University POLITEHNICA of Bucharest

Applied Mathematics in Optimization Problems

Course

Chapter
Transportation Problem

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1. **Problem formulation. Properties**
2. **Methods for determining an initial solution**
3. **Potential method**
4. **Parametric transportation problem**
5. **Special transportation problems**

1. Problem formulation. Properties

The transportation problem is the most important category of optimal distribution problems.

Formulation of a classical transport problem:

In m manufacturing centers, marked with A_1, A_2 ,..., A_m, there is a certain product, in quantities respectively equal to a_i (i = 1,..., m). The product is required in n consumer centers, marked with B_1, B_2 ,..., B_n, in the quantities equal to b_j (j = 1,…, n) respectively.

The unit transport costs are known c_{ij} from each production center i to each consumer center j.

The cost of transport between any two centers is proportional to the quantity of product transported, being possible the transport of any quantity within the limits of the problem.

It is required to establish a transport plan that ensures as much as possible the quantities needed in the consumer centers, at a minimum total cost of the entire transport .

The data of a transport problem can be summarized in a table in the form below :

$A_i \setminus B_j$	B_1	B_2	...	B_n	a_i
A_1	c_{11}	c_{12}	...	c_{1n}	a_1
A_2	c_{21}	c_{22}	...	c_{2n}	a_2
...	...	...	...	...	...
A_m	c_{m1}	c_{m2}	...	$c_{m\,n}$	a_m
b_j	b_1	b_2	...	b_n	

The graphics network associated with a transport problem is illustrated in the Fig.1:

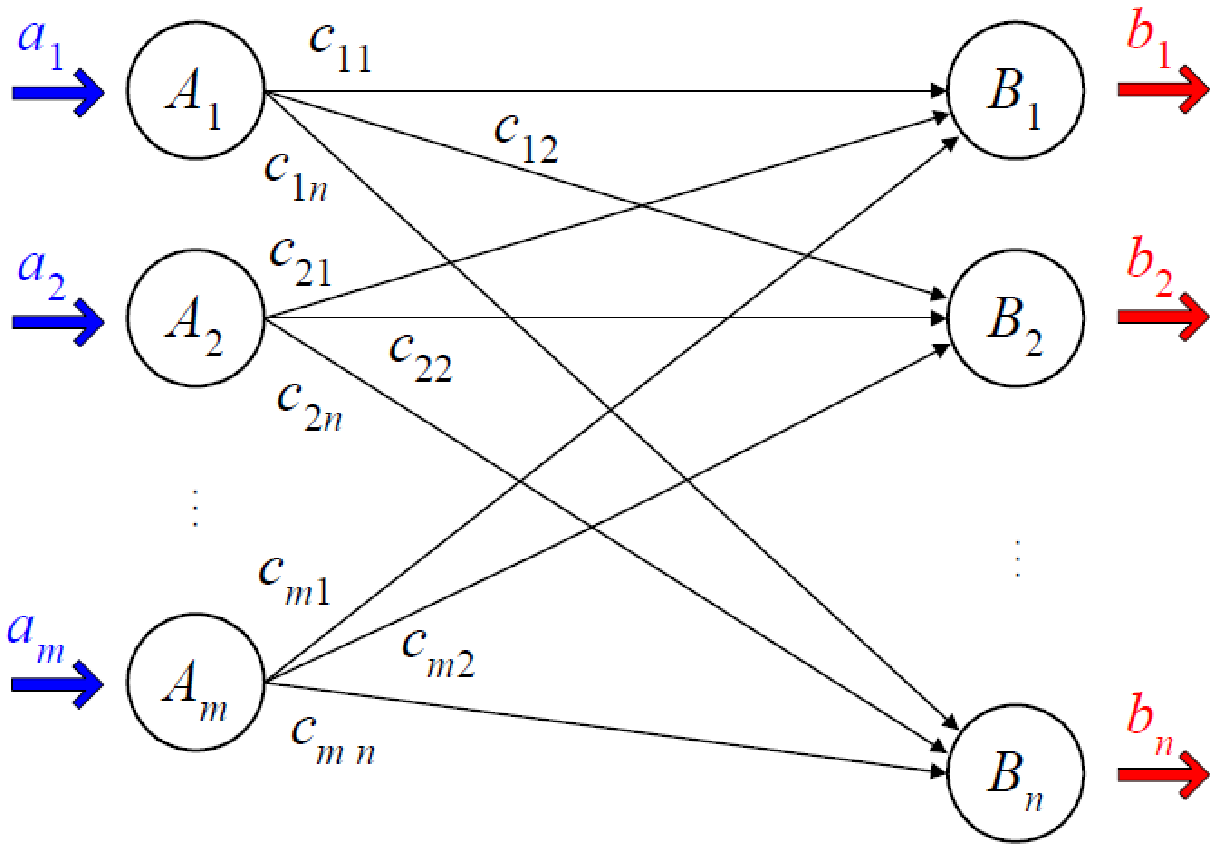

Fig.1

Definition: A transportation problem is *balanced* if the total quantity of product available (at the producing centers) is equal to the total quantity required (at the consuming centers):

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \overset{\text{not}}{=} \gamma$$

Remark. Any unbalanced transport problem can be brought into the form of a balanced transport problem.

If $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$, that is, the total available is greater than the total product requirement, means that certain quantities remain untransported in the production centers, certainly in at least one center.

