University POLITEHNICA of Bucharest

Applied Mathematics in Optimization Problems

Course

Chapter

Assignment Problem

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1. General considerations

2. Solving Hungarian method

1. General considerations

In economic practice there are many problems of assignment such as the assignment of several tasks on machines, modules, personnel, et al. Tasks must be performed simultaneously by means that can perform them with different efficiencies .

If the costs or times of the assignment of the tasks of each of the available means are known, the optimal distribution must be found so that the total cost, respectively the total time required is minimal.

If the productivity or profit of the assignment is known, the optimal distribution must be determined so that the productivity, respectively, the total profit, is maximum.

From a mathematical point of view, a problem of assignment consists in the coupling of n elements of one set with n elements of another set. Basically there are n! different coupling possibilities.

The known data of these problems, of the type mentioned, can be considered the elements of a quadratic matrix $C = (c_{ij}) \in \mathcal{M}_{n \times n}$, with $c_{ij} \geq 0$, also called the *coupling matrix*.

Remarks: If the coupling between an element i in one set and the element j in the other set is not possible, then in the matrix C it takes $c_{ij} = \infty$.

Note the unknowns with x_{ij} $(i, j = 1,\ldots, n)$:

$$x_{ij} = \begin{cases} 1, & \text{if the mean } i \text{ is assigned to the task } j \\ 0, & \text{otherwise} \end{cases}$$

The unknowns can be considered the elements of a square matrix $X = (x_{ij}) \in \mathcal{M}_{n \times n}$.

The mathematical model of an assignment problem has the form :

$$\begin{cases} [\min]([\max])\ z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \sum_{i=1}^{n} x_{ij} = 1,\ \ j = 1,\dots, n \\ \sum_{j=1}^{n} x_{ij} = 1,\ \ i = 1,\dots, n \\ x_{ij} \in \{0, 1\},\ \ i,\ j = 1,\dots, n \end{cases}$$

Remark: Each row and column of the matrix X contains a single element equal to 1, the others being null.

From the matrix C, n elements c_{ij}, must be chosen, so that any two elements are not in the same row or column in C. The sum of the chosen elements must be minimum or maximum depending on the objective of the problem. .

The empirical method of establishing the optimal distribution: calculating the sums of those of the $n!$ variants of choice, depending on which the optimal variant is chosen.

The disadvantage of the empirical method: even for small dimensions of the problem a significant volume of calculations is required.

The most efficient method of solving the assignment problems is the algorithm proposed by H. W. Kuhn, known as *Hungarian method*.

2. Solving Hungarian method

The set of rows and columns that contain all the zeros of the matrix C represent the *support of the matrix*.

The support that contains a minimum number of rows and columns is called the *minimum support*.

Algorithm *for the minimization problems*:

Step 1. (*Creating zeros*)

Transform the matrix C into a matrix $C^{(1)}$ containing at least one zero on each row and on each column so:

1) For each column, the smallest element is subtracted from all the other elements of that column.

2) For each line, which contains no zeros, subtract its smallest element from all the other elements of that line.

Step 2. (*Finding an optimal solution*)

a) It is considered the line with the smallest number of zeros. One of the zeros falls (with $\square$). Cross (with $\diagup$) all zeros located in the line and column of the framed zero.

b) Of the remaining *n* - 1 lines, the line with the lowest number of zeros is considered and proceed as at a).

If the number of framed zeros is *n*, then the solution is the coupling defined by these zeros, STOP. Otherwise, go to step 3 .

Step 3. (*Determining a minimum support*)

It is marked laterally with ”* ” :

a) lines that do not contain any framed zeros;

b) columns containing crossed zeros in marked lines ;

c) lines containing at least one zero framed in the marked columns;

d) repeat operations b) and c) until no more rows or columns can be marked.

The minimum support consists of the set of unmarked lines and marked columns. It is highlighted by drawing dashed straight lines on unmarked lines and marked columns.

Step 4. (*Possible movement of zeros*)

a) From the elements that are not crossed by dashed lines, the smallest number is chosen.

b) Subtract that number from items not placed on lines and columns crossed by dashed lines.

c) Add that number to the elements at the intersection of the lines with the columns crossed by dashed lines.

d) The other elements of the matrix do not change.

Thus a new matrix $C^{(2)}$, with a different configuration of the zeros position , is obtained.

Step 5. (*Performing a new iteration*)

Repeat the algorithm from step 1 (if appropriate), otherwise from step 2, for the matrix $C^{(2)}$ until the optimal solution is obtained.

Remarks:

1) The optimal solution may not be unique.

2) If the matrix of coupling C is not square, proceed as follows:

- Fill the matrix, as appropriate, with rows or columns with all elements equal to zero to obtain a square matrix;

➢ The algorithm is applied to the new obtained matrix.

