### **Markov Chains and Their Applications**

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Week 2, University of Debrecen



# Linear algebra

Markov chains

Pongrácz



Input:  $n \times n$  matrices A, B





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For n = 2, possible with 7 multiplications rather than 8. If  $2^{k-1} \le n \le 2^k$ , dynamically build up in *k* iterative steps. E.g., if n = 4, then apply the 2 × 2 Strassen method to the 2 × 2 blocks.



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$$\begin{array}{ll} X_1 = (A_{1,1} + A_{2,2})(B_{1,1} + B_{2,2}); & X_2 = (A_{2,1} + A_{2,2})B_{1,1} \\ X_3 = A_{1,1}(B_{1,2} - B_{2,2}); & X_4 = A_{2,2}(B_{2,1} - B_{1,1}) \\ X_5 = (A_{1,1} + A_{1,2})B_{2,2}; & X_6 = (A_{2,1} - A_{1,1})(B_{1,1} + B_{1,2}) \\ X_7 = (A_{1,2} - A_{2,2})(B_{2,1} + B_{2,2}). \end{array}$$



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Conjecture : best algorithm has runtime  $\approx \Theta(n^2)$ . The lower bound  $2n^2$  is trivial (we have to read the input), necessity of  $Cn^2$  multiplications with large *C* can be proven by advanced algebraic methods.



#### Definition

Let *K* be a field, and let  $x^m + a_{m-1}x^{m-2} + \cdots + a_1x + a_0 = f(x) \in K[x]$  be an arbitrary degree *m* monic polynomial,  $m \in \mathbb{N}$ .



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| /0 | 0 | 0 | ••• | 0 | 0 | $-a_0$                      |
|----|---|---|-----|---|---|-----------------------------|
| 1  | 0 | 0 | ••• | 0 | 0 | $-a_1$                      |
| 0  | 1 | 0 | ••• | 0 | 0 | $-a_2$                      |
| :  |   | · |     |   |   |                             |
| :  |   |   | ۰.  |   |   |                             |
| 0  | 0 | 0 |     | 1 | 0 | $-a_{m-2}$                  |
| /0 | 0 | 0 |     | 0 | 1 | - <i>a</i> <sub>m-1</sub> / |



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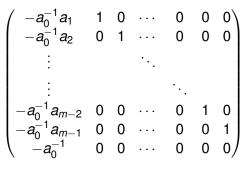
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## Frobenius normal form



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Remark: the 0 polynomial is allowed in the series  $f_1(x), \ldots, f_k(x)$  any number of times (at the end).





The polynomials  $f_1(x), \ldots, f_k(x)$  can be read from A (in linear time). Then  $\chi_A(x) = f_1(x) \cdots f_k(x)$ ,  $m_A(x) = f_k(x)$  (yielding a proof to the Cayley-Hamilton theorem).



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The inverse matrix  $A^{-1} = (SFS^{-1})^{-1} = SF^{-1}S^{-1}$  can be determined by replacing each block of *F* by its inverse; see an earlier slide.



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- The algorithm is not yet fully de-randomized.

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- 6. What is dim ker(A) for the matrix A constructed in problem 3?