The quantities not shipped may be considered as destined for a fictive consumer center, marked with B_{n+1}, with a necessity:

$$b_{n+1} = \sum_{i=1}^{m} a_i - \sum_{j=1}^{n} b_j$$

The unit transportation costs for those quantities are considered zero, as in reality those quantities remain at the production centers.

Example. Let the unbalanced transportation problem:

$A_i \setminus B_j$	B_1	B_2	B_3	a_i
A_1	4	3	2	300
A_2	2	1	3	400
b_j	200	150	250	

The balanced transportation problem has the form:

$A_i \setminus B_j$	B_1	B_2	B_3	B_4	a_i
A_1	4	3	2	0	300
A_2	2	1	3	0	400
b_j	200	150	250	100	

❑

If $\sum_{i=1}^{m} a_i < \sum_{j=1}^{n} b_j$, that is, the total available is less than the total required product, meaning that some consumer centers do not receive all the required quantities of product. The quantities that meet demand can be considered as coming from a fictive manufacturing center, A_{m+1}, with an available:

$$a_{m+1} = \sum_{j=1}^{n} b_j - \sum_{i=1}^{m} a_i$$

and having the unit transport costs equal to zero. In this way the initial problem becomes balanced.

Example. Let the unbalanced transportation problem:

$A_i \setminus B_j$	B_1	B_2	B_3	a_i
A_1	3	5	4	250
A_2	2	3	2	150
b_j	200	100	200	

The balanced transportation problem has the form:

$A_i \setminus B_j$	B_1	B_2	B_3	a_i
A_1	3	5	4	250
A_2	2	3	2	150
A_3	0	0	0	100
b_j	200	100	200	

❑

Let note with x_{ij} the quantity of product transported from the center A_i at the center B_j.

The mathematical model of the transportation problem:

$$\begin{cases} [\min]\, z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} x_{ij} \\ \sum_{j=1}^{n} x_{ij} = a_i,\ i = \overline{1, m} \\ \sum_{i=1}^{m} x_{ij} = b_j,\ j = \overline{1, n} \\ x_{ij} \geq 0,\ i = \overline{1, m},\ j = \overline{1, n} \end{cases} \qquad (1)$$

The model (1) is a particular model of LP problem. Indeed, the model can be rewritten in matrix form:

$$\begin{cases} [\min] \, z = c^{\mathrm{T}} x \\ Ax = b \\ x \geq 0 \end{cases}$$

where:

$c \in \mathbf{R}^{m \times n}, \quad c = [c_{11}, c_{12}, \ldots, c_{1n}, c_{21}, \ldots, c_{2n}, \ldots, c_{m1}, \ldots, c_{mn}]^{\mathrm{T}}$

$x \in \mathbf{R}^{m \times n}, \quad x = [x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, \ldots, x_{2n}, \ldots, x_{m1}, \ldots, x_{mn}]^{\mathrm{T}}$

$b \in \mathbf{R}^{m+n}, \quad b = [a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n]^{\mathrm{T}}$

$A \in \mathcal{M}_{m+n,\, m \times n}$,

$$A = \underbrace{\begin{bmatrix} I & 0 & \cdots & 0 \\ 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & I \\ E & E & \cdots & E \end{bmatrix}}_{m\times n \text{ columns}} \begin{matrix} \} m \text{ lines} \\ \} n \text{ lines} \end{matrix}$$

where $I = [1, 1, \ldots, 1] \in \mathbf{R}^n$, $0 = [0, 0, \ldots, 0] \in \mathbf{R}^n$, E is the unit matrix of order n.

Remarks:

1) There are problems of transport type in which the objective pursued consists, not in the minimization of costs, but of other indicators such as distances, times, et al.

2) Economic applications of transportation problems:

- transport of goods from producers to consumers so that total fuel consumption is minimal;
- design of energy transmission networks (electricity, heat, information, etc.), so that the distances from the transmission stations to the reception stations to be minimal;

- location of material warehouses within the manufacturing companies, so that the supply can be made as quickly as possible and with a minimum effort;
- optimal distribution of production tasks on equipment, operators, et al.
- optimization of certain production and storage problems .

Definition. A solution that satisfies the system of constraints (equations) and non-negativity conditions is called a *feasible solution* to a balanced transport problem (1) .

Theorem. *The set of feasible solutions to a transportation problem is empty.*

Dem.: We will show that values

$$x_{ij} = \frac{1}{\gamma} a_i b_j,\ i = \overline{1,m},\ j = \overline{1,n}$$

is a feasible solution.

Indeed, the system of restrictions is being checked:

$$\sum_{i=1}^{m} x_{ij} = \sum_{i=1}^{m} \frac{1}{\gamma} a_i b_j = \frac{b_j}{\gamma} \sum_{i=1}^{m} a_i = b_j,\ \ j = \overline{1,n}$$

$$\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n} \frac{1}{\gamma} a_i b_j = \frac{a_i}{\gamma} \sum_{j=1}^{n} b_j = a_i,\ i = \overline{1,m}$$

$$x_{ij} > 0,\ i = \overline{1,m},\ j = \overline{1,n}$$

❑

2. Methods for determining an initial solution

2.1. Northwest corner method

2.2. Minimum element in the table method

The optimal solution of a transportation problem is one of its basic feasible solutions.

To determine a basic initial solution we will study two methods:

- Northwest corner method;
- Minimum element in the table method.

2.1. Northwest corner method

This method does not take into account unit transport costs c_{ij}, but only by the quantities available a_i from the production centers and the quantities required bj at consumer centers.

