**Ordinary Differential Equations**

**Instructor:** Prof.Dr.Ayşe Hümeyra Bilge

e-mail:ayse.bilge@khas.edu.tr:phone:0 533 267 44 12

**Textbook:**

1. Elementary Differential Equations and Boundary Value Problems.Ed8,BoyceR.C.DiPrima.JWS

**Foreward:**

* These lecture notes aim to review the material of the standard 2nd year differential equations course of engineering curricula and supplement it with an introduction to the study of partial differential equations.
* Ordinary differential equations are used in many engineering applications.
  + **First order ODE’s** describe many growth and decay phenomena.
  + **Second order ODE’s** describe osciallatory phenomena.
  + **Series solutions** are needed for the study of for example Bessel funcions that describe propagation of waves.
  + The method of **Laplace transform** is used in taking into account initial conditions.
  + **Systems of ODE’s** are used as models for all linear systems and find important applications in control theory. re are many fine textbooks part on ordinary differential equations is based
* Partial differential equations solvable by the method of separation of variables are also studied in a first course in differential equations. These topics are presented under the PDE headings.

**The notes on ODE’S consist of the following parts**

* Part 1: First order ordinary differential equations
* Part 2: Higher order ordinary differential equations
* Part 3: Series solutions
* Part 4: Laplace transform
* Part 5: Systems of ODE’s

**Part 1**

**First Order Ordinary Differential Equations**

In this section we present the following topics

* Classification of ODE’s
* Linear first order equations
* Separable equations
* Modeling with first order equations
* Population dynamics

The general formula for the integration of linear first order equation is the basis of most of engineering applications and it should be very well understood.

Separable equations are nonlinear first order equations that can be solved by a simple integration procedure. The examples given in the modeling part and in population dynamics are typical applications of first order ordinary differential equations.

**Classification of differential equations, elementary solution methods.**

* Differential equations are broadly classified as ordinary differential equations involving a single independent variable, and partial differential equations involving more than one independent variable.
* In simple cases the solution of ordinary differential equations can be reduced to “quadrature”, that means integration. In this section we will discuss these cases.
* Linear first order equation is a very important group of equations tahr appear in many areas. Solution methods are given and examples are discussed.

**Classification of Differential Equations**

* A **differential equation** is a relation between the derivatives of the dependent variables
* If only **ordinary derivatives** appear it is called an **ordinary differential equation (**ODE)

Here t is the independent variable, and x is the dependent variable.

* If there are **partial derivatives**, it is called a **partial differential equation** (PDE)

Here in the first equation, x and y are independent variables, u is the dependent variable. In the second equation t and x are independent variables, y is the dependent variable.

* If there is **more than one dependent variable** , we have a **system** of ODE or PDE:

This is a system of ODE, where t is the independent variable, x and y are the dependent variables

* The **order** of a differential equation is the order of the highest derivatives that appear in the equations
* **Linear** and **nonlinear** equations: An equation is said to be lineer, if the dependent variable and its derivatives appear linearly, there is no product or nonlineer functions present in the expression:

is **lineer**, y is the dependent variable, the coefficent functions may have nonlinear dependency on t.

is **lineer,** coefficient function are nonlinear but this has no effect on deciding on linearity.

is **nonlinear** because of the term

is **nonlinear** because of the siny term

* **A solution** of a differential equation is a function that satisfies the differential equation. It is given on an interval and it has to be sufficiently differentiable. In the examples below, we just **“check”** that the given function satisfies the differential equation.

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| 1. is a solution, because 2. Let find the first and second derivatives and substitute |

**Linear first order equations**

The general form of a linear first order equation is as below

General form:

**Remark:** We have to be careful when using the term “linear”. A “linear” differential equation is a differential equation whose solutions form a “linear subspace” of the vector space consisting of functions (with certain differentiability properties).

