4.2. Matrix exponentials: properties and examples

PROPOSITION 4.10. *Let A and B be real $n \times n$ matrices. Then*

(i) if $AB = BA$, then $e^A B = B e^A$.

(ii) if $AB = BA$, then $e^{A+B} = e^A e^B$.

(iii) $\left(e^A\right)^{-1} = e^{-A}$.

$e^{x+y} = e^x e^y$ $\quad x, y \in \mathbb{R}$

Exercise 4.11. *Prove Proposition 4.10.*

Example 4.12. *Consider a diagonal $n \times n$ real matrix $A = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$, where $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$. Similar conclusion as in Example 4.4 gives*

$$e^A = \begin{pmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{pmatrix} \tag{4.14}$$

COROLLARY 4.13. *If follows from Example 4.12 (take $\lambda_1 = \cdots = \lambda_n = 0$) that if A is the zero matrix, then $e^A = I$, where I is the identity matrix.*

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$$

$$\underbrace{e^{At} x_0}_{x(t)}$$

A $$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$\underbrace{e^{A}}_{} = I + \frac{A}{1!} + \frac{A^2}{2!} + \cdots$$

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix} \rightsquigarrow e^{A} = \begin{pmatrix} e^{\lambda_1} & & 0 \\ & e^{\lambda_2} & \\ 0 & & \ddots \\ & & & e^{\lambda_n} \end{pmatrix}$$

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \qquad \begin{cases} \dot{x}_1 = \lambda_1 x \\ \dot{x}_2 = \lambda_2 x_2 \end{cases}$$

$$e^{At} = \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{pmatrix}$$

Example 4.14. *Let λ and γ be real numbers and consider*

$\lambda I M = M \lambda I$

$$\boxed{A = \begin{pmatrix} \lambda & \gamma \\ 0 & \lambda \end{pmatrix}.} \tag{4.15}$$

e^A $A = \underbrace{\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}}_{\lambda I} + \underbrace{\begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix}}_{M}$

In this example, we show that

$$\boxed{e^A = e^\lambda \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}.} \tag{4.16}$$

Write $A = \lambda I + M$, where I is the identity matrix and $M = \left(\begin{smallmatrix} 0 & \gamma \\ 0 & 0 \end{smallmatrix}\right)$. The matrices M and λI commute (we say two matrices P and Q commute if $PQ = QP$). By Proposition 4.10, we have $e^A = e^{\lambda I + M} = e^{\lambda I} e^M$.

In Example 4.4, we have shown that

$$e^{\lambda I} = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix}. \tag{4.17}$$

On the other hand, we have $M^2 = 0$, and therefore $M^k = 0$ for all integer $k \geq 0$. This yields

$$e^M = \sum_{k=0}^{\infty} \frac{M^k}{k!} = I + M = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}. \tag{4.18}$$

$M^2 = 0$

We have

$$e^A = e^{\lambda I} e^M = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix} \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix} = e^\lambda \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}. \tag{4.19}$$

$M^3 = 0$

$\underbrace{e^{\lambda I + M}}_{} = e^{\lambda I} e^M$

$e^A = \begin{pmatrix} e^\lambda & 0 \\ 0 & e^\lambda \end{pmatrix} \cdot e^M$

$e^M = I + M + \frac{M^2}{2!} + \cdots = I + M$

$e^M = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}$

Example 4.15. *Let a and b be real numbers and consider*

$a, b \in \mathbb{R}$

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}. \tag{4.20}$$

$b \neq 0$

We show that

$\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$

$$e^A = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}. \tag{4.21}$$

Let $\lambda = a + bi$, where $i = \sqrt{-1}$. Thus

$$A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,(\lambda) & -\mathrm{Im}\,(\lambda) \\ \mathrm{Im}\,(\lambda) & \mathrm{Re}\,(\lambda) \end{pmatrix}. \tag{4.22}$$

Note that, $\lambda^2 = (a + bi)^2 = a^2 - b^2 + 2abi$. *Therefore*

$$A^2 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} a^2 - b^2 & -2ab \\ 2ab & a^2 - b^2 \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\,(\lambda^2) & -\mathrm{Im}\,(\lambda^2) \\ \mathrm{Im}\,(\lambda^2) & \mathrm{Re}\,(\lambda^2) \end{pmatrix}. \tag{4.23}$$

Inductively, for any integer $k > 0$, we can show that

$$A^k = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}^k = \begin{pmatrix} \mathrm{Re}\left(\lambda^k\right) & -\mathrm{Im}\left(\lambda^k\right) \\ \mathrm{Im}\left(\lambda^k\right) & \mathrm{Re}\left(\lambda^k\right) \end{pmatrix}. \tag{4.24}$$

We have

$$\begin{aligned} e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} = \sum_{k=0}^{\infty} \begin{pmatrix} \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) & -\mathrm{Im}\left(\frac{\lambda^k}{k!}\right) \\ \mathrm{Im}\left(\frac{\lambda^k}{k!}\right) & \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) \end{pmatrix} &= \begin{pmatrix} \sum_{k=0}^{\infty} \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) & -\sum_{k=0}^{\infty} \mathrm{Im}\left(\frac{\lambda^k}{k!}\right) \\ \sum_{k=0}^{\infty} \mathrm{Im}\left(\frac{\lambda^k}{k!}\right) & \sum_{k=0}^{\infty} \mathrm{Re}\left(\frac{\lambda^k}{k!}\right) \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{Re}\left(e^{\lambda}\right) & -\mathrm{Im}\left(e^{\lambda}\right) \\ \mathrm{Im}\left(e^{\lambda}\right) & \mathrm{Re}\left(e^{\lambda}\right) \end{pmatrix} = e^a \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}. \end{aligned} \tag{4.25}$$

Note that, in the last equality, we used the Euler's formula: for any real number x, we have $e^{ix} = \cos x + i \sin x$. Thus, for $\lambda = a + ib$, we get $e^{\lambda} = e^a e^{ib} = e^a (\cos b + i \sin b)$.

