**Partial Differential Equations**

**Part 1: Overview and Classification**

**Part 2: The Cauchy Problem**

**Part 3: Boundary Value Problems for PDE’s**

**Textbook:**

L. Debnath- Nonlinear Partial Differential Equations for Scientists and Engineers, Springer, 2012

**Classical Textbooks:**

1. Weinberger, H. F. – A First Course in Partial Differential Equations with Complex Variables and Transform Methods, Dover 1965.

2. John, F., - Partial Differential Equations, Springer 1971.

**Foreward**

* **The study of partial differential equations is an advanced topic that goes beyond an intermediate level course.**
* **Special methods are needed for each type of partial differential equation, even for each type of boundary conditions**
* **The notes below aim to give an idea about the problems involved and the methods that are applied.**
* **In Part1, we give the basic definition, discuss the notion of linearity, superposition principle and linear operators**
* **In Part 2, we define the Cauchy problem on the propogation oof initial datafor hyperbolic equations**
* **In Part 3, we overview basic results for boundary value problems of elliptic equations.**

**Partial Differential Equations**

* Partial differential equations are those differential equations that involve more than one independent variable.
* Fundamental laws of nature and many engineering problems are modeled in terms of partial differential equations
* The study of partial differential equations was initiated in the second half of the $19^{th}$ century; many mathematicians were actively involved in the investigation of partial differential equations and various problems in science and engineering.
	+ **Examples:**Euler’s equations for the dynamics of rigid bodies and for the motion of an ideal fluid,
	+ Lagrange’s equations of motion,
	+ Hamilton’s equations of motion in analytical mechanics,
	+ Fourier’s equation for the diffusion of heat,
	+ Cauchy’s equation of motion
	+ Navier’s equation of motion in elasticity,
	+ Navier–Stokes equations for the motion of viscous fluids,
	+ Cauchy–Riemann equations in complex function theory,
	+ Cauchy–Green equations for the static and dynamic behavior of elastic solids,
	+ Kirchhoff’s equations for electrical circuits,
	+ Maxwell’s equations for electromagnetic fields,
	+ Schrödinger’s equation
	+ Dirac’s equation in quantum mechanics.

A partial differential equation includes one or more partial derivatives of the dependent variable in addition to the dependent variable and the independent variables.

$$f\left(x,y,u,u\_{x},u\_{y},u\_{xx},u\_{xy},u\_{xxx},…\right)=0.$$

$x,y$ are independent variables, $u$ is the function of these variables, $u\_{x},u\_{y},u\_{xx},u\_{xy},u\_{xxx},…$, are and the partial derivatives of the function.

For $u=u\left(x,y\right),$

$$\frac{∂u}{∂x}=u\_{x}, \frac{∂u}{∂y}=u\_{y}, \frac{∂^{2}}{∂y∂x}=u\_{xy}.$$

For example,

$$\begin{matrix}u\_{xy}+uu\_{x}& =y\\5yu\_{xy}+3xu\_{yy}& =7sin⁡x\\\left(u\_{x}\right)^{2}+\left(u\_{y}\right)^{2}& =1\\u\_{xxx}-u\_{yy}& =0\end{matrix}$$

are partial differential equations.

The highest-ordered partial derivative that appears in a partial differential equation determines the equation's **order**. For example;

Second-order partial differential equation:

$$5yu\_{xy}+3xu\_{yy}=7sin⁡x\rightarrow $$

Third-order partial differential equation:

$$u\_{xxx}-u\_{yy}=0 $$

When the coefficients of a partial differential equation depend only on the independent variables and the unknown function and all of its derivatives are **linear,** the equation is said to be linear; when the highest-ordered derivative of the unknown function is linear, the equation is said to be **quasi-linear**.

Second-order linear partial differential equation

$$yu\_{xx}+5xyu\_{yy}+u=3$$

Second-order quasi-linear partial differential equation

$$u\_{x}u\_{xx}+xuu\_{y}=\sin(y) $$

A family of functions based on $n$ separate arbitrary constants serves as the generic solution of a linear ordinary differential equation of $n^{th}$ order. As opposed to arbitrary constants, the general solution of partial differential equations depends on arbitrary functions.

