Markov Chains and Their Applications

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Week 3, University of Debrecen

Graphs, adjacency matrices, spectra

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Corollary : Cayley's theorem follows immediately, i.e., the number of spanning trees of the complete graph K_n is n^{n-2} (cf. the exercises).

Cauchy interlace theorem

Definition

Given $f(x), g(x) \in \mathbb{R}[x]$, all roots are real, $\deg f = n = \deg g + 1$. Roots of f are $\alpha_1 \leq \cdots \leq \alpha_n$, roots of g are $\beta_1 \leq \cdots \leq \beta_{n-1}$. Then f and g interlace if

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Interlacing polynomials

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Theorem (Rahman and Schmeisser)

The polynomials $f(x), g(x) \in \mathbb{R}[x]$ with all real roots and $\deg f = n = \deg g + 1$ interlace if and only if for all $\lambda \in \mathbb{R}$ the polynomial $f + \lambda g$ has all real roots.

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If P_i is the projection $\mathbb{R}^n \to \mathbb{R}^n$ defined by the annulation of the i-th coordinate, then the i-th minor is the restriction of $P_i^* A P_i$ to the image of P_i.

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$$\det \begin{pmatrix} A_n - xI_{n-1} & \underline{c} \\ \underline{c}^* & d + \lambda - x \end{pmatrix} =$$

$$\det \begin{pmatrix} A_n - xI_{n-1} & \underline{c} \\ \underline{c}^* & d - x \end{pmatrix} + \det \begin{pmatrix} A_n - xI_{n-1} & \underline{c} \\ \underline{0}^* & \lambda \end{pmatrix} = f(x) + \lambda g(x).$$

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Then for all $1 \leq j \leq m$ we have $\alpha_j \leq b_j \leq \alpha_{n-m+j}$.

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Corollary: The following are equivalent for a graph G.

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Proof (sketch): (1) $\Rightarrow$ (2) $\Rightarrow$ (1) $\Rightarrow$ (3) $\Rightarrow$ (2) is easy by using Hoffman's theorem. See more details on the next class.

Exercises

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6. Prove Cayley's theorem on the number of labelled trees on n vertices.