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$$

$$e^{a+bi} = e^a e^{bi}$$

$$\lambda = a + bi$$

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \quad \begin{pmatrix} \lambda & \gamma \\ 0 & \lambda \end{pmatrix}, \quad \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

4.3. Matrix exponentials: the key idea of calculation

A natural question that may arise here is that how we can calculate e^A for an arbitrary matrix A. The key idea is as follows. Let P be an invertible matrix, and consider $B := P^{-1}AP$. Then, for any integer $k > 0$, we have

$$B^k = \left(P^{-1}AP\right)^k = \overbrace{\left(P^{-1}AP\right)\cdots\left(P^{-1}AP\right)}^{k \text{ times}} = P^{-1}A\cancel{P}\cancel{P^{-1}}A\cancel{P}\cancel{P^{-1}}A\cancel{P}\cdots\cancel{P^{-1}}AP = P^{-1}A^kP \tag{4.26}$$

which implies $A^k = PB^kP^{-1}$. Thus,

$$e^B$$

$$e^A = \sum_{k=0}^{\infty}\frac{A^k}{k!} = \sum_{k=0}^{\infty}\frac{PB^kP^{-1}}{k!} = P\left(\sum_{k=0}^{\infty}\frac{B^k}{k!}\right)P^{-1} = Pe^BP^{-1}. \tag{4.27}$$

What relation (4.27) suggests is that if, for a given A, we can find B such that $B = P^{-1}AP$, for some invertible matrix P, and computing e^B be easy, then we can find e^A using relation (4.27), i.e. $e^A = Pe^BP^{-1}$. For example, if A is diagonalizable, we can choose B to be a diagonal matrix and then use Example 4.12.

Remark 4.16. *Most of the matrices are diagonalizable. For non-diagonalizable matrices, the matrix B can be chosen to be the Jordan form of A. In this course, we deal with non-diagonalizable case for 2×2 matrices and refer the reader to [VS18] for higher dimensional case.*

$$A \xrightarrow[P]{} \boxed{B = P^{-1}AP} \Rightarrow$$

$$B^2 = P^{-1}A\cancel{P}\cancel{P^{-1}}AP = P^{-1}A^2P$$

$$B^3 = P^{-1}A\cancel{P}\cancel{P^{-1}}A\cancel{P}\cancel{P^{-1}}AP = P^{-1}A^3P$$

$$B^k = P^{-1}A^kP \Rightarrow \boxed{A^k = PB^kP^{-1}}$$

$$A \sim \begin{pmatrix}\lambda_1 & & 0\\ & \lambda_2 & \\ 0 & & \lambda_n\end{pmatrix}$$

4.4. Planar linear systems

In this section, we study the dynamics of

$$\boxed{\dot{x} = Ax,} \qquad A = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \tag{4.28}$$

where A is a 2×2 real matrix. Our approach is based on the following lemma

LEMMA 4.17. *For a given $A \in \mathbb{R}^{2\times 2}$, there exists an invertible $P \in \mathbb{R}^{2\times 2}$ such that $B = P^{-1}AP$ has one of the following forms*

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \qquad B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \qquad or \qquad B = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \tag{4.29}$$

where λ, μ, a and $b \neq 0$ are real.

Proof. This lemma is the Jordan form theorem for the particular case of 2-dimensional matrices. See [Per01], Jordan canonical form theorem (Section 1.8). □

Let B and P be as in Lemma 4.17, and define the change of variables $y = P^{-1}x$. Thus, $y \in \mathbb{R}^2$ and $x = Py$. Then

$$\dot{y} = P^{-1}\dot{x} = P^{-1}Ax = P^{-1}APy = By. \tag{4.30}$$

This relation together with Lemma 4.17 suggests that by a linear change of variables, any given linear planar system $\dot{x} = Ax$ can be reduced to a system $\dot{y} = By$, where B is one of the three matrices given by Lemma 4.17.

$\dot{y} = By$

$y \in \mathbb{R}^2$

$\dot{x} = Ax$

$y = P^{-1}x \Rightarrow \dot{y} = P^{-1}\dot{x} = P^{-1}Ax = \underbrace{P^{-1}AP}_{B}y$

$\hookrightarrow x = Py$

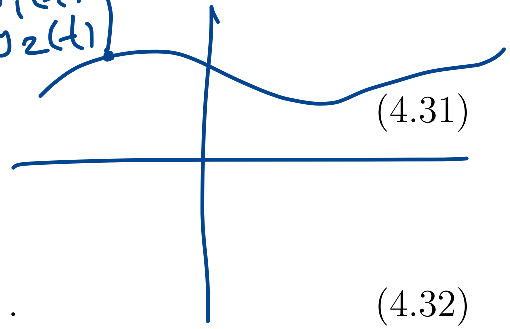
$\hookrightarrow \dot{y} = By$

4.4.1 Case I: $\boxed{B = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)}$

$\dot{y} = By$ $\quad e^{Bt}$ $\quad y(0) = (y_{10}, y_{20})$

In this section, we study the system $\dot{y} = By$, where $B = \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \mu \end{smallmatrix}\right)$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Consider an initial point $(y_{10}, y_{20}) \in \mathbb{R}^2$. The solution of $\dot{y} = By$ passing through this initial point at $t = 0$ is

$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$



$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{Bt} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{pmatrix} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} y_{10} \\ e^{\mu t} y_{20} \end{pmatrix}. \tag{4.31}$$

Assume λ, y_{10} and y_{20} are non-zero. Then,

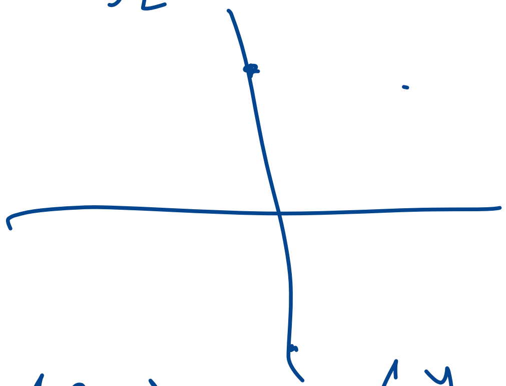
$$y_2(t) = e^{\mu t} y_{20} = \left(e^{\lambda t}\right)^{\frac{\mu}{\lambda}} y_{20} = \left(e^{\lambda t} y_{10}\right)^{\frac{\mu}{\lambda}} y_{10}^{-\frac{\mu}{\lambda}} y_{20} = y_{10}^{-\frac{\mu}{\lambda}} y_{20} \left[y_1(t)\right]^{\frac{\mu}{\lambda}}. \tag{4.32}$$

This means that for the case that λ, y_{10} and y_{20} are non-zero, the orbit of (y_{10}, y_{20}) lies in the set

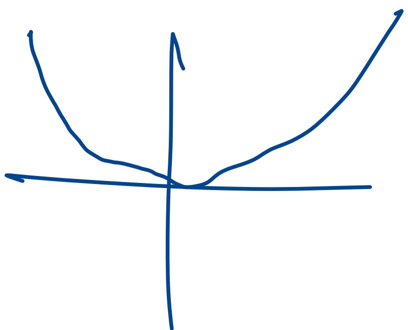
$$\{(y_1, y_2): \quad y_2 = y_{10}^{-\frac{\mu}{\lambda}} y_{20} y_1^{\frac{\mu}{\lambda}}\}. \tag{4.33}$$

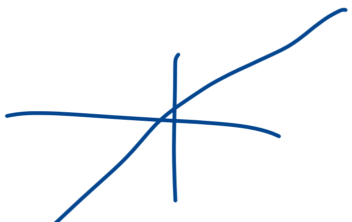
When $\lambda \neq 0$ but $y_{10} = 0$, it follows from (4.31) that $y_1(t) = 0$ for all $t \in \mathbb{R}$. This implies that the orbit of $(0, y_{20})$ is the positive side of y_2-axis if $y_{20} > 0$, the negative side of y_2-axis if $y_{20} < 0$, and the origin if $y_{20} = 0$. Similarly, when $\lambda \neq 0$ but $y_{20} = 0$, it follows from (4.31) that $y_2(t) = 0$ for all $t \in \mathbb{R}$. Thus, the orbit of $(y_{10}, 0)$ is the positive side of y_1-axis if $y_{10} > 0$, and the negative side of y_1-axis if $y_{10} < 0$. This analysis also implies that the vertical and horizontal axes are invariant with respect to the dynamics.