**We will give examples of very simple partial differential equations and illusteate their solutions:**

**Example 1.1:** Consider the equation $u\_{xy}=0$ and $u=u(x,y)$,

integrate this equation with respect to $y$,

$$u\_{x}\left(x,y\right)=f\left(x\right)$$

Then, integrate with respect to $x$,

$$u(x,y)=g(x)+h(y),$$

$g(x)$ and $h(y)$ are arbitrary functions.

**Example 1.2:** Consider the equation $u\_{yy}=2$ and $u=u\left(x,y,z\right),$

integrate this equation with respect to $y$,

$$u\_{y}\left(x,y,z\right)=2y+f\left(x,z\right)$$

Then, integrate with respect to $y$,

$$u(x,y,z)=y^{2}+yf(x,z)+g(x,z),$$

$f(x,z)$ and $g\left(x,z\right)$ are arbitrary functions.

**Example 1.3:** Consider the equation

$$u\_{xx}+u=0$$

and $u=u\left(x,y\right),$

Since there is no derivative with respect to y, we can solve it simply by integrating with respect to x. The only difference is to replace the intecration constant by a function of y.

$=> u^{''}+u=0 =>u\left(x,y\right)= c\_{1}\left(y\right).cosx+ c\_{2}\left(y\right).sinx$.

**In the case of ordinary differential equations, we impose initial conditions of boundary conditions by assigning certain “constants” as initial or boundary values. For partial differential equations these are replaced by “functions”.**

The problem of finding the unknown function of a partial differential equation is defined by the **initial condition** (IC) or **boundary condition** (BC).

**Example:**

Differential equation: $u\_{t}-u\_{xx}=0, $

Domain: $0<x<m, t>0,$

Initial Condition: $u\left(x,0\right)=cosx, 0<x<m, $

 Boundary Conditions: $u\left(0,t\right)=0, u\left(m,t\right)=0, t\geq 0, \rightarrow $

**A mathematical problem has to be “well posed”, this requires the following**

1. **Existence:** There exists at least one solution of the problem .
2. **Uniqueness:** There is at most one solution.
3. **Continuity:** The unique solution depends continuously on the initial data. If the data are changed a little, the corresponding solution also changes only a little.

**An important class of partial differential equations is linear equations. These equations can be viewed by an operator acting on the depenedent variable. In this section we review properties of differential operators and present basic examples.**

**1.1 Differential operators:**

 $L=\left(\frac{∂^{2}}{∂x^{2}}+\frac{∂^{2}}{∂y^{2}}\right)$,

$$M=\left(\frac{∂^{2}}{∂x^{2}}-\frac{∂}{∂x}\right)+x\frac{∂}{∂y}.$$

$$\begin{matrix}L[u]& =\frac{∂^{2}u}{∂x^{2}}+\frac{∂^{2}u}{∂y^{2}}\\M[u]& =\frac{∂^{2}u}{∂x^{2}}-\frac{∂u}{∂x}+x\frac{∂u}{∂y}\end{matrix}$$

**1.2 Linear operators:**

Conditions for an operator to be **linear**:

* $L\left[cu\right]=cL\left[u\right], c$ is constant.
* $L\left[u\_{1}+u\_{2}\right]=L\left[u\_{1}\right]+L\left[u\_{2}\right] =>L\left[c\_{1}u\_{1}+c\_{2}u\_{2}\right]=c\_{1}L\left[u\_{1}\right]+c\_{2}L\left[u\_{2}\right], c\_{1} $and $c\_{2 }$ are constants.

The sum of two linear operators$ L$ and $M$ :

$$(L+M)[u]=L[u]+M[u]$$

The product of two linear differential operators $L$ and $M$ : $ $

$$ LM[u]=L(M[u])$$

The following criteria are generally met by linear differential operators:

1. Commutativity: $L+M=M+L, $ and $LM=ML,$
2. Associativity: $ \left(L+M\right)+N=L+\left(M+N\right), $ and

$$\left(LM\right)N=L\left(MN\right),$$

1. Distributivity: $ L\left(c\_{1}M+c\_{2}N\right)=c\_{1}LM+c\_{2}LN$,

where $c\_{1} $and $c\_{2 }$ are constants.

