Markov Chains and Their Applications

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Week 4, University of Debrecen

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2. There is no other positive eigenvector of P than the positive multiples of the Perron vector.
3. By normalizing $\underline{u}$, that is, picking the constant multiple so that the sum of entries is 1, and similarly picking a normalized Perron vector $\underline{w}$ for P^*, we have $\lim_{k\to\infty} \frac{1}{r^k} P^k = \underline{w}^* \underline{u}$.

Definition

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$\chi_P(x) = x^2 - x - 1$ with roots $\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}$, so the Perron root is $r = \frac{1+\sqrt{5}}{2}$. The corresponding (right) eigenvector is $\underline{u} = (1, \frac{1+\sqrt{5}}{2})^*$.

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Definition

Given a non-negative matrix $A \in M_n(\mathbb{R})$, the associated digraph is $G_A = (V, D)$, where V is an n-element set identified by the rows and columns, and $(v_i, v_j) \in D$ iff $A[i, j] > 0$.

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A reducible matrix thus must have a proper subset of vertices $U \subseteq V$ such that there is no edge (u, v) with $u \in U$ and $v \notin U$. By rearranging the identification of vertices with rows and columns so that vertices in U come first, we obtain a matrix similar to A (by conjugation with a permutation matrix) of the form $\begin{pmatrix} E & F \\ 0 & G \end{pmatrix}$

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Note that in item 1, the condition is about the non-existence of invariant coordinate subspaces, not invariant subspaces in general. The latter would be impossible for $n \geq 3$, as every real matrix with $n \geq 3$ has an at most 2-dimensional proper invariant subspace.

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This observation leads to the extra property that together with irreducibility guarantees a slightly weaker version of the Perron-Frobenius theorem: see Class 6 for further details.

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Let $A = (a_{i,j}) \in M_n(\mathbb{C})$ be a matrix. Then the n closed discs on the complex plane for all $1 \leq i \leq n$ with center $a_{i,i}$ and radius $\sum_{j \neq i} |a_{i,j}|$ are called Gershgorin discs.

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It was observed by Semyon Aronovich Gershgorin that the union of these discs contain the spectrum of the matrix.

Example

$$A = \begin{pmatrix} 7 & 0 & -1.1 & 2.6 \\ 2 & -11 & 0.7 & 1.4 \\ -0.9 & 1.1 & i & 1 \\ 0.7 & -0.3 & 1.6 & i \end{pmatrix}$$

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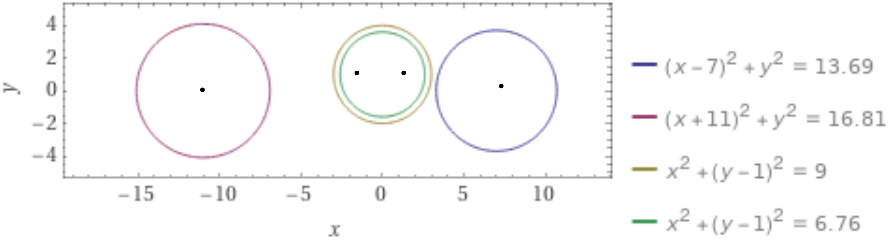

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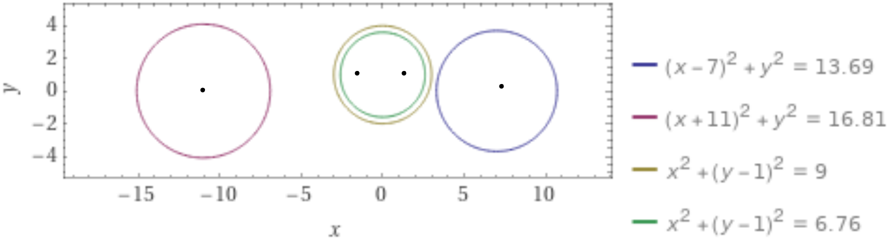

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Note that the union of circles can be divided into three connected components, consisting of (from left to right) one, two and one discs, respectively.

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By taking the modulus of both sides, and using the triangle inequality, we obtain $\sum_{j \neq i} |a_{i,j}||u[j]| \geq |\lambda - a_{i,i}||u[i]|$.

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By taking the modulus of both sides, and using the triangle inequality, we obtain $\sum_{j \neq i} |a_{i,j}||u[j]| \geq |\lambda - a_{i,i}||u[i]|$. As $|u[j]| \leq |u[i]|$ for all $i \neq j$, we have $\sum_{j \neq i} |a_{i,j}||u[i]| \geq |\lambda - a_{i,i}||u[i]|$,

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By taking the modulus of both sides, and using the triangle inequality, we obtain $\sum_{j \neq i} |a_{i,j}||u[j]| \geq |\lambda - a_{i,i}||u[i]|$. As $|u[j]| \leq |u[i]|$ for all $i \neq j$, we have $\sum_{j \neq i} |a_{i,j}||u[i]| \geq |\lambda - a_{i,i}||u[i]|$, and then after dividing by $|u[i]|$, the desired inequality $\sum_{j \neq i} |a_{i,j}| \geq |\lambda - a_{i,i}|$ follows.

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Theorem

If a connected region of the union of discs is the union of k discs, then it contains exactly k eigenvalues with multiplicity (cf. the example).

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The Gershgorin discs of $M(t)$ have the same centers for all t, and the radii are linear functions starting from 0 and ending up at the radii of the Gershgorin circles of A.

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By the Gershgorin circle theorem, the spectrum of $M(t)$ is contained in $K \cup K'$ for all t. As K and K' have a positive distance and the labelled spectrum is a continuous function, there are exactly k entries of the labelled spectrum of $M(t)$ in K for all t, and then in particular for $t=1$.

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4. Observe that the series of vectors $\underline{u}_n = (F_n, F_{n+1})^*$, where F_n is the n-th Fibonacci number, satisfies the relations $\underline{u}_0 = (0,1)^*$ and $\underline{u}_n = P\underline{u}_{n-1}$, with P as in the previous exercise. Deduce that $\underline{u}_n = P^n\underline{u}_0$. By the previous exercise, find the exact formula for F_n.

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5. Show that the union of the Gershgorin discs coincide with the spectrum iff the matrix is diagonal.