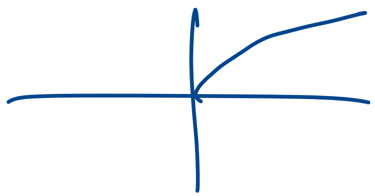
$y_2 = y_1^2$ $\qquad y_2 = \text{constant} \cdot y_1^{\boxed{\mu/\lambda}}$



$y_2 = y_1$ $\qquad y_2 = y_1^{\frac{1}{2}}$







$\begin{pmatrix} 0 \\ y_{20} \end{pmatrix}$ $\qquad \begin{pmatrix} y_{10} \\ 0 \end{pmatrix}$

In order to figure out the phase portrait of the system $\dot{y} = By$, where $B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, we consider the following scenarios:

(i) $\lambda < 0 < \mu$ or $\mu < 0 < \lambda$.

(ii) $\lambda = \mu > 0$ or $\lambda = \mu < 0$.

(iii) $\mu > \lambda > 0$ or $\mu < \lambda < 0$.

(iv) $\lambda > \mu > 0$ or $\lambda < \mu < 0$.

(v) $\lambda = 0$ or $\mu = 0$.

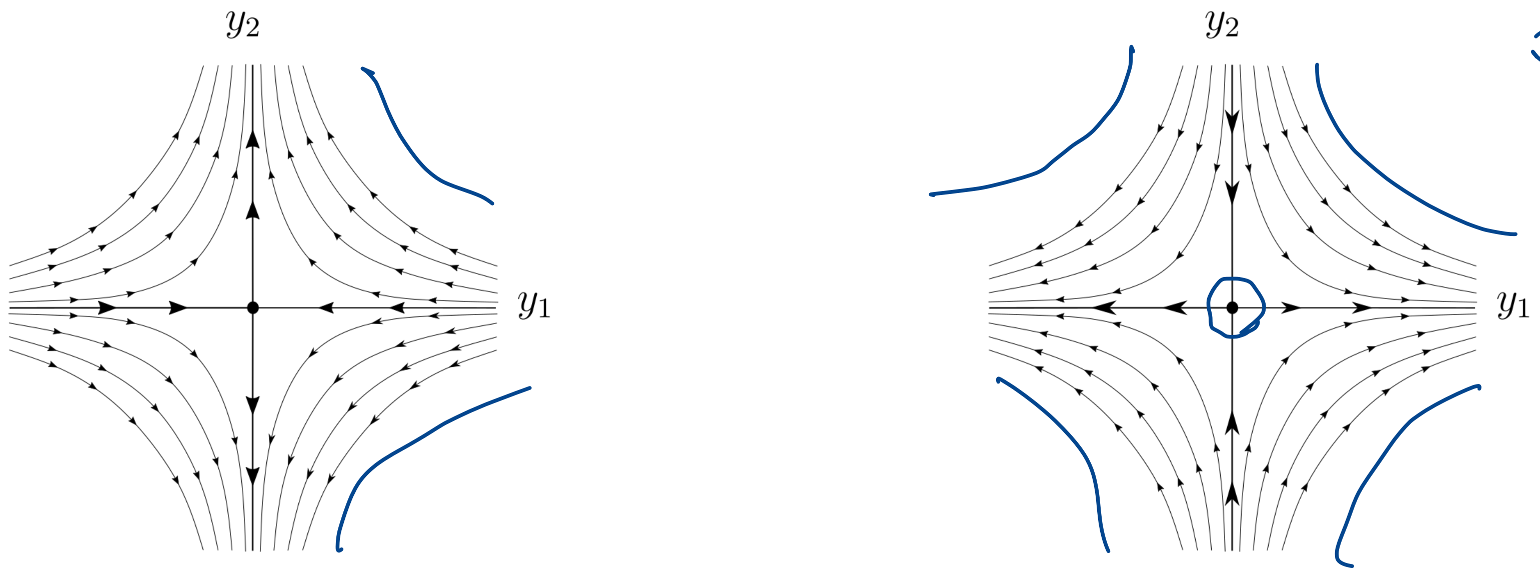
- **Scenario (i):** $\lambda < 0 < \mu$ **or** $\mu < 0 < \lambda$.

In this scenario, $\frac{\mu}{\lambda} < 0$. Define $\beta = \frac{\mu}{\lambda}$. According to (4.33), we need to plot the curves of the form $y_2 = \text{constant} \cdot y_1^{\beta}$, where $\beta < 0$. Taking into account that the horizontal and vertical axes are invariant, we can plot the phase portrait of the system for this scenario (see Figure 12).

- In this scenario, two orbits approach the origin as $t \to \infty$ and two other orbits approach the origin as $t \to -\infty$.
- The equilibrium point at the origin in such scenarios is called a *saddle* point.

$y_2 = Cnt\ y_1^{\mu/\lambda}$

$\dot{y}_2 = \mu y_2$



$t \to -\infty$

$t \to \infty$

(a) $\lambda < 0 < \mu$. (b) $\mu < 0 < \lambda$.

Figure 12: Phase portrait of scenario (i).

- **Scenario (ii):** $\lambda = \mu > 0$ **or** $\lambda = \mu < 0$.