The method consists in assigning, in turn, values to the variables, starting with the one in the upper left corner of the problem table.

It is chosen $x_{11} = \min (a_1, b_1)$. Three alternatives are possible :

(i) $\min(a_1, b_1) = a_1 \Rightarrow x_{11} = a_1$, and $x_{1j} = 0$ ($j = 1,.., n$);
in center B_1 the necessary becomes $b_1 - a_1$.

(ii) $\min(a_1, b_1) = b_1 \Rightarrow x_{11} = b_1$, and $x_{i1} = 0$ ($i = 1,.., m$);
in center A_1 the availability becomes $a_1 - b_1$.

(iii) $a_1 = b_1 \Rightarrow x_{11} = a_1 = b_1$, and $x_{1j} = 0$ ($j = 1,\ldots, n$),
$x_{i1} = 0$ ($i = 1,.., m$).

It deletes the first row / first column of the table from top to bottom / from left to right. The second strictly positive component of the basic starting solution is determined analogously, continuing with the box in the upper left corner of the remaining subtable.

The procedure is repeated until all the boxes in the table are occupied.

Example. Establish an initial solution with the north-west corner method for the transportation problem :

$A_i \setminus B_j$	B_1	B_2	B_3	a_i
A_1	5	3	2	40
A_2	3	2	4	30
b_j	15	35	20	

Let x_{11} = min (40, 15) = 15 $\Rightarrow$ x_{21} = 0, a_1 = 40-15 = = 25; x_{12} = min (25, 35) = 25 $\Rightarrow$ x_{13} = 0, b_2 = 35-25 = =10; x_{22} = min (30, 10) = 10 $\Rightarrow$ a_2 = 30-10 = 20; x_{23} = min (20, 20) = 20.

The above calculations can be expressed synthetically in tabular form :

$A_i \setminus B_j$	B_1	B_2	B_3	a_i		
A_1	15	25	0	40	25	0
A_2	0	10	20	30	20	
b_j	15	35	20			
	0	10				
		0				

The value of the objective function - the total cost of transport - corresponding to this solution :

$$z = 5 \cdot 15 + 3 \cdot 25 + 2 \cdot 10 + 4 \cdot 20 = \mathbf{250} \text{ m.u.} \qquad \square$$

2.2. Minimum element in the table method

This method consists in the successive assignment of values to the variables, starting with the variable in which the unit cost of transport is minimal.

The value of the variable is given (as in the northwest corner method) by the minimum between the available and the necessary in the line, respectively, the corresponding column.

The available and necessary values corresponding to the respective variable are updated.

It deletes the row/column from top to bottom/ left to right .

Then the process continues with the minimum cost variable in the remaining subtable.

Remarks:

1) If there are several equal minimum costs, then the variable that can take the maximum value is considered first.

2) Usually, this method offers a better initial solution than the northwest corner method.

Example. Establish an initial solution with the minimum element in the table method for the problem :

$A_i \setminus B_j$	B_1	B_2	B_3	a_i
A_1	5	3	2	40
A_2	3	2	4	30
b_j	15	35	20	

The minimum cost in the table is $c_{13} = c_{22} = 2$. The values of the variables are $x_{13} = \min(40, 20) = 20$ şi $x_{22} = \min(30, 35) = 30$.

Because $x_{22} > x_{13}$, it is chosen $x_{22} = 30 \Rightarrow x_{21} = x_{23} = 0$; $b_2 = 35 - 30 = 5$.

Then $x_{13} = 20 \Rightarrow a_1 = 40 - 20 = 20$. The minimum remaining cost is $c_{12} = 3$; $x_{12} = \min(20, 5) = 5 \Rightarrow a_1 = 20\text{-}5 = 15$; $x_{12} = \min(15, 15) = 15$.

$$z = 5\cdot15 + 3\cdot5 + 2\cdot20 + 2\cdot30 = \mathbf{190} \text{ m.u.}$$

The calculations can be expressed in the tabular form:

$A_i \backslash B_j$	B_1	B_2	B_3	a_i		
A_1	15	5	20	40	20	15
A_2	0	30	0	30	0	
b_j	15	35	20			
		5	0			
		0				

❑

3. Potential method

Solving the transport problem by the simplex method is generally inefficient. Some of the reasons are as follows :

- the large number of variables, equal to $m \cdot n$;
- the absence of an initial basis; the solution by considering auxiliary variables for each row and column is laborious .

The problem have $m \cdot n$ unknows of which $(m+ n\text{-}1)$ are main unknowns main or basic variables, and the other (m-1)(n-1) are secondary unknows.

The potential method allows testing and improving the solution of a transport problem.

This method uses the dual problem associated with the transport problem (1):

$$\begin{cases} \max\left(\sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j\right) \\ u_i + v_j \le c_{ij},\ i = \overline{1,m},\ j = \overline{1,n} \\ u_i \ \text{ arbitrary, } i = \overline{1,m} \\ v_j \ \text{ arbitrary, } j = \overline{1,n} \end{cases}$$

Potential method algorithm:

Step 1. An initial basic solution $(\overline{x}_{ij})$ is determined.

Step 2. The values of the dual variables u_i and v_j are calculated, as solutions of the system of equations:

$$u_i + v_j = c_{ij}, \ \forall (i, j) \in \mathcal{J}$$

where $\mathcal{J}$ is the set of indices of the basic variables.