**1.3 The principle of linear superposition:**

A linear second-order partial differential equation with two independent variables:

$$\begin{matrix}&A\left(x,y\right)u\_{xx}+B\left(x,y\right)u\_{xy}+C\left(x,y\right)u\_{yy}+D(x,y)u\_{x}+E(x,y)u\_{y}+F(x,y)u=G(x,y)\\&\end{matrix}$$

$A\left(x,y\right), B\left(x,y\right), C\left(x,y\right), D\left(x,y\right), E\left(x,y\right), F\left(x,y\right)$ are coefficients, and $G\left(x,y\right)$ is the nonhomogeneous term.

$$L\left[u\right]=G => L=A\frac{∂^{2}}{∂x^{2}}+B\frac{∂^{2}}{∂x∂y}+C\frac{∂^{2}}{∂y^{2}}+D\frac{∂}{∂x}+E\frac{∂}{∂y}+F,$$

Let $v\_{1},v\_{2},…,v\_{n}$ and $w\_{1},w\_{2},…,w\_{n}$ be functions,

$$L\left[v\_{j}\right]=G\_{j}, j=1,2,…,n$$

$$L\left[w\_{j}\right]=0, j=1,2,…,n.$$

 $ u\_{j}=v\_{j}+w\_{j}=> $ the function $u=\sum\_{j=1}^{n} u\_{j}$ satisfies the equation $L[u]=\sum\_{j=1}^{n} G\_{j}.$

* The general linear second-order partial differential equation in one dependent variable $u:$

$$\sum\_{i,j=1}^{n}A\_{ij}u\_{x\_{i}x\_{j}}+ \sum\_{i=1}^{n}B\_{i}u\_{x\_{i}}+Fu=G .$$

* The second-order equations in the dependent variable $u$ and the independent variables $x, y:$

$$Au\_{xx} + Bu\_{xy} + Cu\_{yy} + Du\_{x} + Eu\_{y} + F u = G. (1)$$

* Three basic second-order partial differential equations:
1. The Laplace Equation:$ u\_{xx}+u\_{yy}+u\_{zz}=0.$
2. The Wave Equation: $ u\_{tt}-c^{2}\left(u\_{xx}+u\_{yy}+u\_{zz}\right)=0.$
3. The Heat Equation: $u\_{t}-k\left(u\_{xx}+u\_{yy}+u\_{zz}\right)=0.$

Convert the Equation (1) into a canonical form.

Let $ φ= φ(x,y)$ and $ω= ω(x,y)$ be new variables, also $φ= φ(x,y)$ and $ω= ω(x,y)$ be twice continuously differentiable, and let the Jacobian hold.

$$J=\left|\begin{matrix}φ\_{x}&φ\_{y}\\ω\_{x}&ω\_{y}\end{matrix}\right|$$

$$u\_{x}= u\_{φ}φ\_{x}+u\_{ω}ω\_{x},$$

$$u\_{y}= u\_{φ}φ\_{y}+u\_{ω}ω\_{y},$$

$$u\_{xx}= u\_{φφ}φ\_{x}^{2}+2u\_{φω}φ\_{x}ω\_{x}+u\_{ωω}ω\_{x}^{2}+u\_{φ}+u\_{ω}ω\_{xx},$$

$$u\_{xy}= u\_{φφ}φ\_{x}φ\_{y}+u\_{φω}(φ\_{x}ω\_{y}+φ\_{y}ω\_{x})+u\_{ωω}ω\_{x}ω\_{y}+u\_{φ}φ\_{xy}+u\_{ω}ω\_{xy},$$

$$u\_{yy}= u\_{φφ}φ\_{y}^{2}+2u\_{φω}φ\_{y}ω\_{y}+u\_{ωω}ω\_{y}^{2}+u\_{φ}φ\_{yy}+u\_{ω}ω\_{yy}.$$

$$=> A^{\*}u\_{φφ} + B^{\*}u\_{φω} + C^{\*}u\_{ωω} + D^{\*}u\_{φ} + E^{\*}u\_{ω} + F^{\*} u = G^{\*}. (2)$$