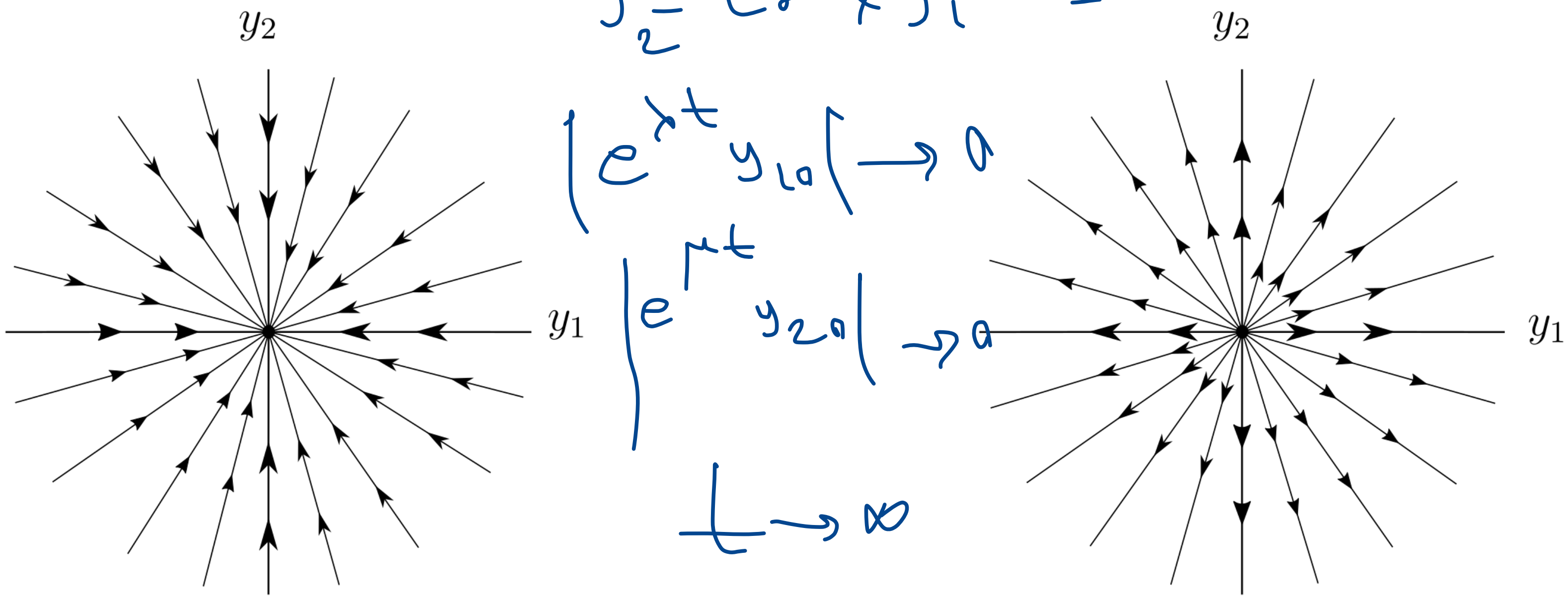
In this scenario, $\frac{\mu}{\lambda} = 1$. According to (4.33), we need to plot the straight lines $y_2 = \text{constant} \cdot y_1$. Taking into account that the horizontal and vertical axes are invariant, we can plot the phase portrait of the system for this scenario (see Figure 13).

$y_2 = \text{c} \times y_1^{\mu/\lambda} = \text{con} \times y_1$

$|e^{\lambda t} y_{10}| \to 0$

$|e^{\mu t} y_{20}| \to 0$

$t \to \infty$



(a) $\lambda = \mu < 0$.

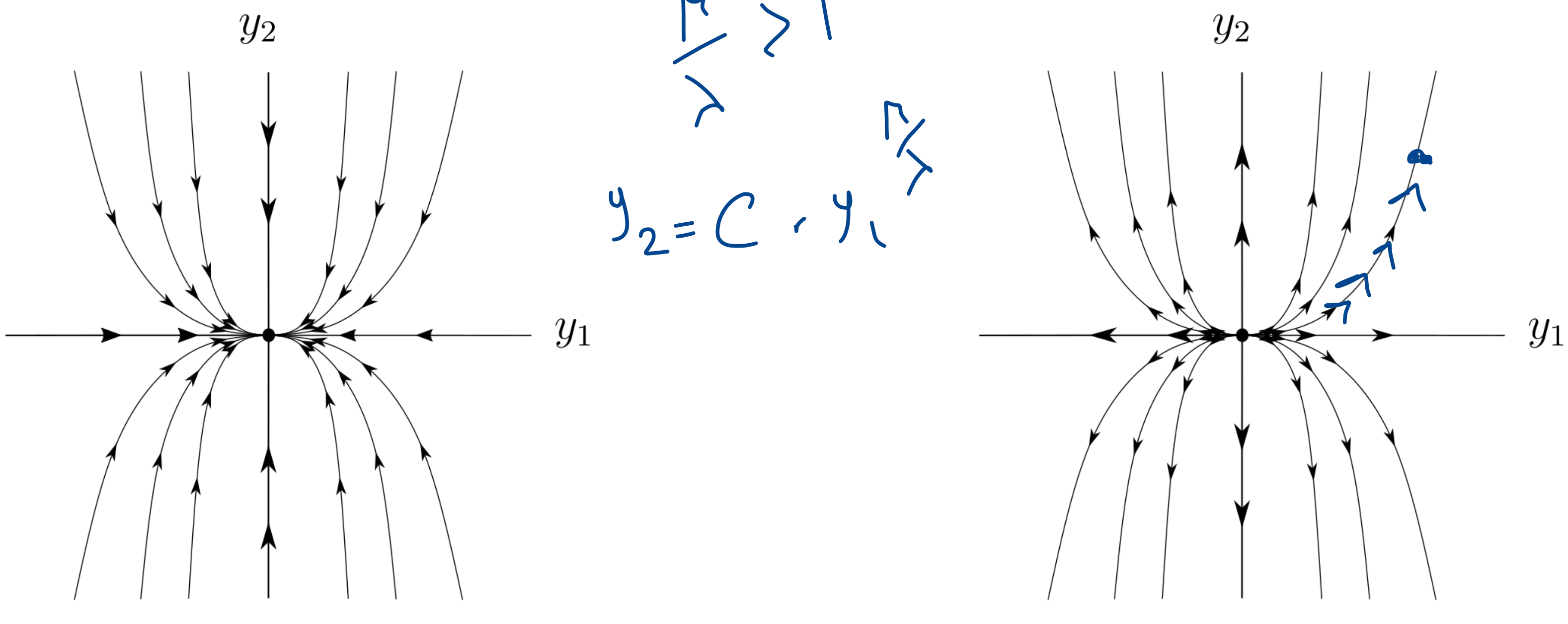
(b) $\lambda = \mu > 0$.

Figure 13: Phase portrait of scenario (ii).

$t \to -\infty$

- **Scenario (iii): $\mu > \lambda > 0$ or $\mu < \lambda < 0$.**

In this scenario, $\frac{\mu}{\lambda} > 1$. Define $\beta = \frac{\mu}{\lambda}$. According to (4.33), we need to plot the curves of the form $y_2 = \text{constant} \cdot y_1^{\beta}$, where $\beta > 1$. Taking into account that the horizontal and vertical axes are invariant, we can plot the phase portrait of the system for this scenario (see Figure 14).