Step 3. It calculates the values:

$$\delta_{ij} = u_i + v_j - c_{ij}, \ \forall (i, j) \notin \mathcal{J}$$

If $\delta_{ij} \le 0, \ \forall (i, j) \notin \mathcal{J}$, then STOP: the solution is optimal; else go to step 4.

Step 4. It calculates with the *entering in base criterion*

$$\delta_{lk} = \max\{\delta_{ij} \mid \forall (i, j) \notin \mathcal{J}\}$$

the variable x_{lk} that will enter the base.

It constructs for box (l, k) a circuit that starts from box (l, k), with untransported quantity, which passes only through boxes (i, j) occupied with values $x_{ij} > 0$, with alternating passages on rows and columns, and returning to box (l, k).

It adopts a certain direction of travel of this circuit and number its boxes starting with the box (l, k).

Step 5. It determines by the *exit from base criterion*

$$\theta = \bar{x}_{st} = \min\{\bar{x}_{ij} \mid (i, j) \text{ par rank box in the circuit}\},$$

and the variable x_{st} will leave the base.

Step 6. It determines a new admisible basic solution $(\tilde{x}_{ij})$ with *formulas of the base change*:

$$\tilde{x}_{ij} = \begin{cases} \bar{x}_{ij} - \theta, & (i, j) \text{ has even rank in circuit} \\ \bar{x}_{ij} + \theta, & (i, j) \text{ has odd rank in circuit} \\ \bar{x}_{ij}, & (i, j) \text{ does not belong to the circuit} \end{cases}$$

It returns to step 2 with the obtained new basic solution.

Remarks:

1) The initial basic solution obtained in step 1 of the algorithm depends on the method used to determine it. This does not influence the optimal solution, but only (possibly) the number of iterations required.

2) The tables that are built when applying the algorithm consist of boxes (i, j) that can have one of the configurations:

c_{ij}	
+[-]	x_{ij}

c_{ij}	δ_{ij}
	$x_{ij}=0$

The first box corresponds to a basic variable ($x_{ij} > 0$), and the second to a non-basic variable ($x_{ij} = 0$), for which the value δ_{ij} is calculated.

The "+" or "-" symbols appear if the box is part of the circuit built in step 4. The "+" symbol indicates an odd-numbered box, and the "-" symbol indicates an even-numbered box.

Example. Solving the transportation problem:

$A_i \setminus B_j$	B_1	B_2	B_3	B_4	a_i
A_1	8	7	3	6	19
A_2	4	6	7	3	20
A_3	3	4	2	5	16
b_j	15	12	14	14	

The problem is balanced.

The initial basic solution obtained with the minimum element method : x_{11} = 7, x_{12} = 12, x_{21} = 6, x_{24} = = 14, x_{31}= 2, x_{33} = 14. (see next tabel)

The value of the corresponding objective function

$$z = 8\cdot7 + 7\cdot12 + 4\cdot6 + 3\cdot14 + 3\cdot2 + 2\cdot14 = 240 \text{ m.u.}$$

7	12	0	0	19	12	
6	0	0	14	20	6	0
2	0	14	0	16	2	0
15	12	14	14			
13		0	0			
7						
0						

Iteration 1

8 7	7 12	3 0	6 0
4 6	6 0	7 0	3 14
3 2	4 0	2 14	5 0

	$v_1 = 8$	$v_2 = 7$	$v_3 = 7$	$v_4 = 7$
$u_1 = 0$	8 - 7	7 12	3 [4] + 0	6 [1] 0
$u_2 = -4$	4 6	6 [-3] 0	7 [-4] 0	3 14
$u_3 = -5$	3 + 2	4 [-2] 0	2 14	5 [-3] 0

Because $\delta_{ij} \not\leq 0, \ \forall (i, j) \notin \mathcal{J}$ the solution is not optimal.

$\delta_{lk} = \delta_{13} = 4 \Rightarrow$ the variable x_{13} will enter in base.

Let the circuit: (1, 3), (1, 1), (3, 1), (3, 3), (1, 3).

$\theta = \min \{7, 14\} = 7 \Rightarrow x_{11}$ will come out of the base.

It modifies the values of the circuit boxes: $x_{13} = 0 + 7 = 7$, $x_{11} = 7 - 7 = 0$, $x_{31} = 2 + 7 = 9$, $x_{33} = 14 - 7 = 7$.

$z = 7{\cdot}12 + 3{\cdot}7 + 4{\cdot}6 + 3{\cdot}14 + 3{\cdot}9 + 2{\cdot}7 = 212$ u.m.

Iteration 2

	$v_1 = 4$	$v_2 = 7$	$v_3 = 3$	$v_4 = 3$
$u_1 = 0$	8 | -4 ; 0	7 ; - 12	3 ; + 7	6 | -3 ; 0
$u_2 = 0$	4 ; 6	6 | 1 ; 0	7 | -4 ; 0	3 ; 14
$u_3 = -1$	3 ; 9	4 | 2 ; + 0	2 ; 7	5 | -3 ; 0

Because $\delta_{ij} \nleq 0, \ \forall (i, j) \notin \mathcal{J}$ the solution is not optimal.

$\delta_{lk} = \delta_{32} = 2 \Rightarrow$ the variable x_{32} will enter in base.

Let the circuit: (3, 2), (3, 3), (1, 3), (1, 2), (3, 2).

$\theta = \min \{7, 12\} = 7 \Rightarrow x_{33}$ will come out of the base. It modifies the values of the circuit boxes: $x_{32} = 0 + 7 = 7$, $x_{33} = 7 - 7 = 0$, $x_{13} = 7 + 7 = 14$, $x_{12} = 12 - 7 = 5$.

$$z = 7 \cdot 5 + 3 \cdot 14 + 4 \cdot 6 + 3 \cdot 14 + 3 \cdot 9 + 4 \cdot 7 = 198 \text{ u.m.}$$

Iteration 3

	$v_1 = 6$	$v_2 = 7$	$v_3 = 3$	$v_4 = 5$
$u_1 = 0$	8 -2 0	7 5	3 14	6 -1 0
$u_2 = -2$	4 6	6 -1 0	7 -6 0	3 14
$u_3 = -3$	3 9	4 7	2 -2 0	5 -3 0

Because $\delta_{ij} < 0,\ \forall (i,\ j) \notin \mathcal{J}$, it results that the last solution optimal:

$$x_{12} = 5,\ x_{13} = 4,\ x_{21} = 6,\ x_{24} = 14,\ x_{31} = 9,\ x_{32} = 7.$$

The total transport cost corresponding to the optimal solution is $z_{\min} = 198$ u.m. ❑

4. Parametric transportation problem

The transportation problem in which at least one of its elements (cost matrix C; available quantities a_i; required quantities b_j) depends, linearly or nonlinearly, on one or more parameters is a *parametric transportation problem.*

By analogy with the problems of parametric linear programming, in case of solving the parametric transportation problems it is necessary to go through two stages :

- Determining the optimal solution to the problem for specified value of the parameter, or for a system of values if there are several parameters .
- Study of the sensitivity of the determined optimal solution at variation of the parameter(s).

5. Special transportation problems

5.1. Transportation problem with intermediate centers

5.2. Transfer problem

5.3. Transportation problem with limited capacities

5.4. Transportation problem with linked centers

5.1. Transportation problem with intermediate centers

In m production centers, A_1, ..., A_m, there is a certain product in the quantities a_i (i = 1, ..., m) respectively. The product is transported to p warehouses, D_1, ..., D_p, of capacities d_k (k = 1, ..., p), from where, subsequently, it is transported to n consumption centers, B_1, ..., B_n , which is required in the quantities b_j (j = 1,…, n) respectively.

The unit cost of transport from the center A_i to the warehouse D_k is c_{ik} (i = 1, ..., m; k = 1,…, p). The unit cost of transport from the warehouse D_k to the center B_j is q_{kj} ($k = 1, ..., p$; $j = 1,\dots, n$).

Determine an optimal transportation plan of the product from the production centers A_i ($i = 1, ..., m$) to the consumption centers B_j ($j = 1, ..., n$) so that the total transport cost to be minimal.

The graphic network associated with a transportation problem with intermediate centers is presented in the Fig.2.

The mathematical model of the transportation problem with intermediate centers is written:

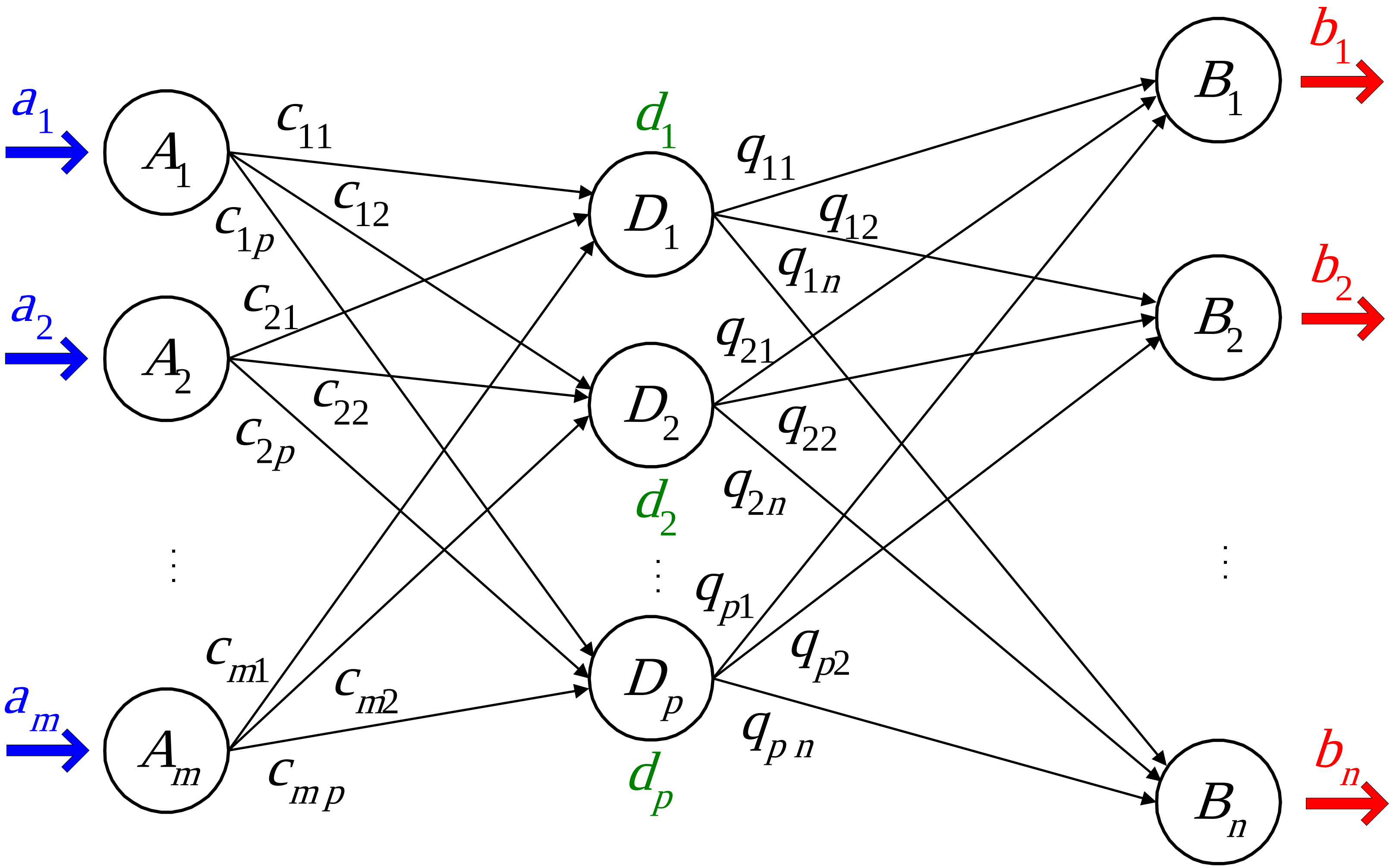
a_1
a_2
a_m
A_1
A_2
A_m
c_{11}
c_{12}
c_{1p}
c_{21}
c_{22}
c_{2p}
c_{m1}
c_{m2}
c_{mp}
d_1
d_2
d_p
D_1
D_2
D_p
q_{11}
q_{12}
q_{1n}
q_{21}
q_{22}
q_{2n}
q_{p1}
q_{p2}
q_{pn}
B_1
B_2
B_n
b_1
b_2
b_n