where

$$A^{\*}=Aφ\_{x}^{2}+Bφ\_{x}φ\_{y}+Cφ\_{y}^{2},$$

$$B^{\*}=2Aφ\_{x}ω\_{x}+B\left(φ\_{x}ω\_{y}+φ\_{y}ω\_{x}\right)+2Cφ\_{y}ω\_{y},$$

$$C^{\*}=Aω\_{x}^{2}+Bω\_{x}ω\_{y}+Cω\_{y}^{2}, $$

$$D^{\*}=Aφ\_{xx}+Bφ\_{xy}+Cφ\_{yy}+Dφ\_{x}+Eφ\_{y},$$

$$E^{\*}=Aω\_{xx}+Bω\_{xy}+Cω\_{yy}+Dω\_{x}+Eω\_{y},$$

$$F^{\*}=F,$$

$$G^{\*}=G.$$

It is seen that

$$B^{\*}^{2}-4A^{\*}C^{\*}=J^{2}\left(B^{2}-4AC\right).$$

Equation (1) is also expressed that

$Au\_{xx} + Bu\_{xy} + Cu\_{yy}=H(x,y,u,$ $u\_{x},u\_{y}). (3)$

Equation (2) is also expressed that

$A^{\*}u\_{φφ} + B^{\*}u\_{φω} + Cu\_{ωω}=H(φ,ω,u,$ $u\_{φ},u\_{ω}). (4)$

**1.4 Canonical Forms:**

$A^{\*}=Aφ\_{x}^{2}+Bφ\_{x}φ\_{y}+Cφ\_{y}^{2}=0 $ and $C^{\*}=Aω\_{x}^{2}+Bω\_{x}ω\_{y}+Cω\_{y}^{2}=0$ are can be expressed as

$$Aδ\_{x}^{2}+Bδ\_{x}δ\_{y}+Cδ\_{y}^{2}=0,$$

where $δ$ represents any of the $φ$ or $ω$ functions. Divide by $δ\_{y}^{2}$,

$$A\left(\frac{δ\_{x}}{δ\_{y}}\right)^{2}+B\left(\frac{δ\_{x}}{δ\_{y}}\right)+C=0. (5)$$

Along the curve $δ$ = constant,

$$dδ=δ\_{x}dx+δ\_{y}dy=0.$$

$$=> \frac{dy}{dx}=-\frac{δ\_{x}}{δ\_{y}},$$

From the Equation (5)

$$A\left(\frac{dy}{dx}\right)^{2}-B\left(\frac{dy}{dx}\right)+C=0. (6)$$

Characteristic Equations:

$$\frac{dy}{dx}=\frac{\left(B+\sqrt{B^{2}-4AC}\right)}{2A}, (7)$$

$$\frac{dy}{dx}=\frac{\left(B-\sqrt{B^{2}-4AC}\right)}{2A}, (8)$$

The characteristic equations are ordinary differential equations for families of curves on the $xy-plane$ along $φ$ =constant and $ω$ =constant. The integrals of the characteristic equations are referred to as the characteristic curves. As a result of the equations, the solutions to first-order ordinary differential equations can be expressed as

$$μ\_{1}\left(x,y\right)=c\_{1},$$

$$μ\_{2}\left(x,y\right)=c\_{2}$$

where $c\_{1}$ and $c\_{2 }$are constants.

The equations $φ=μ\_{1}\left(x,y\right) $and $ω=μ\_{2}\left(x,y\right)$ will convert the Equation (3) to canonical form.

**1.4.1 Hyperbolic Type**

$$B^{2}-4AC>0$$

* By integrating Equations (7) and (8), two independent families of real characteristic curves are obtained. Thus, Equation (5) reduces to

$u\_{φω}=\frac{H^{\*}}{B^{\*}}=H\_{1}, B^{\*}\ne 0, \left(9\right).$

The form of Equation (9) is the *first canonical form of the hyperbolic equation*.

$$α=φ+ω,$$

$$β= φ-ω $$

* Equation (9) is transformed to

$$u\_{αα}-u\_{ββ} =H\_{2}\left( α, β,u,u\_{α}, u\_{β}\right), \left(10\right).$$

The form of Equation (10) is the *second canonical form of the hyperbolic equation*.