$\frac{\mu}{\lambda} > 1$

$y_2 = C \cdot y_1^{\mu/\lambda}$

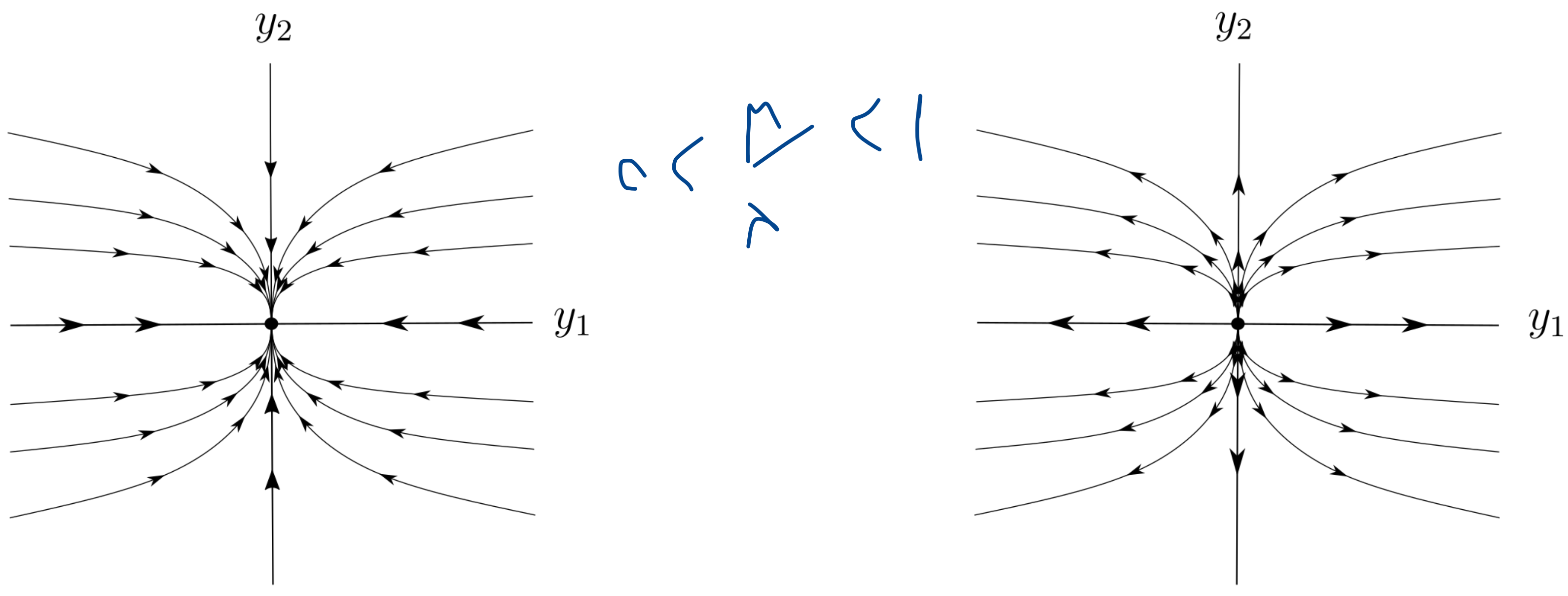
(a) Case $\mu < \lambda < 0$. (b) Case $\mu > \lambda > 0$.

Figure 14: Phase portrait of scenario (iii).

$t \to \infty$

- **Scenario (iv): $\lambda > \mu > 0$ or $\lambda < \mu < 0$.**

In this scenario, $0 < \frac{\mu}{\lambda} < 1$. Define $\beta = \frac{\mu}{\lambda}$. According to (4.33), we need to plot the curves of the form $y_2 = \text{constant} \cdot y_1^{\beta}$, where $0 < \beta < 1$. Taking into account that the horizontal and vertical axes are invariant, we can plot the phase portrait of the system for this scenario (see Figure 15).



$0 < \frac{\mu}{\lambda} < 1$

(a) Case $\lambda < \mu < 0$.

(b) Case $\lambda > \mu > 0$.

Figure 15: Phase portrait of scenario (iv).

- **Scenario (v): $\lambda = 0$ or $\mu = 0$.**

$e^{\lambda t} = e^{0 t} = 1$

Assume $\lambda = 0$ and $\mu \neq 0$. Recall from (4.31) that

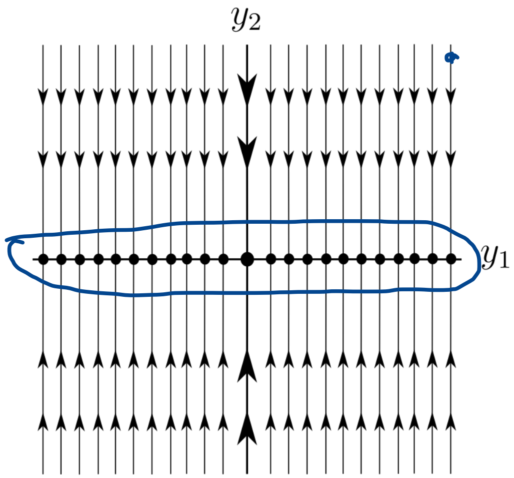
$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} e^{\lambda t} y_{10} \\ e^{\mu t} y_{20} \end{pmatrix} = \begin{pmatrix} y_{10} \\ e^{\mu t} y_{20} \end{pmatrix} \tag{4.34}$$

Suppose $\lambda = 0$ and observe that any point on the y_1-axis is an equilibrium. Moreover, by (4.34), we have $(y_1(t), y_2(t)) = (y_{10}, e^{\mu t} y_{20})$. This suggests that when $\lambda = 0$, the orbit of (y_{10}, y_{20}) is the positive side of the vertical line $y_1 = y_{10}$ if $y_{20} > 0$, the negative side of the vertical line $y_1 = y_{10}$ if $y_{20} < 0$, and the point $(y_{10}, 0)$ if $y_{20} = 0$. By this analysis, we have the phase portrait for the case $\lambda = 0$ and $\mu \neq 0$ as in Figure 16. By an analogous analysis, we obtain the phase portrait of the case $\lambda \neq 0$ and $\mu = 0$ as in Figure 17.