Fig.2

$$
\begin{cases}
[\min]\, z = \sum_{i=1}^{m}\sum_{k=1}^{p} c_{ik} x'_{ik} + \sum_{k=1}^{p}\sum_{j=1}^{n} q_{kj} x''_{kj} \\
\sum_{k=1}^{p} x'_{ik} = a_i,\ i = \overline{1,m} \\
\sum_{i=1}^{m} x'_{ik} = d_k,\ k = \overline{1,p} \\
\sum_{j=1}^{n} x''_{kj} = d_k,\ k = \overline{1,p} \\
\sum_{k=1}^{p} x''_{kj} = b_j,\ j = \overline{1,n} \\
x'_{ik}, x''_{kj} \geq 0,\ i = \overline{1,m},\ k = \overline{1,p},\ j = \overline{1,n}
\end{cases} \qquad (2)
$$

where:

- x'_{ik} the quantity to be transported from the source center i to the intermediate center k;
- x''_{kj} the quantity to be transported from the inter-mediate center k to the destination center j;

The mathematical model (2) is a LP model.

The necessary and sufficient condition for the existence of solutions of the problem (2) is:

$$\sum_{i=1}^{m} a_i = \sum_{k=1}^{p} d_k = \sum_{j=1}^{n} b_j$$

Because each intermediate center is both destination center and source center, two transportation problems can be considered:

$$\begin{cases} [\min]\, z = \sum_{i=1}^{m}\sum_{k=1}^{p} c_{ik}x'_{ik} \\ \sum_{k=1}^{p} x'_{ik} = a_i,\ i=\overline{1,m} \\ \sum_{i=1}^{m} x'_{ik} = d_k,\ k=\overline{1,p} \\ x'_{ik} \geq 0,\ i=\overline{1,m}, k=\overline{1,p} \end{cases} \quad (3)$$

$$\begin{cases} [\min]\, z = \sum_{k=1}^{p}\sum_{j=1}^{n} q_{kj}x''_{kj} \\ \sum_{j=1}^{n} x''_{kj} = d_k,\ k=\overline{1,p} \\ \sum_{k=1}^{p} x''_{kj} = b_j,\ j=\overline{1,n} \\ x''_{kj} \geq 0,\ k=\overline{1,p},\ j=\overline{1,n} \end{cases} \quad (4)$$

It can be shown that $(\overline{x'_{ik}})$ and $(\overline{x''_{kj}})$ represent an optimal solution of the transportation problem (2) if and only if $(\overline{x'_{ik}})$ is an optimal solution of the transportation problem (3), and $(\overline{x''_{kj}})$ is an optimal solution of the transportation problem (4).

Consequently, solving problem (2) is equivalent to solving the two problems (3) and (4).

5.2. Transfer problem

The transfer problem is a special transportation problem due to the following aspects:

• In the graphics network associated with the problem there are source centers, destination centers and transit centers.

• Some source (destinations) centers may also function as transit centers for certain quantities of product coming from other centers in the network.

• The transport between two centers between which there is a direct connection, can be done in one direction or both; the unit cost of transport may depend on the direction of travel.

An example of a problem is shown in Fig.3.

Remarks:

1) The centers 1 and 3 have the quantities of product a_1, respectively, a_3.

2) The centers 2 and 7 request the product quantities b_2, respectively, b_7.

