**Partial Differential Equations**

**The method of Separation of Variables**

**Foreward:**

* The general theory of partial differential equation is an advanced topic. Nevertheless, the method of solution by **separation of variables** is closely related to the boundary value problems for ordinary differential equations.
* The method of separation of variables is applicable to **linear partial differential equations;** a typical example is the wave equation

uxx=utt.

* In the application of the method of separation of variables one assumes that the unknown function is of the form u(x,t)=X(x)T(t). Then the equation uxx=utt reduces to

X’’T=XT’’,

where prime denotes ordinary differentiation with respect to the relevant independent variable. Dividing by X(x)T(t) we obtain

X’’/X=T’’T

* As both sides of this equation are functions of different variables, they should be equal to a constant, say k. Thus we finally get ODE’s

X’’=kX, T’’=kT.

Thus the problem is reduced to a set of ODE’s.

* The boundary conditions are crucial in solving these ODE’s. Thus one needs to learn about the **boundary value problems for ODE’s**.
* The possibility of solving the problem by separation of variables depends both on the differential equation and on the boundary conditions. It is necessary to choose appropriate coordinate systems, adopted to the problem and write the differential equation in these coordinates. In these notes we consider **ODE’s in Cartesian coordinates** only.
* Since the equations are linear, superposition of solutions is again a solution. For the periodic case, superposition of solutions leads to a Fourier series. Thus one needs to study **Fourier series.**
* The Hear equation, the wave equation and Laplace’s equation are typical examples that are common and important applications. We give on outline of the solutions and **refer the reader to the textbook for a complete solution**.

**Textbook:**

**Background pattern

Description automatically generated**Elementary differential equations and boundary value problems

William . Boyce, Richard C. DiPrima.

8th Edition

**Two-Point Boundary Value Problems *(Textbook Ch 10.1)***

* For a **Two-Point Boundary Value Problem** dependent variable *y* or its derivative *y’* is specified at two different points.
* *Finding function y solution,* with the given functions  *and additional two boundary conditions, is a two-point boundary value problem.:*

,

***(boundary conditions).***

**Example :**

Consider with boundary conditions .

The solution has the following general form

The first condition requires that .

The second condition requires that .

Now since , so it requires .

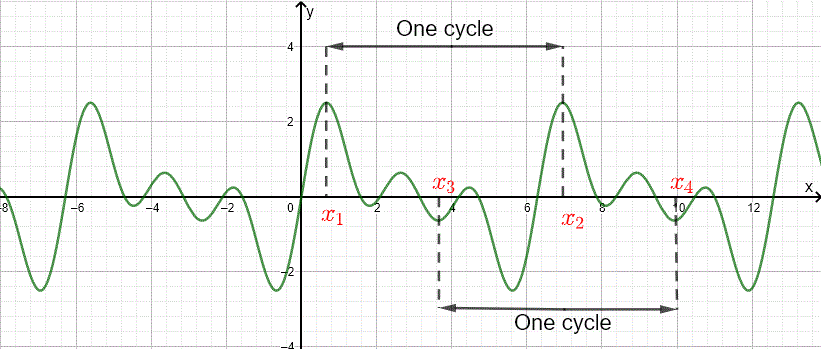
The entire solution is .

A boundary problem is **homogeneous** if the equation is homogeneous, and the two boundary conditions involve **zero**. That is, homogeneous boundary conditions might be one of these types:

On the other hand, if the equation is **nonhomogeneous** or any of the boundary conditions do **not equal zero**, then the boundary value problem is nonhomogeneous or **inhomogeneous**.

**Fourier Series *(Textbook Ch 10.2)***

* On the set of points where the series (1) **converges**, it defines a function , whose value at each point is the sum of the series for that value of .
* In this case the series (1) is said to be the **Fourier series** for . Main focus is determining what functions can be represented as a sum of a Fourier series and finding some means of computing the coefficients in the series corresponding to a given function.
* Periodicity of the Sine and Cosine functions. To discuss Fourier series, it is necessary to develop certain properties of the trigonometric functions and , where is a positive integer.
* The first property is their **periodic character**. A function is said to be periodic with period if the domain of contains whenever it contains , and if for every value of . An example of a periodic function is shown in **Figure 1. 2**.. It follows immediately from the definition that if is a period of , then is also a period, and so indeed is any integral multiple of .



