Markov chains and applications

Dr. András Pongrácz

Week 9, University of Debrecen

Absorbing Chains

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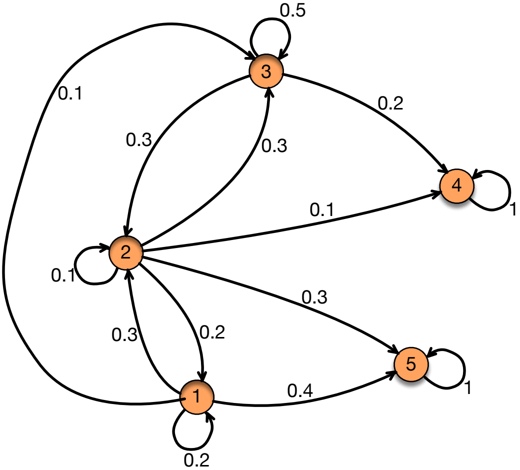
The non-absorbing states in such chains are called transient; cf. the last class for a more precise introduction to transient states.


0.5
3
0.1
0.2
0.3
0.3
4
0.1
1
2
0.1
0.3
0.3
0.2
5
1
0.4
1
0.2

Markov chains

Transition matrix:

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\begin{array}{c}
\begin{array}{ccccc} Q & & & & R \end{array} \\
\begin{pmatrix}
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0 & 0.3 & 0.5 & 0.2 & 0 \\
0 & 0 & 0 & 1 & 0 \\
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\end{pmatrix}
\end{array}
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fundamental matrix

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(Given an initial state) what is the expected time to absorption?
Answer: $N\underline{\mathbf{1}}$

(Given an initial state) what is the probability of absorption at each absorbing state? Answer: NR

Sketch of arguments

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Corollary II

The probability of absorption in a finite number of steps from any initial state is 1.

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The entry $Q^k[i,j]$ is the probability that, starting from the i-th state, we are at the j-th state after k steps.

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yielding $\underline{x} = N\underline{1}$.

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The proof is based on the law of total probability. (Cf. the exrecizes.)

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yielding $\underline{x} = NR$.

Higher moments

Moment generating funciton

Given a random variable X, its moment generating function is

$$t \mapsto E(e^{tX})$$

(wherever it is defined).

Theorem

If the moment generating function is defined in a neighborhood of 0, then

$$\left(\frac{d^n}{dt^n} E(e^{tX}) \right)_{t=0} = E(X^n)$$

is the n-th moment of X.

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We obtain the system of linear equations

$$\underline{v}(t) = e^t(Q\underline{v}(t) + (I - Q)\underline{1})$$

Rearranging:

$$(I - e^t Q)\underline{v}(t) = e^t (I - Q)\underline{1}$$

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The first derivative at zero is
$\underline{v}'(0) = (I + 2Q + 3Q^2 + 4Q^3 + \cdots)(I - Q)\underline{1} = (I - Q)^{-2}(I - Q)\underline{1} = N\underline{1}$.

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$f_1(N) = N$, $f_2(N) = 2N^2 - N$, $f_3(N) = 6N^3 - 6N^2 + N$, $f_4(N) = 24N^4 - 36N^2 + 14N - N$ can be easily computed from the above.

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4. Prove the general formula about $N\underline{v}$ being the expected value of the sum of entries on $\underline{v}$ during a walk until absorption.
5. Observe that the hypergeometrical distribution is a special absorbing Markov chain with 2 states. Deduce the formula $1/p$ for the expected value as a special case of the expected runtime of an absorbing Markov chain.