3) Except for sections (2, 4) and (7, 6), the transport in both directions is allowed on the other sections; they are represented by unoriented arcs, the value of the unit transport cost being the same in both

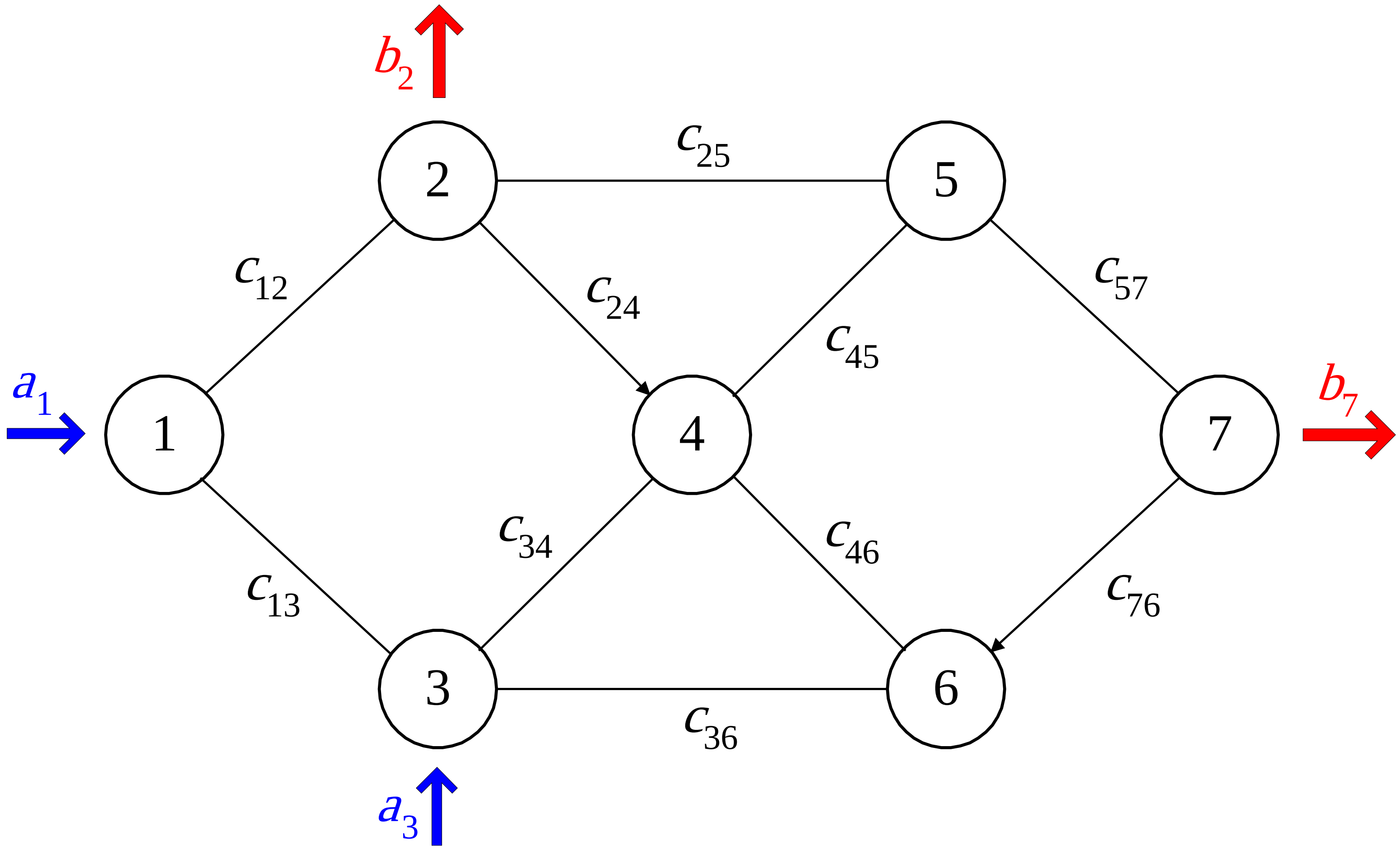

Fig.3

directions.

It is considered a transfer problem with n centers. We will use the following notations for known data:

a_i - the quantity of product available in the center i ($a_i \geq 0$, $i = 1, ..., n$; $a_i > 0$ if i is a source center);

b_j - the quantity of product requested by the center j ($b_j \geq 0$, $j = 1, ..., n$; $b_j > 0$ if j is a destination center);

c_{ij} - the unit cost of transporting the product from the center i to center j ($c_{ij} > 0$, $i, j = 1, ..., n$, with $i \neq$ j; $c_{ii} = 0$; $c_{ij} = \infty$ if the transport between the centers i and j is not possible).

The unknowns of the problem are noted as follows:

x_{ij} - the quantity of product to be transported from center i to center j ($i \neq j$);

$\tilde{x}_{ii}$ - the quantity of product in transit in center i ($i = 1, ..., n$).

The mathematical model of the transfer problem has the following form:

$$
\begin{cases}
[\min] z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot x_{ij} \\
\sum_{\substack{k=1 \\ k \neq i}}^{n} x_{ik} - \tilde{x}_{ii} = a_i, \quad i = 1, \ldots, n \\
\sum_{\substack{k=1 \\ k \neq j}}^{n} x_{kj} - \tilde{x}_{kk} = b_j, \quad j = 1, \ldots, n \\
x_{ij} \geq 0, \; \tilde{x}_{ii} \geq 0, \quad i, j = 1, \ldots, n; \; i \neq j
\end{cases} \tag{5}
$$

Remark: The model (5) differs from the mathematical model of a classic transportation problem in that it has the variables $\tilde{x}_{ii}$ ($i = 1, ..., n$) with negative coefficients in constraints.

