Markov Chains and Their Applications

Dr. András Pongrácz

Week 2, University of Debrecen

Linear algebra

Matrix multiplication

Input: $n \times n$ matrices A, B

Matrix multiplication

Input: $n \times n$ matrices A, B

Find AB

Matrix multiplication

Input: $n \times n$ matrices A, B

Find AB

By definition: n^3 multiplications (and some additions, less importantly).

Matrix multiplication

Input: $n \times n$ matrices A, B

Find AB

By definition: n^3 multiplications (and some additions, less importantly).

STRASSEN

For $n = 2$, possible with 7 multiplications rather than 8.

Matrix multiplication

Input: $n \times n$ matrices A, B

Find AB

By definition: n^3 multiplications (and some additions, less importantly).

STRASSEN

For $n = 2$, possible with 7 multiplications rather than 8.

If $2^{k-1} \leq n \leq 2^k$, dynamically build up in k iterative steps.

Matrix multiplication

Input: $n \times n$ matrices A, B

Find AB

By definition: n^3 multiplications (and some additions, less importantly).

STRASSEN

For $n = 2$, possible with 7 multiplications rather than 8.

If $2^{k-1} \leq n \leq 2^k$, dynamically build up in k iterative steps. E.g., if $n = 4$, then apply the 2×2 Strassen method to the 2×2 blocks.

Strassen, $n = 2$

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \qquad B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$$

the entries can be square matrices of equal size.

Strassen, $n = 2$

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \qquad B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$$

the entries can be square matrices of equal size. Then

$$AB = \begin{pmatrix} X_1 + X_4 - X_5 + X_7 & X_3 + X_5 \\ X_2 + X_4 & X_1 - X_2 + X_3 + X_6 \end{pmatrix}$$

Strassen, $n = 2$

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} \\ A_{2,1} & A_{2,2} \end{pmatrix} \qquad B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \end{pmatrix}$$

the entries can be square matrices of equal size. Then

$$AB = \begin{pmatrix} X_1 + X_4 - X_5 + X_7 & X_3 + X_5 \\ X_2 + X_4 & X_1 - X_2 + X_3 + X_6 \end{pmatrix}$$

$$X_1 = (A_{1,1} + A_{2,2})(B_{1,1} + B_{2,2}); \quad X_2 = (A_{2,1} + A_{2,2})B_{1,1}$$
$$X_3 = A_{1,1}(B_{1,2} - B_{2,2}); \quad X_4 = A_{2,2}(B_{2,1} - B_{1,1})$$
$$X_5 = (A_{1,1} + A_{1,2})B_{2,2}; \quad X_6 = (A_{2,1} - A_{1,1})(B_{1,1} + B_{1,2})$$
$$X_7 = (A_{1,2} - A_{2,2})(B_{2,1} + B_{2,2}).$$

Strassen, Coppersmith–Winograd, etc

Runtime (measured in the number of multiplications) is $\Theta(n^{\log_2 7}) \approx \Theta(n^{2.807})$.

Strassen, Coppersmith–Winograd, etc

Runtime (measured in the number of multiplications) is $\Theta(n^{\log_2 7}) \approx \Theta(n^{2.807})$.

Currently best improvement: $\approx \Theta(n^{2.373})$

Strassen, Coppersmith–Winograd, etc

Runtime (measured in the number of multiplications) is $\Theta(n^{\log_2 7}) \approx \Theta(n^{2.807})$.

Currently best improvement: $\approx \Theta(n^{2.373})$

(Coppersmith–Winograd, Stothers, Williams, Le Gall) Similar ideas, more complex expressions.

Strassen, Coppersmith–Winograd, etc

Runtime (measured in the number of multiplications) is $\Theta(n^{\log_2 7}) \approx \Theta(n^{2.807})$.

Currently best improvement: $\approx \Theta(n^{2.373})$

(Coppersmith–Winograd, Stothers, Williams, Le Gall) Similar ideas, more complex expressions.

Conjecture: best algorithm has runtime $\approx \Theta(n^2)$.

Strassen, Coppersmith–Winograd, etc

Runtime (measured in the number of multiplications) is $\Theta(n^{\log_2 7}) \approx \Theta(n^{2.807})$.

Currently best improvement: $\approx \Theta(n^{2.373})$

(Coppersmith–Winograd, Stothers, Williams, Le Gall) Similar ideas, more complex expressions.

Conjecture: best algorithm has runtime $\approx \Theta(n^2)$. The lower bound $2n^2$ is trivial (we have to read the input),

Strassen, Coppersmith–Winograd, etc

Runtime (measured in the number of multiplications) is $\Theta(n^{\log_2 7}) \approx \Theta(n^{2.807})$.