**1.4.2 Parabolic Type**

$$B^{2}-4AC=0$$

=> Since Equation (7) and Equation (8) coincide, there exists only one real family of characteristics. $φ=constant, ω=constant.$

$$B^{2}=4AC=> A^{\*}=0 $$

It was mentioned that

$$A^{\*}=Aφ\_{x}^{2}+Bφ\_{x}φ\_{y}+Cφ\_{y}^{2}$$

$$= (\sqrt{A}φ\_{x}+\sqrt{C}φ\_{y}) ^{2}=0. $$

It was mentioned that

$$B^{\*}=2Aφ\_{x}ω\_{x}+B\left(φ\_{x}ω\_{y}+φ\_{y}ω\_{x}\right)+2Cφ\_{y}ω\_{y}$$

$$=2\left(\sqrt{A}φ\_{x}+\sqrt{C}φ\_{y}\right)\left(\sqrt{A}ω\_{x}+\sqrt{C}ω\_{y}\right)=0$$

Divide Equation (4) by $C^{\*}$,

$$u\_{ωω}=H\_{3}\left( φ, ω,u,u\_{φ}, u\_{ω}\right), C^{\*}\ne 0. (11)$$

The form of Equation (11) is the *canonical form of the parabolic equation*.

If we choose $ω$ = constant as the integral of Equation (7), Equation (4) can be

$$u\_{φφ}=H\_{3}\left( φ, ω,u,u\_{φ}, u\_{ω}\right). $$

**1.4.3 Elliptic Type**

$$B^{2}-4AC<0$$

=> Equation (6) has no real solution, but has two complexes conjugate solutions that are continuous complex valued functions real variables $x$ and $y$. Real characteristic curves don't exist. However, one can take into account equation (6) for complex $x$ and $y$ if the coefficients $A, B, $and $C$ are analytic functions of $x$ and $y$.

$φ$ and $ω$ are complex, let $α=\frac{1}{2}(φ+ω)$ and $β= \frac{1}{2i}(φ-ω)$be real variables.

* $φ=α+iβ$, $ω=α-iβ$ . (12)

Transform the Equation (3):

$A\_{\*}\left(α, β\right)u\_{αα}+B\_{\*}\left(α, β\right)u\_{αβ}+C\_{\*}\left(α, β\right)u\_{ββ}= H\_{4}\left( α, β,u,u\_{α}, u\_{β}\right)$ (13)

with the use of Equation (12), the equations $A^{\*}= C^{\*}=0 $become

$$\left(Aα\_{x}^{2}+Bα\_{x}α\_{y}+Cα\_{y}^{2}\right)-\left(Aβ\_{x}^{2}+Bβ\_{x}β\_{y}+Cβ\_{y}^{2}\right)+i\left[2Aα\_{x}β\_{x}+B\left(α\_{x}β\_{y}+α\_{y}β\_{x}\right)+2Cα\_{y}β\_{y} \right]=0,$$

$$\left(Aα\_{x}^{2}+Bα\_{x}α\_{y}+Cα\_{y}^{2}\right)-\left(Aβ\_{x}^{2}+Bβ\_{x}β\_{y}+Cβ\_{y}^{2}\right)-i\left[2Aα\_{x}β\_{x}+B\left(α\_{x}β\_{y}+α\_{y}β\_{x}\right)+2Cα\_{y}β\_{y} \right]=0,$$

$$=>\left(A\_{\*}-C\_{\*}\right)+iB\_{\*}=0, $$

$$\left(A\_{\*}-C\_{\*}\right)-iB\_{\*}=0. $$

$=>A\_{\*}=C\_{\*}$ and $B\_{\*}=0.$

From Equation (13),

$$A\_{\*}\left(α, β\right)u\_{αα}+A\_{\*}\left(α, β\right)u\_{ββ}= H\_{4}\left( α, β,u,u\_{α}, u\_{β}\right).$$

Divide by $A\_{\*}$

$$u\_{αα}+u\_{ββ}= \frac{H\_{4}}{A\_{\*}}\left( α, β,u,u\_{α}, u\_{β}\right)= H\_{5}\left( α, β,u,u\_{α}, u\_{β}\right). (14)$$

The form of Equation (14) is the *canonical form of the elliptic equation*.

**---------END OF THE PART 1--------------**