**Figure 1.1: A periodic function**

* If and are any two periodic functions with common period , then their product and any linear combination are also periodic with period .
* For proving the statement, let ; then for any
* Orthogonality of the **Sine and Cosine Functions.** The standard inner product of two real-valued functions and on the interval is defined by
* Let, and are said to be **orthogonal** on if **inner product** is 0 that is, if
* A set of functions is said to be mutually orthogonal if each distinct pair of functions in the set is orthogonal.
* The functions and resulted to a **mutually** **orthogonal** set of functions on the interval .
* The following orthogonality relations is satisfied:
* To derive Eq. (8) by direct integration:
* On the other hand, if , then , and the integral must be evaluated in  
  series in a different way.
* The Euler-Fourier Formulas. Now let us suppose that a series of the form (1) converges, and let us call its :
* The coefficients and can be related with as a result of **orthogonality conditions**.
* First **multiplication** of (9) with , where , and **integration** with respect to from to . The integration assumed to be progressive **term by term**,
* is fixed when ranges over the positive integers, it follows from the orthogonality relations (6) and (7) that the only **nonzero term** on the right side of Eq. (10) is the one for which in the **first summation**. Then,
* The value can be represented by multiplying (9) by . Then, integrating term wise from to and using the orthogonality relations from (7) and (8);
* The Equation (14) is known as the Euler-Fourier formula for the **coefficients** in a Fourier series.

**Example :**

Fourier series converging to the function is

Determine the coefficients.

* The function is an example of a triangular wave (see Figure 1.2) Thus in this case , and the function periodic with 4.
* Fourier series has the initial form (16):

Chart, line chart

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Figure 1.2. A Triangular wave.

* After determination of the coefficients, the Fournier series have the final version in the following

**Heat Conduction in a Rod (Textbook Ch 10.5)**

Linear equations of second order:

* The Heat Equation
* The Wave Equation
* The Potential Equation ( also Laplace’s Equation)
* Consider a conduction heat equation issue for a straight bar of homogenous material with a uniform cross-section. Let the x-axis be set up to follow the axis of the bar (and . The bar is entirely heat-insulating, preventing any heat transfer. Temperature, can be considered constant on the tiny cross sections. Then, the function relies simply on the and time.

Diagram

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* The variation of temperature in the bar is governed by the heat conduction equation, and has the form
* where is a constant known as the thermal diffusivity.
* *T*1 at *x* = 0 and *T*2 at *x* = *L, are fixed*.
* Initial temperature distribution:
* Let consider *T*1 = *T*2 = 0. Thus, there will be boundary conditions:
* Chart

  Description automatically generatedFocus of the heat equation to find satisfies  
   all conditions below:

**Heat Equation: Separation of Variables**

* Initial form:
* Substitution into differential equation:
* Let’s use the for the separation of variables:
* For nontrivial solutions, Then it follows the following boundary problem:
* The eigenfunctions are nontrivial solutions to this boundary value problem
* The solution for the first order equation of is:

* Therefore, the fundamental solutions:

***There are many applications of Heat Equation; please see textbook.***

**Summary:**

* Heat Conduction Problem
* The solution is:

where

Diagram

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**Wave Equation: Vibrating String *(Textbook Ch 10.7)***

* and denote the two ends of the string and the x-axis limits.
* vertical displacement function of the string at the point *x* at time *t*.

Shape

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* Wave equation problem is:
* The constant coefficient is given where *T* is the tension, *ρ* is the mass.

1. **Separation of Variables**

* Initial form:
* Substitution into differential equation:
* Let’s use the for the separation of variables:

1. **Boundary Conditions**

* For nontrivial solutions, Then it follows the following boundary problem:

1. **Eigenvalues and Eigenfunctions**

* The eigenfunctions are nontrivial solutions to this boundary value problem
* The solution for the equation of is:

where are constants. Since , and hence

1. **Fundamental Solution**

* Fundamental solutions have the form
* Assume

where the are chosen so that the initial condition is satisfied:

***There are many applications of Wave Equation; please see textbook.***

**Summary**

* Vibrating string problem
* Solution

where

.

Shape

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**Laplace’s Equation *(Textbook Ch 10.8)***

* The equation initial forms (2-dimensions and 3-dimensions respectively):
* No initial conditions to be satisfied.
* Some boundary conditions according to bounding curve or surface of the region can be occur.
* Laplace’s equation is of second order. Then, two boundary conditions are required.

**Dirichlet Problem vs Neumann Problem**

* **Dirichlet problem**: Given boundary conditions is known for Laplace’s equation.
* **Neumann problem**: Values of the normal derivative are prescribed on the boundary known for Laplace’s equation.

**Dirichlet Problem for a Rectangle :**

* Drichlet Problem:
* Solution is

where

Chart

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Chart, pie chart

Description automatically generated**Dirichlet Problem on a Circle :**

* Drichlet Problem:
* Solution is

***There are many applications of Laplace’s Equation; please see textbook.***