3) For maximization assignment problems, the previous algorithm is applied to the matrix $C' = (c'_{ij}) \in \mathcal{M}_{n \times n}$ whose elements are calculated with the relation:

$$c'_{ij} = \max_{i,\, j} c_{ij} - c_{ij}$$

Example. Let consider the assignment problem of five operations $y_j,\ j=\overline{1,5}$, for execution at five workers $x_i,\ i=\overline{1,5}$. The table contains the necessary times t_{ij} for the worker x_i to perform the operation y_j. If a worker cannot perform a certain operation, $t_{ij}=\infty$.

Determine the assignment of operations among workers so that the total time required for execution is minimal.

	y_1	y_2	y_3	y_4	y_5
x_1	14	8	6	7	6
x_2	5	9	7	4	8
x_3	8	∞	10	9	5
x_4	9	3	11	∞	3
x_5	8	10	7	12	6

Solution. The algorithm for minimizing assignment problems is applied to solve .

Iteration 1

Step 1. Subtract the minimum element from each column and it obtains the table:

	y_1	y_2	y_3	y_4	y_5
x_1	9	5	0	3	3
x_2	0	6	1	0	5
x_3	3	∞	4	5	2
x_4	4	0	5	∞	0
x_5	3	7	1	8	3

In each of lines 3 and 5 subtract the most element (2, respectively, 1) from the rest of the line elements and it obtains the table :

	y_1	y_2	y_3	y_4	y_5
x_1	9	5	0	3	3
x_2	0	6	1	0	5
x_3	1	∞	2	3	0
x_4	4	0	5	∞	0
x_5	2	6	0	7	2

Step 2

Drop the zero from the position c_{13} and cross the zero from the position c_{53}.

Drop the zero from the position c_{35} and cross the zero from the position c_{45}.

Drop the zero from the position c_{42}.

Drop the zero from the position c_{21} and cross the zero from the position c_{24}.

Only four framed zeros are resulted (see next table), so the obtained solution is not optimal.

	y_1	y_2	y_3	y_4	y_5	
x_1	9	5	$\boxed{0}$	3	3	*
x_2	$\boxed{0}$	6	1	$\cancel{0}$	5	
x_3	1	∞	2	3	$\boxed{0}$	
x_4	4	$\boxed{0}$	5	∞	$\cancel{0}$	
x_5	2	6	$\cancel{0}$	7	2	*
			*			

Step 3. It marks with * the line 5, the column 3, then the line 1 (see above tabel).

It cuts the lines 2, 3, 4 and the column 3 with dashed lines (see above table).

Step 4. In the partial table of uncut elements, the smallest element is $c_{51} = 2$. Subtract c_{51} from those elements and add to the elements located at the intersections of dashed lines. It obtains the table:

	y_1	y_2	y_3	y_4	y_5
x_1	7	3	0	1	1
x_2	0	6	3	0	5
x_3	1	∞	4	3	0
x_4	4	0	7	∞	0
x_5	0	4	0	5	0

Iteration 2

Step 1. It is not necessary in this case because each row and each column of the previous table contains at least one zero.

Step 2

Drop the zero from the position c_{13} and cross the zero from the position c_{53}.

Drop the zero from the position c_{35} and cross the zeros from the positions c_{45} and c_{55}.

Drop the zero from the position c_{42}.

Drop the zero from the position c_{51} and cross the zero from the position c_{21}.

Drop the zero from the position c_{24}.

The table has the configuration:

	y_1	y_2	y_3	y_4	y_5
x_1	7	3	$\boxed{0}$	1	1
x_2	$\cancel{0}$	6	3	$\boxed{0}$	5
x_3	1	∞	4	3	$\boxed{0}$
x_4	4	$\boxed{0}$	7	∞	$\cancel{0}$
x_5	$\boxed{0}$	4	$\cancel{0}$	5	$\cancel{0}$

Five framed zeros are obtained, so this assignment is optimal: (x_1, y_3), (x_2, y_4), (x_3, y_5), (x_4, y_2), (x_5, y_1).

The minimum total time, required to perform the operations, corresponding to this assignment, is:

$$t_{\min} = t_{13} + t_{24} + t_{35} + t_{42} + t_{51} = 26 \text{ u.}$$ ❑

Example. A manager has four teams of workers at his disposal to assign them to the execution of three products. The time required (in hours) for each team to execute a product is indicated in the table below.

Choose three teams so that the total execution time of the products is minimal .

	y_1	y_2	y_3	y_4
x_1	15	7	9	8
x_2	11	6	10	8
x_3	12	10	13	5

Solution. The table associated with the problem is not quadratic, and the algorithm cannot be applied.