* If the right-hand side is zero, i.e, g(t)=0, such equations are called “**homogeneous**” and the solutions form indeed a linear subspace. In these cases, the “superposition principle” is valid, this means if y1(t) and y2(t) are solutions, then their linear combination c1y1(t)+c2y2(t) is again a solution.
* If g(t) is not zero, the equation is called “**inhomogeneous**” and the superposition principle no longer holds. Nevertheless, differential equations whose left hand side is linear in the independent variable and its derivatives are called “linear” equations.

**Method of solution:** We multiply both sides of the equation by an “integrating factor” to obtain an expression that can be solved by an integration.

Next example is a linear first order ODE where the ingomogeneous term is a constant.

**Example 1.1:**

Multiply by

integrate

* We obtained the solution by multiplying with an “integrating factor” then integrating. Note that there is an “integration constant” reflecting the fact that indefinite integrals are defined up to a constant.
* Next example illustrates the case whete the inhomogeneous term is not a constant. In this example we have also shown incorporation of initial conditions to the solution.

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| **Example 1.2:**  multiply by  Initial conditions : |

We now give the general formula

**General formula for**  . Integrating factor is

In terms of indefinite integrals, the solution is:

In terms of definite integrals the solution is written as below.

, if we use definite integrals

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| **Example 1.3:**  We need to integrate and . We use integration by parts,  (  t+=t  , |

* In this example, the difficulty was the need for integration by parts. In some cases the integral may be non-elementary, i.e, it may not be expressed in terms of known functions, but the ODE is still considered as “solved”, because the solution is reduced to a “quadrature”.
* The next step is to solve equations where p is not a constant, i.e, it is a function of t. In this case we need to integrate p(t) to obtain the integrating factor

**General case:**

Multiply by

The general solution is formally obtained in terms of indefinite integrals. Whether these integrals are elementary of not, the ODE is considered as “solved”.

We illustrate this procedure by the examples below.

|  |
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| **Example 1.4:** ,    Multiply the differential equation by use form:  If we are given an initial condition  **Remark**: t should be non zero. The domain can be Since the initial condition is given at , the domain is , |

|  |
| --- |
| **Example 1.5:** Solve the given initial value problem  ,  multiply by use =1    Using the initial condition, |

* We have seen that the solution of first order linear equations always reduces to a “quadrature”, i.e, to integration, althouh the integrals that we encounter may be elementary, i.e, they may not be evaluated in terms of known functions.
* A group of nonlinear ODE’s that can be solved by quadrature, is the so-called, “separable” equations. In these equations the right hand side can be written as a product of functions of 2 functions, one is a function of the independent variable, the other is a function of the dependent variable. The solution is obtained by dividing the differential equation by the function og the dependent variable and integrate the left hand side with respect to the dependent variable and integrate the left hand side with respect to the independent variable.

**Separable Equations**

These equations are of the form

We solve them by integrating

These integrals may be non-elementary; even if both integrals are obtained in terms of known functions, the result is an implicit expression of the dependent variable.

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| **Example 1.6** |

|  |  |  |
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| **Example 1.7**  Put | | |
| **Example 1.8** |

**Modeling with first order equations**

**Example 1.9**  Continuously compounded interest is modeled as where S0 is the initial investment, r is the interest rate and S(t) is the balance at time t. The solution is obtained by a simple integration, as

We compare this method with discrete compounding, namely with annual, semi-annual quarterly and monthly compounding.

annual

semi annual

quarterly

monthly

A constant deposit of withdrawal at fixed intervals can be modeled by the differential equation below, in the setting of continous compounding. The solution is obtained similarly.

, after rearranging, we obtain,

We illustrate this last case with a specific example. Assume that a person starts depositing at a fixed rate k, and with no initial investment, at age 25. Thus and let the deposit and the interest rates be Find S(t) at age 65.

The duration of the deposit years. We use the formula to compute S(t)

**Example 1.10 Population dynamics: Exponential growth**

In the absence of any environmental limitation, the growth rate of a population is proportional to the present amount. This is the basic law of population growth, and leads to an exponential increase in the population.