The problem is compatible if the total quantities shipped from the sources is equal to the total quantities arrived at the destinations, ie:

$$\sum_{i=1}^{n} a_i = \sum_{j=1}^{n} b_j = L$$

Obviously, in any center, the quantity in transit will not exceed the total available:

$$\tilde{x}_{ii} \le L, \quad i = 1,\dots,n$$

from wich:

$$\tilde{x}_{ii} = L - x_{ii}, \quad i = 1,\dots,n \qquad (*)$$

By substituting the relations (*) in the model (5) a model of balanced classic transportation problem is obtained:

$$\begin{cases} [\min] z = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{ij} \cdot x_{ij} \\ \sum_{k=1}^{n} x_{ik} = L + a_i, \quad i = 1, \ldots, n \\ \sum_{k=1}^{n} x_{kj} = L + b_j, \quad j = 1, \ldots, n \\ x_{ij} \geq 0, \quad i, j = 1, \ldots, n \end{cases} \tag{6}$$

In order to obtain the solution of the original problem, the values of the variables $\tilde{x}_{ii}$ (i = 1, ..., n) are determined with the relation (*) .

5.3. Transportation problem with limited capacities

In many economic situations the capacities of transport routes are limited for technical and economic reasons. The mathematical model of the problem is:

$$\begin{cases} [\min] z = \sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \cdot x_{ij} \\ \sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, \ldots, m \\ \sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, \ldots, n \\ 0 \le x_{ij} \le \alpha_{ij}, \quad i = 1, \ldots, m;\ j = 1, \ldots, n \end{cases} \tag{7}$$

The problem allows solutions if the following conditions are met:

$$a_i \geq 0, b_j \geq 0, \alpha_{ij} \geq 0, \quad i = 1, \dots, m;\ j = 1, \dots, n$$

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

$$\sum_{j=1}^{n} \alpha_{ij} \geq a_i, \quad i = 1, \dots, m$$

$$\sum_{i=1}^{m} \alpha_{ij} \geq b_j, \quad j = 1, \dots, n$$

5.4. Transportation problem with linked centers

In the previously studied models, the availability a_i of each source center (producer) A_i and the necessary b_j of each destination center (consumer) B_j were known. These are also called transportation problems with *independent centers*.

There are also situations in which the individual availability and / or individual needs of certain centers are not known, but only their total availability and / or their total needs. These are called transportation pro-blems with *linked centers*.

1^0 The problem with linked source centers

They are known for source centers $A_1, A_2, \ldots, A_k$ the available individual quantities, a_1, a_2, ..., respectively, a_k, and for the other source centers, A_{k+1}, A_{k+2}, ..., A_m, their total availability a is known.

To solve, we proceed to the "aggregation" of the linked source centers in a single one, A, having the available a.

The unit transportation costs related to the center A, noted γ_j ($j = 1, \ldots, n$), are given by the relationship:

$$\gamma_j = \min\left\{c_{ij} \;\middle|\; i = \overline{k+1, m}\right\}, \quad j = \overline{1, n}$$

The table of the transportation problem thus obtained has the form:

$A_i \setminus B_j$	B_1		B_j		B_n	a_i
A_1	c_{11} x_{11}	...	c_{1j} x_{1j}	...	c_{1n} x_{1n}	a_1
A_2	c_{21} x_{21}	...	c_{2j} x_{2j}	...	c_{2n} x_{2n}	a_2
...	...	...	...	...	...	...
A_k	c_{k1} x_{k1}	...	c_{kj} x_{kj}	...	c_{kn} x_{kn}	a_k
A	γ_1 x_1	...	γ_j x_j	...	γ_n x_n	a
b_j	b_1		b_j		b_n	

It solves the previous transportation problem and the solutions $x_1, x_2, \ldots, x_n$, from the last line, are distributed on the linked source centers $A_{k+1}, A_{k+2}, \ldots, A_m$.

2^0 The problem with destination linked centers

They are known for destination centers $B_1, B_2, \ldots, B_k$ the required individual quantities, $b_1, b_2, \ldots,$ respectively, b_k, and for the other destination centers, $B_{k+1}, B_{k+2}, \ldots, B_n$, their total need b is known.

To solve, the "aggregation" of the linked destination centers is performed in one, noted by B, having with the necessary b.

The unit transportation costs related to center *B*, noted γ_i ($i = 1,\ldots, m$), are given by the relation:

$$\gamma_i = \min\left\{ c_{ij} \;\middle|\; j = \overline{k+1, n} \right\}, \quad i = \overline{1, m}$$

The table of the transportation problem thus obtained has the form :

$A_i \setminus B_j$	B_1	B_2	...	B_k	B	a_i
A_1	c_{11} x_{11}	c_{12} x_{12}	...	c_{1k} x_{1k}	γ_1 x_1	a_1
...	...	...	...	...	...	...
A_i	c_{i1} x_{i1}	c_{i2} x_{i2}	...	c_{ik} x_{ik}	γ_i x_i	a_i
...	...	...	...	...	...	...
A_m	c_{m1} x_{m1}	c_{m2} x_{m2}	...	c_{mk} x_{mk}	γ_n x_n	a_m
b_j	b_1	b_2	...	b_k	b	

It solves the previous problem and the solutions x_1, ..., x_n, from the last column, are distributed on the linked destination centers B_{k+1}, ..., B_n.

3^0 The problem with linked source and destination centers

The "aggregation" of the linked centers in one, denoted by *A*, as well as of the destination centers linked in one, denoted by *B*, is performed.

Then thus obtained transportation problem is solved.