To reduce the problem to a common case, an additional line, with all elements equal to zero, is added to the table so as not to change the final solution. It obtains the square table below, to which the solving algorithm can be applied .

	y_1	y_2	y_3	y_4
x_1	15	7	9	8
x_2	11	6	10	8
x_3	12	10	13	5
x_4	0	0	0	0

Iteration 1

Step 1. By subtracting, in each of the first three lines of the table, the minimum element from the rest of the elements of that line, it obtains the table:

	y_1	y_2	y_3	y_4
x_1	8	0	2	1
x_2	5	0	4	2
x_3	7	5	8	0
x_4	0	0	0	0

Step 2

Drop the zero from the position c_{12} and cross the zeros from the positions c_{22}, c_{42}.

Drop the zero from the position c_{34} and cross the zero from the position c_{44} .

Drop the zero from the position c_{41} and cross the zero from the position c_{43}.

Only three framed zeros are resulted, so the obtained solution is not optimal:

	y_1	y_2	y_3	y_4
x_1	8	$\boxed{0}$	2	1
x_2	5	$\cancel{0}$	4	2
x_3	7	5	8	$\boxed{0}$
x_4	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$

Step 3

It marks with * the line 2 and the column 2.

It marks with * the line 1.

It cuts the lines 3, 4, and the column 2 with dashed lines (see table below).

	y_1	y_2	y_3	y_4	
x_1	8	$\boxed{0}$	2	1	*
x_2	5	$\cancel{0}$	4	2	*
x_3	7	5	8	$\boxed{0}$	—
x_4	$\boxed{0}$	$\cancel{0}$	$\cancel{0}$	$\cancel{0}$	—
		*			

Step 4

From the partial table of uncut elements, it chooses the smallest $c_{14} = 1$. Subtract it from the uncut columns, add it to the cut lines; it obtains the table:

	y_1	y_2	y_3	y_4
x_1	7	0	1	0
x_2	4	0	3	1
x_3	7	6	8	0
x_4	0	1	0	0

Iteration 2

Step 1. It is not necessary.

Step 2

Drop the zero from the position c_{22} and cross the zero from the position c_{12}.

Drop the zero from the position c_{14} and cross the zeros from the positions c_{34} , c_{44}.

Drop the zero from the position c_{41} and cross the zero from the position c_{43}.

Only three framed zeros are resulted, so the obtained solution is not optimal(see next tabel).

	y_1	y_2	y_3	y_4
x_1	7	$\cancel{0}$	1	$\boxed{0}$
x_2	4	$\boxed{0}$	3	1
x_3	7	6	8	$\cancel{0}$
x_4	$\boxed{0}$	1	$\cancel{0}$	$\cancel{0}$

Step 3

It marks with * the line 3 and the column 4.

It marks with * the line 1 and the column 2.

It marks with * the line 2.

It cuts the line 4 and the columns 2 and 4:

	y_1	y_2	y_3	y_4	
x_1	7	$\cancel{0}$	1	$\boxed{0}$	*
x_2	4	$\boxed{0}$	3	1	*
x_3	7	6	8	$\cancel{0}$	*
x_4	$\boxed{0}$	1	$\cancel{0}$	$\cancel{0}$	
		*		*	

Step 4

From the partial table of uncut elements, it chooses the smallest $c_{13} = 1$. Subtract c_{13} from the uncut columns, add it to the cut lines; it obtains the table :

	y_1	y_2	y_3	y_4
x_1	6	0	0	0
x_2	3	0	2	1
x_3	6	6	7	0
x_4	0	2	0	1

Iteration 3.

Step 1. It is not necessary.

Step 2

Drop the zero from the position c_{22} and cross the zero from the position c_{12}.

Drop the zero from the position c_{34} and cross the zero from the position c_{14}.

Drop the zero from the position c_{13} and cross the zero from the position c_{43}.

Drop the zero from the position c_{41}:

	y_1	y_2	y_3	y_4
x_1	6	$\cancel{0}$	$\boxed{0}$	$\cancel{0}$
x_2	3	$\boxed{0}$	2	1
x_3	6	6	7	$\boxed{0}$
x_4	$\boxed{0}$	2	$\cancel{0}$	1

Four framed zeros are resulted, so the obtained solution is optimal : $(x_1, y_3), (x_2, y_2), (x_3, y_4), (x_4, y_1)$.

How the product x_4 is fictive, it results as optimal solution to the initial problem: choosing teams y_2, y_3 and y_4.

The minimum value of the required total time is:

$$t_{\min} = t_{13} + t_{22} + t_{34} = 20 \text{ hours.} \qquad \square$$