Markov chains and applications

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Week 5, University of Debrecen

Finite Markov chains

... as digraphs

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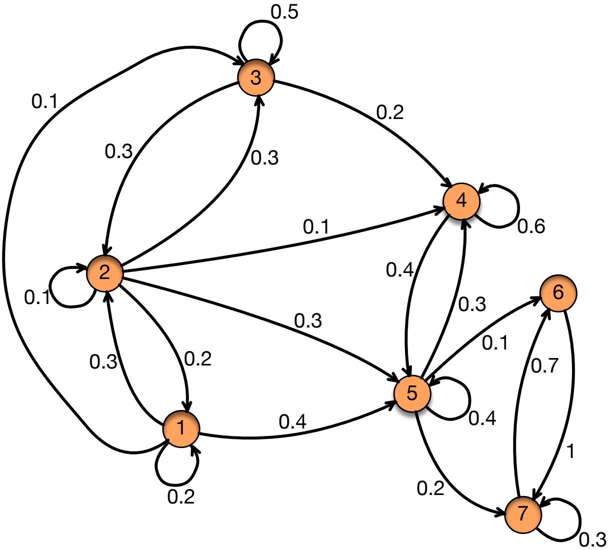
The idea is that we walk on the states, and at every step we decide where to move according to the probability distribution on the outgoing edges.



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Basic properties:

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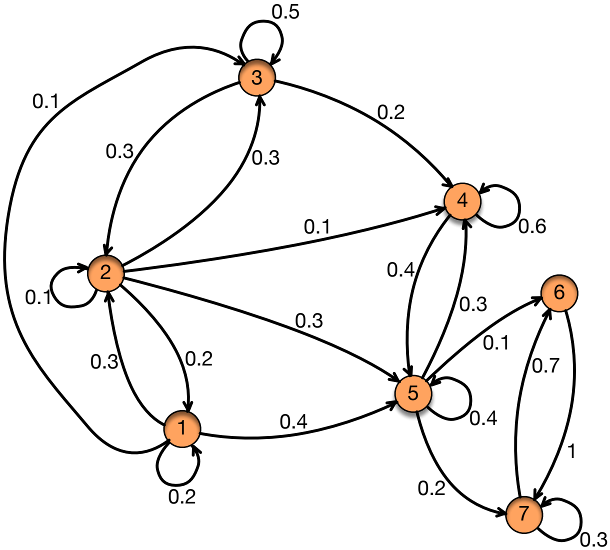
Basic properties :

- $P[i,j] \geq 0$
- $\sum_{j=1}^{n} P[i,j] = 1$ for all $1 \leq i \leq n$. Equivalently $P\underline{1} = \underline{1}$, so the all-1 vector is an eigenvector of P with eigenvalue 1.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$$P = \begin{pmatrix} 0.2 & 0.3 & 0.1 & 0 & 0.4 & 0 & 0 \\ 0.2 & 0.1 & 0.3 & 0.1 & 0.3 & 0 & 0 \\ 0 & 0.3 & 0.5 & 0.2 & 0 & 0 & 0 \\ 0 & 0 & 0.6 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0.4 & 0.1 & 0.2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0.3 \end{pmatrix}$$

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Proposition

The probability of being at s_j after m steps starting from s_i is

$$(P^m)[i,j]$$

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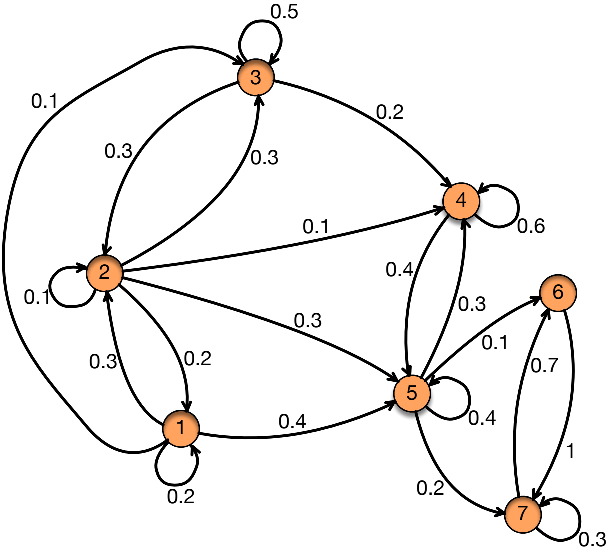
We can deconstruct the walk into two walks: first on this acyclic graph, until we reach a sink, and then a walk on the sink. (Cf. topological sorting of an acyclic digraph.)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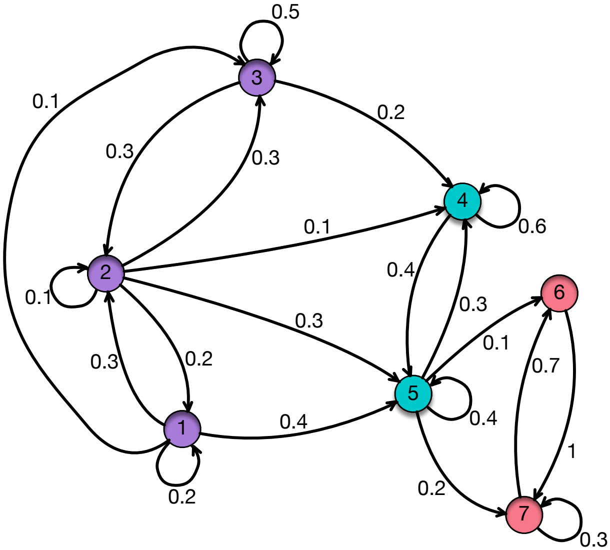
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Absorption

A state in a Markov chain is absorbing if it has only one outgoing edge, and the edge points to the state (and then the transition probability assigned to it is necessarily 1). A finite Markov chain is absorbing if (it has at least one absorbing state and) from any state, we can reach an absorbing state with positive probability in a finite number of steps.

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The natural problems to study on them are completely different, so they are investigated separately on the next classes.

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4. Show that a matrix P is a transition matrix (of some Markov chain) iff P is non-negative and $P\underline{1} = \underline{1}$.
5. What is the difference between acyclic graphs and partial orders?
6. Prove that if P is absorbing, then so is P^m for any m. What does an absorbing, primitive chain look like?