Exponential growth:

Unlimited growth is in general unrealistic. It can be used for example to model bacterial growth. In more realistic cases, there is a threshold called “environmental carrying capacity”, this means the growth rate decreses as the population approaches this threshold

These situations are described by logistic growth models.

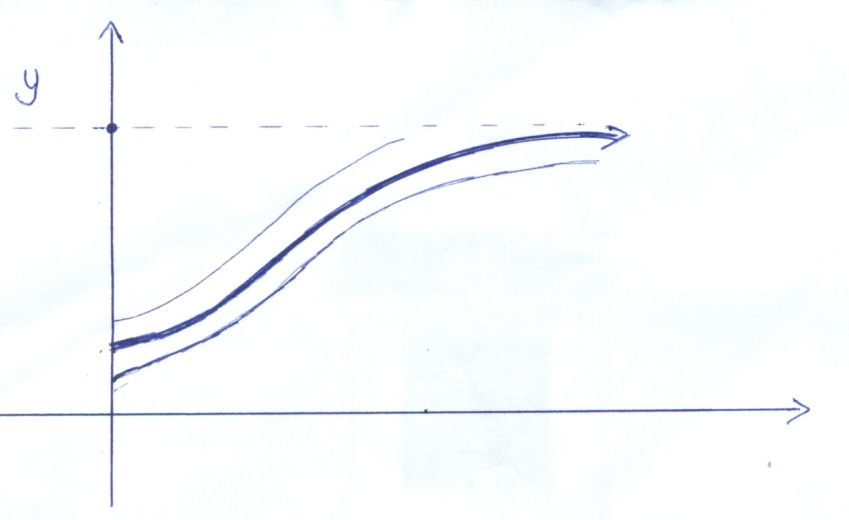
In the equation , if r<0, then the solutions are and the solutions are decaying. This case, described as “exponential decay” is representative of radioactive decay and cooling. These cases are described by the examples below.

**Example 1.11 Population dynamics: Logistic growth**

The logistic growth is characterized by the differential equation below. In this equation, the rate of change of the quantity is proportioal to a product.

Logistic growth:

Note that for y between 0 and k, the rate of change is positive, hence the solution curve is increasing. The rate of change approaches the limit 0, as y gets close to 0 and to k.



The exact solution can be obtained by a straightforward integration, as given below. It is more instructive to use a geometric method. In this approach, one considers the sign of the right hand side. In the region where y>0 and y<k, y(t) is increasing. y(t)=0 and y(t)=k are solutions of the ODE. By the uniquness theorem that will ve given below, solution curves cannot inersect, thus solutions starting in the region (0,k) stay in this region approaching the y(t)=k line asymptotically.

**Solution:** Let k=1 r=1

Use partial fraction expansion to integrate,

Assume

**Example 1.12 Radioactive decay**

Unstable atoms spontaneously decay into smaller atoms by emitting mass or radiation. This process is called radioactive decay. The equation describing the decay of a radioactive substance is as follows. According to the formula below, it can be seen that the concentration of a radioactive substance at time t is y.

Radioactive decay: r > 0.

The period of time needed for half of the radioactive nuclei in a sample to decay is known as the half-life of a radioactive element.

The half-life of the element is the such that .

.

Let the half-life of a radioactive element is = 4350 years. If an element sample contains 12% of the original amount, find the age of the element sample was created.

**Solution:**

**=>**

years.

**Example 1.13 Newton’s law of cooling**

The rate at which the temperature of a material changes at a time t is proportional to the difference between the temperature of this material and the environment temperature. This observation is called Newton's Law of Cooling.

Let be the temperature of the object at time and is the constant environment temperature.

Newton’s law of cooling :

If is accepted, then

, is constant.

.

The solution of is .

where is temperature at .

A container with water at 50 °C is placed in an environment at 10 °C. After 2 minutes the water temperature is 20 C, when will the water temperature be 15 C?

**Solution:**

**Exact Equations and integrating factors:**

Recall that a linear first order differential equation is solved by multiplying with an “integrating factor”. The same method works for certain nonlinear equations, but there is no precise rule for deciding whether this method will work or not, and the is no straightforward algorithm for finding the integration factor. Nevertheless the method is useful and it is closely related to the important notion of “exact equations”.

In this method, we follow the steps below:

* **➀** Decide whether a given equation is exact or not
* **➁** If it is exact, find the solution
* **➂** If the equation is not exact, and you are given an integrating factor, multiply the equation with this integrating factor and solve
* **➃** If the integrating factor is not given we will tell you its form, say or etc. Then you should first find then solve the differential equation.

**Definition** Consider the equation

**(1)**

This can be written in the form

Or equivalently , sometimes in the form

1. is called exact if

**(2)**

WHY? Because if (2) holds, there is a function such that

Then, implicit differentiation of gives

**➀**

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| **Example 1.14**  Is this equation exact?  The equation is exact |

|  |
| --- |
| **Example 1.15:** Is exact?  It is **not** exact |

**➁** How to integrate an exact equation:

1. or

Evaluate the easiest integral. Then differentiate

1. given an equation for g(y)
2. given an equation for h(x)

Solve these to determine up to a constant.

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| **Example 1.16:**  Integrate M with respect to x:  Compare  If we start by integrating N:  Compane :  , Solution is given implicitly by |
| **Example 1.17:**  Solution is given implicitly by | | |

**➂** The equation is not exact, integrating factor is given

|  |
| --- |
| **Example 1.18:**  **(1)**  is not exact is an integrating factor:   * Multiply (1) by   **(2)**   * Check that it is exact :   it is exact   * Integrate (2) |

**➃** The equation is not exact, integrating factor is not given:

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| --- |
| **Example 1.19**  We try an integrating factor of the form ,  **(3)**  Equate them :  Divide by |

**The existance and uniqueness theorem**

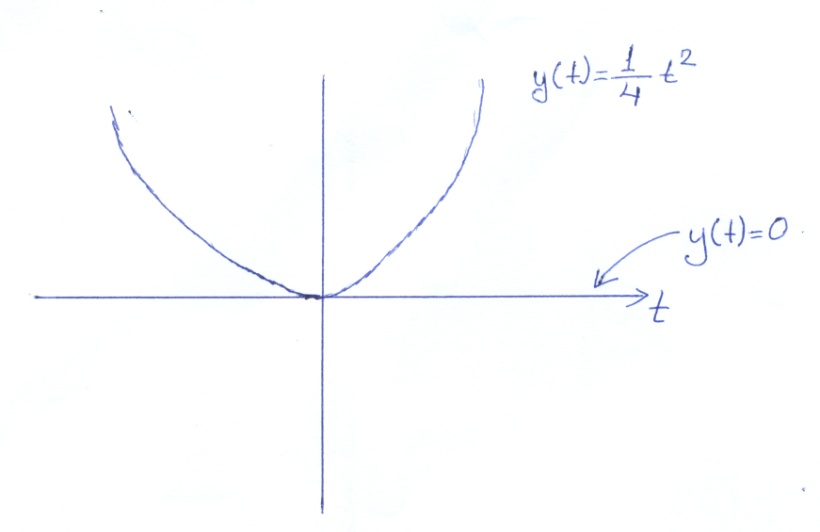
The solution of a first order **ODE**

can be constructed by succession approximations. There are computer programs that construct the solution numerically.

* It is important to know whether there is a solution curve stanting from the point . This is the **existance** problem
* If there is a solution curve stating at , then we ask the question ‘’is it unique’’?

**Ex**: .

1. is a solution
2. If integrate both side to get



Solution 1

Solution 2

Here the solution starting at (0,0) is not unique. If you use a numerical method, you may end up with an unwanted solution.

It is important to be sure about the existence and uniqueness.

**The existence and uniquenes teorem:**

Let

Assume that and aree continuous in a rectangle then there is an inteval in which the solution starting at  **exists and it is unique.**

Solution exists and it is unique here

In the previous example

is not continuous at

Hence the existence and uniqueness theorem cannot be applied.

**END OF PART 1**