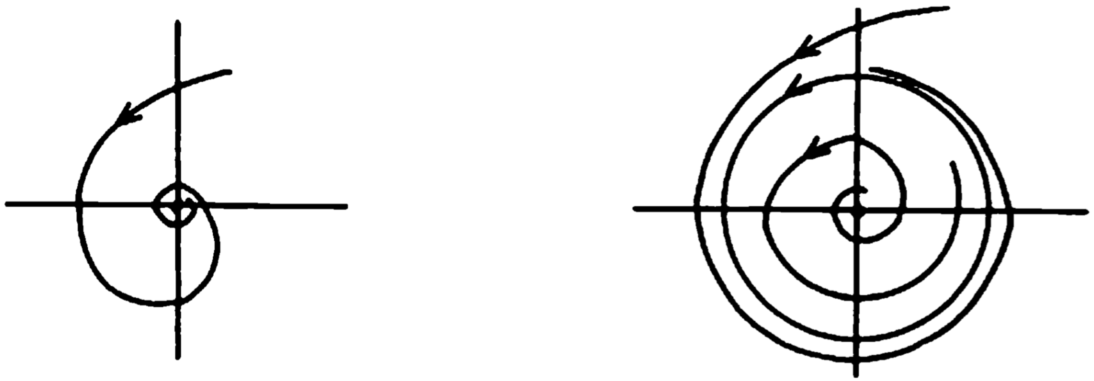
$\dot{r} = 0$

$\cancel{r = 0}$

$\boxed{r = \cancel{-\sqrt{\mu}}, +\sqrt{\mu}}$

- $\dot{r} = 0$ means that $r(t)$ is constant.
- Corresponding to the non-zero roots of $\dot{r}$, we have a periodic orbit for system (5.41), i.e. if $\dot{r}$ vanishes at $r = r_0$ and $r_0 \neq 0$, then $(x(t), y(t))$, such that $[x(t)]^2 + [y(t)]^2 = r_0^2$, is a periodic orbit of system (5.41).
- Analyzing system (5.43), we can draw the phase portrait of system (5.41); see Figure 42.

$\mu > 0$



$\mu \leq 0$ $\mu > 0$

Figure 41: For $\mu \leq 0$, system (5.41) has an asymptotically stable equilibrium at the origin. Once μ becomes positive, the origin loses its stability and an asymptotically stable cycle gets born.

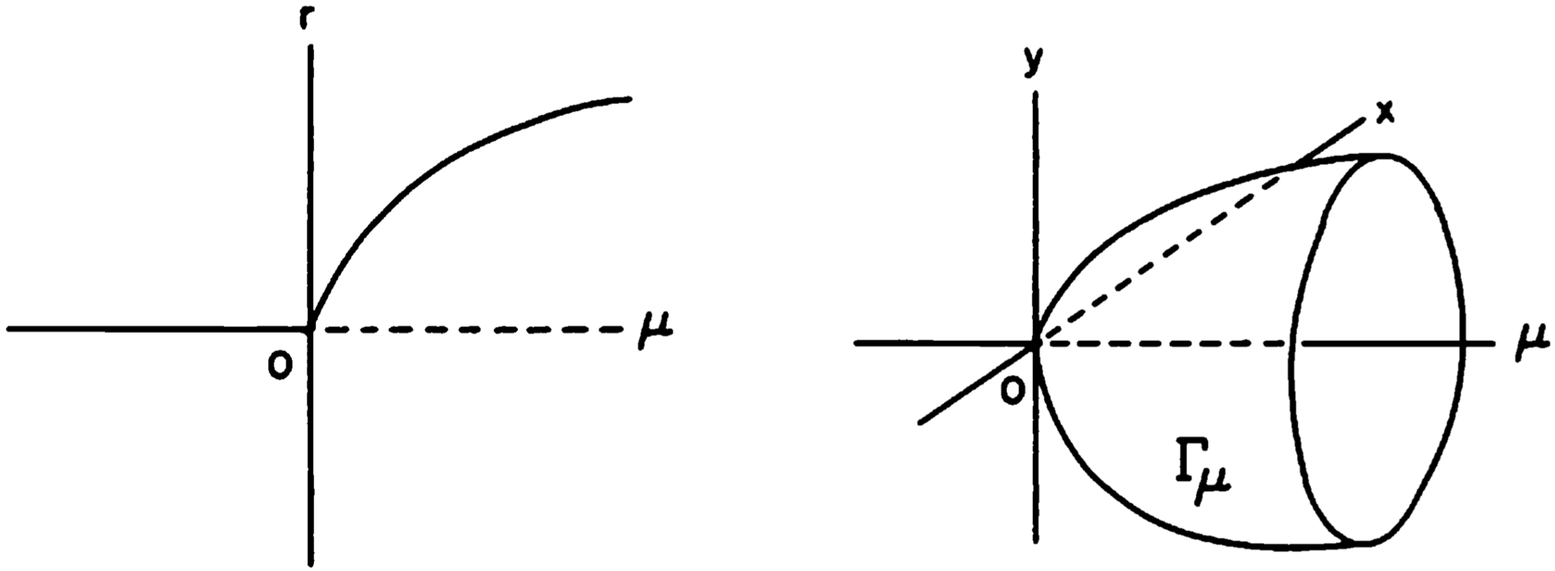


Figure 42: The periodic orbit associated with μ is shown by Γ_μ. This periodic orbit appears only when $\mu > 0$.

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