**Partial Differential Equations**

**Part 1: Overview and Classification**

**Part 2: The Cauchy Problem**

**Part 3: Boundary Value Problems for PDE’s**

**Textbook:**

L. Debnath- Nonlinear Partial Differential Equations for Scientists and Engineers, Springer, 2012

**Classical Textbooks:**

1. Weinberger, H. F. – A First Course in Partial Differential Equations with Complex Variables and Transform Methods, Dover 1965.

2. John, F., - Partial Differential Equations, Springer 1971.

**Foreward**

* **The study of partial differential equations is an advanced topic that goes beyond an intermediate level course.**
* **Special methods are needed for each type of partial differential equation, even for each type of boundary conditions**
* **The notes below aim to give an idea about the problems involved and the methods that are applied.**
* **In Part1, we give the basic definition, discuss the notion of linearity, superposition principle and linear operators**
* **In Part 2, we define the Cauchy problem on the propogation oof initial datafor hyperbolic equations**
* **In Part 3, we overview basic results for boundary value problems of elliptic equations.**

**PART 2**

**The Cauchy Problem**

**2.The Cauchy Problem**

Initial-value problem: The second-order ordinary differential equation

with the initial conditions

We analyze a second-order partial differential equation for the function and assume that it can be represented in the from

For some , initial value of the unknown function,

Finding a solution to Equation (2.1) that is satisfying the *initial-value problem* refers to the initial conditions in Equation (2.2.a) and Equation (2.2.a).

For example, consider the wave equation:

Initial Conditions for wave equation:

* initial displacement,
* initial velocity.

The initial values in initial-value problems typically refer to the data at the value . They may very well be specified along some curve in the direction of the line in the plane. The issue is known as the *Cauchy problem*.

Consider the Euler Equation

Let denote points on a smooth curve in the plane.

The parametric equations of

Let two functions and be defined along the curve. Here, the Cauchy problem is one of determining the solution of in Equation (2.3) adjacent to the curve , where is the direction of the normal to lying to the left of counterclockwise of increasing arc length. The functions and are called *Cauchy data*.

**2.1 The Cauchy-Kowalewskaya Theorem**

The partial differential equation form:

Initial Conditions:

on the noncharacteristic manifold

The Cauchy problem has a unique real analytic solution in some neighborhood of the point if the function is real analytical in some neighborhood of the point and the functions and are analytical in some neighborhood of the point .

This expression is hyperbolic, parabolic, or valid for hyperbolic, parabolic, or elliptic equations, it is difficult to formulate the Cauchy problem for non-hyperbolic equations. For example, Hadamard (1952):

The elliptic equation:

Initial Conditions for ;

Solution:

If increases for any nonzero , the value of is not small. The solution represents an oscillation with unbounded amplitude as for any fixed . This solution is unstable even if is a fixed integer since for every fixed makes as . Therefore, it is clear that the solution does not rely just on the facts. As a result, the issue is not properly posed.

Any continuous function can be precisely approximated by polynomials, as is widely known. Only if a little change in the initial data results in a little change in the solution can we use polynomial approximations to apply the Cauchy Kowalewskaya theorem to continuous data.

**---------END OF THE PART 2--------------**

**PART 3**

**3. Boundary- Value Problems**

Finding a function that solves a given partial differential equation and specific boundary conditions is known mathematically as solving a boundary-value problem.

The second-order partial differential equation of the elliptic type with independent variables

where

Instead of dealing with general elliptic partial differential equations, we will start by presenting the simplest boundary problem value for the two-dimensional Laplace equation.

**Laplace equation:**

**Harmonic Function:**

A function is called a *harmonic* if it satisfies Laplace's equation in domain D and the first two derivatives are continuous in D.

Here, we can state that a linear combination of harmonic functions is harmonic, since Laplace's equation is linear and homogeneous.

**3.1 The First Boundary- Value Problems**

**The Dirichlet Problem:**

Find a harmonic function in satisfying the equation

on such that is a predetermined continuous function on the boundary of the field . is the interior of a simple closed segment smooth curve.

The solution to the Dirichlet problem can be interpreted as the steady-state temperature distribution in an object that does not physically contain source or any sinks of heat, with predicted temperature at all points on the boundary.

**3.2 The Second Boundary- Value Problems**

**The Neumann Problem:**

Find a harmonic function in satisfying the equation

on , and from this equation and Laplace equation,

on *The compatibility function*

The solution of the Neumann problem can be interpreted as the steady-state temperature distribution in an object that does not physically contain source or any sinks of heat, when the heat flux across the boundary is predicted.

Here the compatibility condition may be interpreted physically as the heat requirement that the net heat flux across the boundary be zero.

**3.3 The Maximum Principle Theorem**

Assume that is continuous in and harmonic in a bounded domain . The boundary of is where then reaches its maximum value.

This can be understood to mean that the electrostatic potential in a region without any free charge reaches its maximum (and minimum) values at the region's boundary and that the temperature of a body that was neither a source of heat nor a sink of heat acquires its largest maximum (and minimum values on the surface of the body.

**3.4 The Minimum Principle Theorem**

When is continuous in and harmonic in a bounded domain , it reaches its minimum on the border of .

**3.5 Uniqueness Theorem**

If there is a solution to the Dirichlet problem, it is unique.

**3.6 Continuity Theorem**

Dirichlet's solution the problem constantly depends on the boundary data.

**Remark:** Part 1 of these lecture notes started by giving general information about Partial Differential Equations. Differential operators, the conditions of being a linear operator, The principle of linear superposition, and Hyperbolic, Parabolic, Elliptic Type Canonical Forms are given.In Part 2, Cauchy Problem definition and Cauchy-Kowalewskaya Theorem are detailed. In Part 3, Boundary-Value Problem, The First Boundary- Value Problems, The Second Boundary- Value Problems are given, and some important theorems are mentioned.

* Partial differential equations are those differential equations that involve more than one independent variable.

Differential equation:

Domain:

Initial Condition:

 Boundary Conditions:

* A mathematical problem has to be “well posed”, this requires the **existence, uniqueness, continuity.**
* **Canonical Forms:**

Hyperbolic Type: ,

Parabolic Type:

Elliptic Type: