Markov chains and applications

Dr. András Pongrácz

Week 13, University of Debrecen

Infinite vectors, matrices

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We can multiply infinite row and column vectors (indexed by $\mathbb{N}$ or $\mathbb{N} \cup \{0\}$) in the logical way, by computing an infinite sum.

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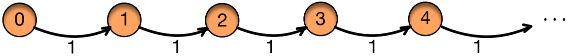

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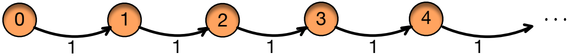

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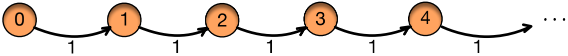

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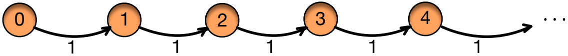

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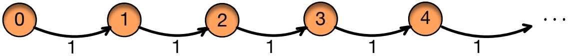

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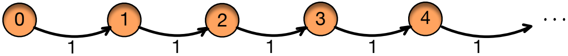

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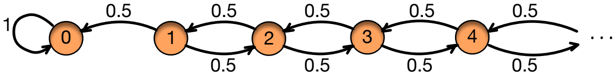
$\lim_{k\to\infty} P^k = 0$ (the walk fades away at infinity)

An absorbing example (infinite fair gambler's ruin) :



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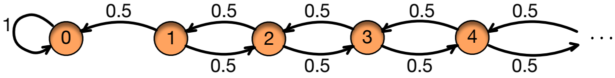
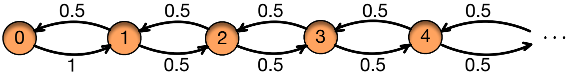


The irreducible version (period is 2):



Classification of states

Recall that m_i is the mean recurrence time, that is, the expected value of the random variable τ_{ii}.

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  - positive recurrent if $m_i = E(\tau_{ii}) < \infty$ (which clearly implies that it is recurrent), and
  - null recurrent if it is recurrent but not positive recurrent.
- transient otherwise, that is, if $\mathbb{P}(\tau_{ii} < \infty) < 1$.

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What about countably infinite chains?

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Given a countable Markov chain with two states i and j in the same strong connected component of the corresponding graph. Then i is transient iff j is.

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Thus with positive probability, a walk starting from i returns to i only finitely many times, a contradiction.

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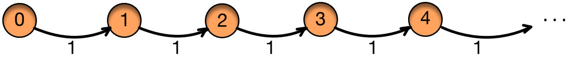
In the right shift, all states are transient:



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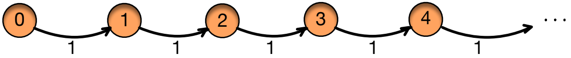

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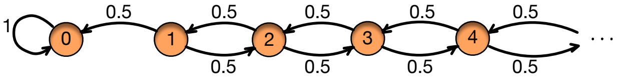

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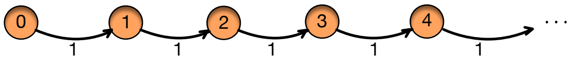

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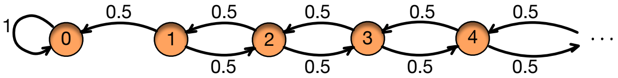
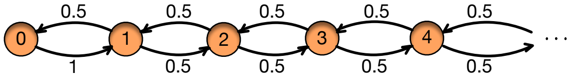


In the irreducible version, all states are null recurrent (cf. the exercises):



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3. The n-lollipop graph consists of a complete graph of $n/2$ vertices with a path of length $n/2$ glued to a vertex. Show that the mean hitting time from any vertex u of the complete graph to the base of the lollipop is at least cubic. Conclude a cubic lower bound for the cover time.

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4. Prove that a connected graph is a two-sided expander iff it is not bipartite. (Hint: recall Hoffman's theorem.)

5. Let $\mathbb{N} \cup \{0\}$ be equipped with the irreducible fair walk structure (third slide) : from every positive integer we move to each neighbor with equal probability $1/2$. Show that 0 is a null-recurrent state. (Hint : show that hitting n from 0 has probability $1/n$.)

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6. Classify all states in the previous example, and also in the unfair versions. That is, stepping to the right has probability p. What if $p > 1/2$? And if $p < 1/2$?

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7. Prove the proposition about communicating recurrent states.