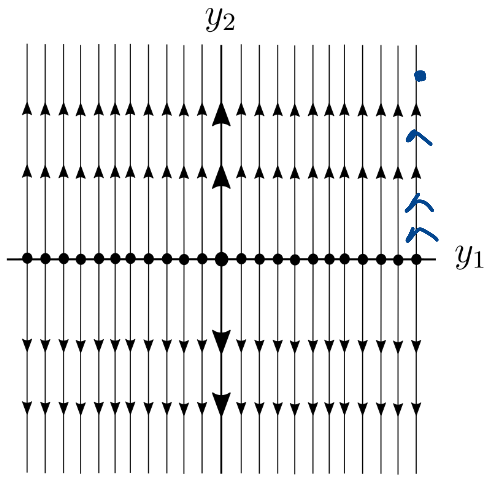
$\lambda = 0 \quad \mu \neq 0$

$B = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

$\begin{pmatrix} y_{10} \\ 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} y_{10} \\ 0 \end{pmatrix} \rightsquigarrow$ equil

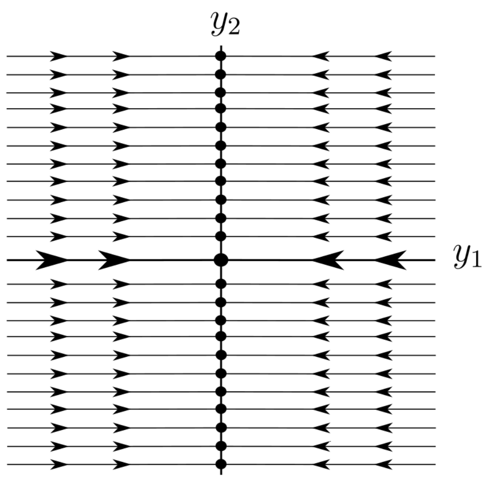


(a) Case $\lambda = 0$ and $\mu < 0$.

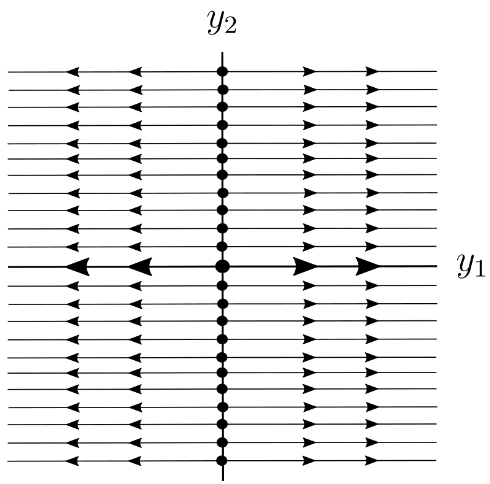


(b) Case $\lambda = 0$ and $\mu > 0$.

Figure 16: Phase portrait of scenario (v).



(a) Case $\mu = 0$ and $\lambda < 0$.



(b) Case $\mu = 0$ and $\lambda > 0$.

$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$

Figure 17: Phase portrait of scenario (v).

4.4.2 Case II: $B = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$

In this section, we study the system $\dot{y} = By$, where $B = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \lambda \end{smallmatrix}\right)$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Consider an initial point $(y_{10}, y_{20}) \in \mathbb{R}^2$. The solution of $\dot{y} = By$ passing through this initial point at $t = 0$ is

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{Bt} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix} = \begin{pmatrix} e^{\lambda t} y_{10} + e^{\lambda t} t y_{20} \\ e^{\lambda t} y_{20} \end{pmatrix}. \tag{4.35}$$

In order to figure out the phase portrait of the system $\dot{y} = By$, where $B = \left(\begin{smallmatrix} \lambda & 1 \\ 0 & \mu \end{smallmatrix}\right)$, we consider the following scenarios:

(i) $\lambda \neq 0$.

(ii) $\lambda = 0$.

- **Scenario (i): $\lambda \neq 0$.**

Assume λ and y_{20} are non-zero. From the equation $y_2(t) = e^{\lambda t} y_{20}$, we obtain

$$t = \frac{1}{\lambda} \ln \frac{y_2(t)}{y_{20}}. \tag{4.36}$$

On the other hand, $\frac{y_1(t)}{y_2(t)} = \frac{y_{10}}{y_{20}} + t$. Thus, by (4.36), we have

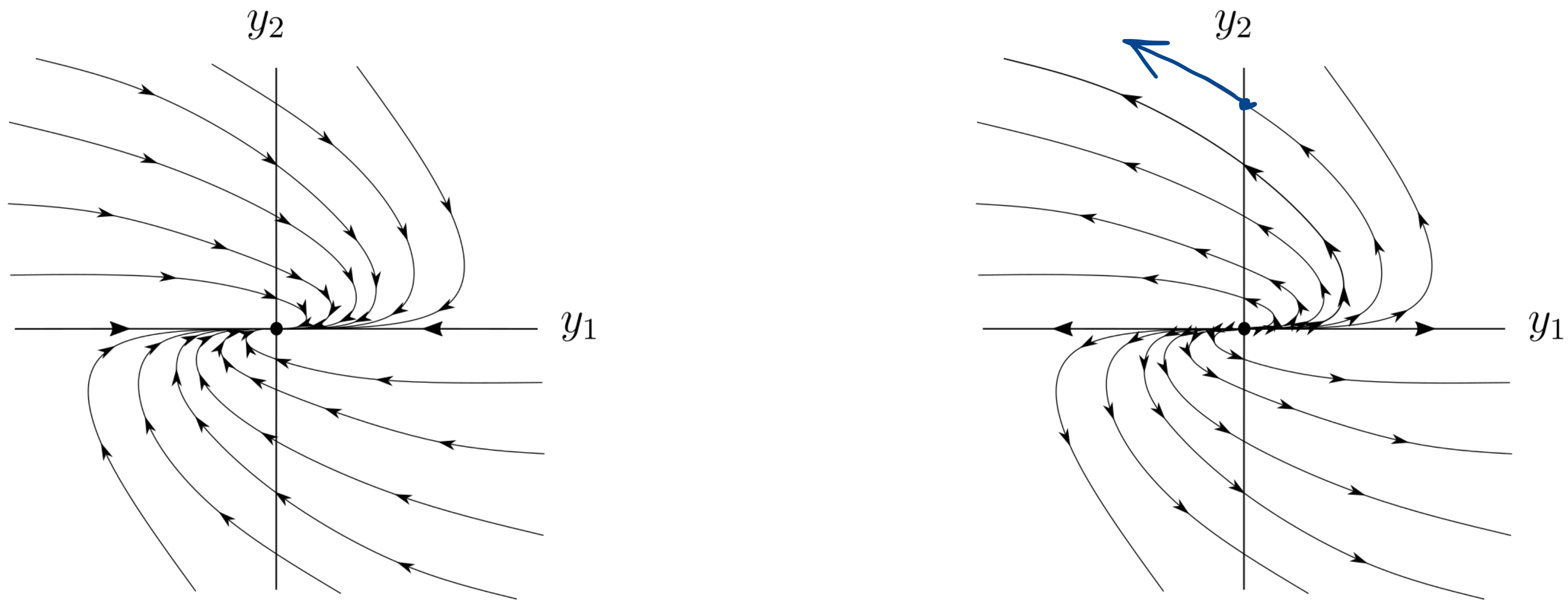
$$\frac{y_1(t)}{y_2(t)} = \frac{y_{10}}{y_{20}} + \frac{1}{\lambda} \ln \frac{y_2(t)}{y_{20}} = \left[\frac{y_{10}}{y_{20}} - \frac{1}{\lambda} \ln y_{20} \right] + \frac{1}{\lambda} \ln y_2(t), \tag{4.37}$$

which gives

$$y_1(t) = \frac{y_{10}}{y_{20}} + \frac{1}{\lambda} \ln \frac{y_2(t)}{y_{20}} = \left[\frac{y_{10}}{y_{20}} - \frac{1}{\lambda} \ln y_{20} \right] y_2(t) + \frac{1}{\lambda} y_2(t) \ln y_2(t). \tag{4.38}$$

