Markov chains and applications

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Frequency of Recurrence

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Thus $\sum_{k=0}^{\infty} (P - W)^k$ exists, and then it has to be $(I - (P - W))^{-1}$.

Fundamental matrix

Definition

Given a regular Markov chain P with stationary distribution $\underline{w}^*$ and $W = \underline{1} \cdot \underline{w}^*$, the fundamental matrix of P is defined as

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Proof: Exercise.

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Note that this returns $M[i,i] = 0$, which is the logical convention. For the more meaningful mean recurrence time, see the previous class: the i-th mean recurrence time is $1/w_i$.

Law of Large Numbers

Classical formulation

Weak form

Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E(X_1) = \mu \in \mathbb{R}$. Then for all $\varepsilon > 0$ we have

$$\lim_{k\to\infty} Pr\left(\left|\frac{X_1 + \cdots + X_k}{k} - \mu\right| > \varepsilon\right) = 0$$

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Strong form: In fact, the random variable $\left|\frac{X_1+\cdots+X_k}{k} - \mu\right|$ converges to 0 with probability 1.

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Given an irreducible (ergodic) chain, let $I_k^{(i)}$ be the indicator that the walk is in state i after k steps. Then for all $\varepsilon > 0$ we have

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Central Limit Theorem

Classical formulation

Theorem

Let $X_1, X_2, \ldots$ be i.i.d. random variables with $E(X_1) = \mu \in \mathbb{R}$ and $\mathrm{Var}(E_1) = \sigma^2 \in \mathbb{R}$. Then

$$\sqrt{k}\left(\frac{X_1 + \cdots + X_k}{k} - \mu\right)$$

converges in distribution to $N(0, \sigma^2)$.

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The logical way to define σ_i^2 is $2w_iZ[i,i] - w_i - w_i^2$.

CLT

Given an irreducible (ergodic) chain, let $I_k^{(i)}$ be the indicator that the walk is in state i after k steps. Define $\sigma_i^2 = 2w_i Z[i,i] - w_i - w_i^2$. Then

$$\sqrt{k}\left(\sum_{m=1}^{k} I_m^{(i)}/k - w_i\right)$$

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6. Compute the mean recurrence times, mean passage times and the σ_i as a continuation of Problem 5.