Thus, to plot the phase portrait of this scenario, we need to consider the curves of the form $y_1 = \alpha y_2 + \frac{1}{\lambda} y_2 \ln y_2$, where α is some constant. This is also easily seen that the horizontal axis $y_2 = 0$ is invariant, and these curves are tangent to the horizontal axis at the origin. This analysis gives Figure 18.

Remark 4.18. *Note that the y_1-axis is invariant while the y_2-axis is not.*



(a) Case $\lambda < 0$.

(b) Case $\lambda > 0$.

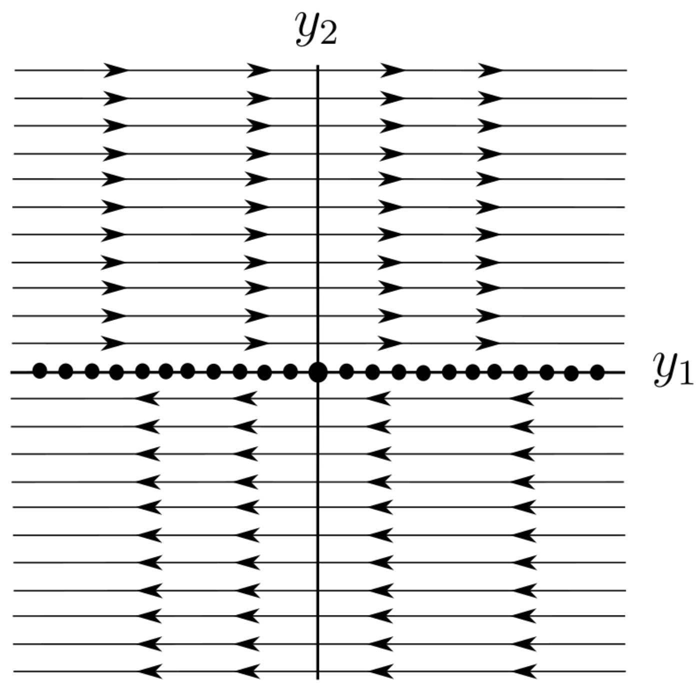
Figure 18: Scenario (i): $\lambda \neq 0$

- **Scenario (ii):** $\lambda = 0$.

By (4.35), when $\lambda = 0$, we have

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} y_{10} + ty_{20} \\ y_{20} \end{pmatrix}. \tag{4.39}$$

This implies that the horizontal lines $y_2 =$ constant are invariant. the phase portrait for this scenario is given in Figure 19.



$$\begin{pmatrix} a & 1 \\ 0 & 0 \end{pmatrix}$$

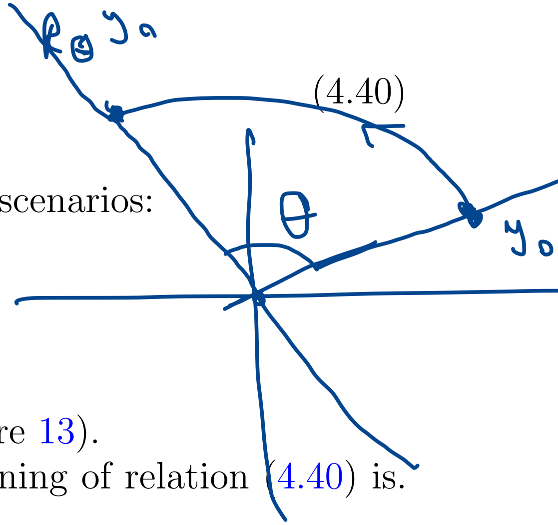
Figure 19: Phase portrait of scenario (ii): $\lambda = 0$.

4.4.3 Case III: $B = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$

In this section, we study the system $\dot{y} = By$, where $B = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$ and $y = (y_1, y_2) \in \mathbb{R}^2$. Consider an initial point $(y_{10}, y_{20}) \in \mathbb{R}^2$. The solution of $\dot{y} = By$ passing through this initial point at $t = 0$ is

Bt

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix} \begin{pmatrix} y_{10} \\ y_{20} \end{pmatrix}. \tag{4.40}$$



In order to figure out the phase portrait of the system $B = \left(\begin{smallmatrix} a & -b \\ b & a \end{smallmatrix}\right)$, we consider the following scenarios:

(i) $a = 0$.

$\|R_\theta y_0\| = \|y_0\|$

(ii) $a \neq 0$.

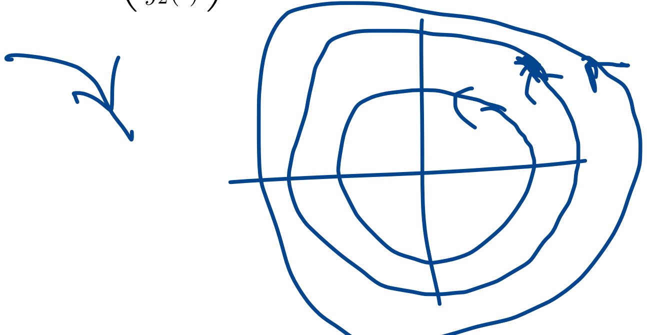
Note that when $b = 0$, we have $B = \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right)$ which is the case that was studied earlier (see Figure 13).

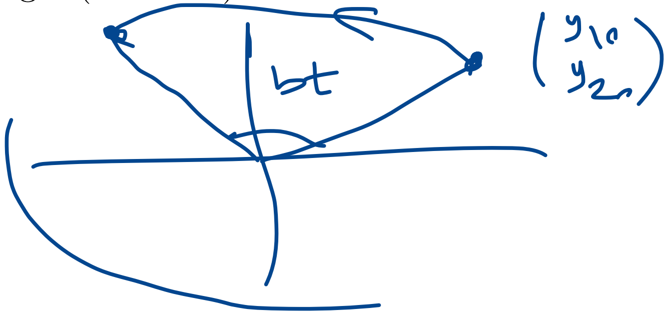
Before we proceed to study the above scenarios, let us first see what the geometrical meaning of relation (4.40) is. Let $\theta \in \mathbb{R}$, and consider the matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{4.41}$$

- The matrix R_θ is called a rotation matrix. This matrix rotates the points in the plane about the origin by the angle θ (see e.g. [Mey00]). The rotation is counter-clockwise when $\theta > 0$, and clockwise when $\theta < 0$.

- In (4.40), the matrix $\left(\begin{smallmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{smallmatrix}\right)$ rotates $\left(\begin{smallmatrix} y_{10} \\ y_{20} \end{smallmatrix}\right)$ by the angle bt. Thus, as t increases, this rotation is counter-clockwise if $b > 0$, and clockwise if $b < 0$. Then, after this rotation, the coefficient e^{at} in (4.40) controls the size of $\left(\begin{smallmatrix} y_1(t) \\ y_2(t) \end{smallmatrix}\right)$. In other words, b controls the angle (rotation) and a controls the size of $\left(\begin{smallmatrix} y_1(t) \\ y_2(t) \end{smallmatrix}\right)$.





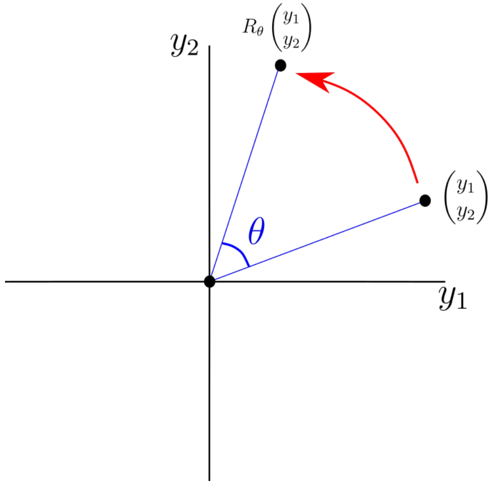
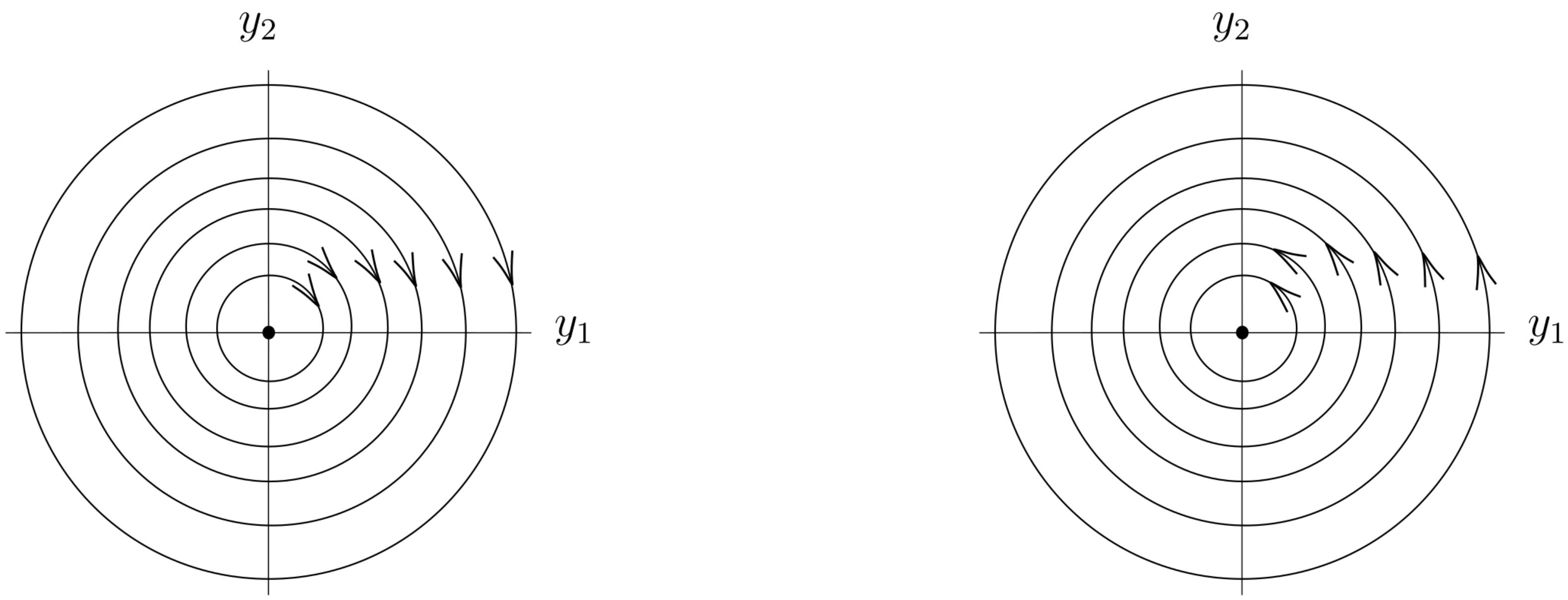


Figure 20: R_θ rotates the points in the plane by the angle θ about the origin.



(a) $b < 0$.

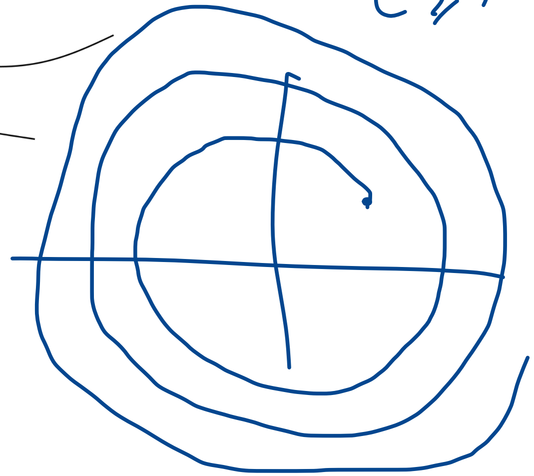
(b) $b > 0$.

Figure 21: Scenario (i): $a = 0$.

(a) $b < 0$. (b) $b > 0$.

Figure 22: Scenario (ii): $a > 0$

$a > 0$
$e^{at} > 1$
$t > 0$



(a) $b < 0$. (b) $b > 0$.

Figure 23: Scenario (ii): $a < 0$

References

[Arn92] V. Arnold. *Ordinary Differential Equations*. Springer Verlag Textbook, third edition, 1992.

[HSD12] M. W. Hirsch, S. Smale, and R. L. Devaney. *Differential equations, dynamical systems, and an introduction to chaos*. Academic press, 2012.

[Mey00] C. D. Meyer. *Matrix analysis and applied linear algebra*, volume 71. SIAM, 2000.

[Per01] L. Perko. *Differential equations and dynamical systems*. Springer-Verlag, third edition, 2001.

[VS18] S. Van Strien. *Lecture notes on ODEs*. available at https://www.ma.imperial.ac.uk/ svanstri/Files/de-4th.pdf, Spring 2018.