Currently best improvement: $\approx \Theta(n^{2.373})$

(Coppersmith–Winograd, Stothers, Williams, Le Gall) Similar ideas, more complex expressions.

Conjecture: best algorithm has runtime $\approx \Theta(n^2)$. The lower bound $2n^2$ is trivial (we have to read the input), necessity of Cn^2 multiplications with large C can be proven by advanced algebraic methods.

Companion matrix

Definition

Let K be a field, and let $x^m + a_{m-1}x^{m-2} + \cdots + a_1 x + a_0 = f(x) \in K[x]$ be an arbitrary degree m monic polynomial, $m \in \mathbb{N}$.

Companion matrix

Definition

Let K be a field, and let $x^m + a_{m-1}x^{m-2} + \cdots + a_1x + a_0 = f(x) \in K[x]$ be an arbitrary degree m monic polynomial, $m \in \mathbb{N}$. Then the following matrix in $M_m(K)$ is called the companion matrix of $f(x)$:

$$\begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & 0 & -a_2 \\ \vdots & & \ddots & & & & \\ \vdots & & & \ddots & & & \\ 0 & 0 & 0 & \cdots & 1 & 0 & -a_{m-2} \\ 0 & 0 & 0 & \cdots & 0 & 1 & -a_{m-1} \end{pmatrix}$$

Companion matrix

Theorem

Let $m \in \mathbb{N}$ and let $A \in M_m(K)$ be the companion matrix of the monic polynomial $f(x) \in K[x]$.

Companion matrix

Theorem

Let $m \in \mathbb{N}$ and let $A \in M_m(K)$ be the companion matrix of the monic polynomial $f(x) \in K[x]$. Then the characteristic polynomial and minimal polynomial of A are both $f(x)$.

Companion matrix

Theorem

Let $m \in \mathbb{N}$ and let $A \in M_m(K)$ be the companion matrix of the monic polynomial $f(x) \in K[x]$. Then the characteristic polynomial and minimal polynomial of A are both $f(x)$. (We have to consider the monic version of the characteristic polynomial, that is, the usual notion should be multiplied by $(-1)^m$.)

Companion matrix

Theorem

Let $m \in \mathbb{N}$ and let $A \in M_m(K)$ be the companion matrix of the monic polynomial $f(x) \in K[x]$. Then the characteristic polynomial and minimal polynomial of A are both $f(x)$. (We have to consider the monic version of the characteristic polynomial, that is, the usual notion should be multiplied by $(-1)^m$.)

Sketch of proof :

Theorem

Let $m \in \mathbb{N}$ and let $A \in M_m(K)$ be the companion matrix of the monic polynomial $f(x) \in K[x]$. Then the characteristic polynomial and minimal polynomial of A are both $f(x)$. (We have to consider the monic version of the characteristic polynomial, that is, the usual notion should be multiplied by $(-1)^m$.)

Sketch of proof: $Ae_1 = e_2, Ae_2 = e_3, \ldots, Ae_{m-1} = e_m$, and $Ae_m = -a_{m-1}e_m - \cdots - a_1 e_2 - a_0 e_1$.

Companion matrix

Theorem

Let $m \in \mathbb{N}$ and let $A \in M_m(K)$ be the companion matrix of the monic polynomial $f(x) \in K[x]$. Then the characteristic polynomial and minimal polynomial of A are both $f(x)$. (We have to consider the monic version of the characteristic polynomial, that is, the usual notion should be multiplied by $(-1)^m$.)

Sketch of proof: $Ae_1 = e_2, Ae_2 = e_3, \dots, Ae_{m-1} = e_m$, and
$Ae_m = -a_{m-1}e_m - \dots - a_1 e_2 - a_0 e_1$.
Thus $f(A)e_1 = 0$,

Companion matrix

Theorem

Let $m \in \mathbb{N}$ and let $A \in M_m(K)$ be the companion matrix of the monic polynomial $f(x) \in K[x]$. Then the characteristic polynomial and minimal polynomial of A are both $f(x)$. (We have to consider the monic version of the characteristic polynomial, that is, the usual notion should be multiplied by $(-1)^m$.)

Sketch of proof: $Ae_1 = e_2, Ae_2 = e_3, \ldots, Ae_{m-1} = e_m$, and
$Ae_m = -a_{m-1}e_m - \cdots - a_1e_2 - a_0e_1$.
Thus $f(A)e_1 = 0$, and $f(A)e_i = 0$ follows for all i,

Companion matrix

Theorem

Let $m \in \mathbb{N}$ and let $A \in M_m(K)$ be the companion matrix of the monic polynomial $f(x) \in K[x]$. Then the characteristic polynomial and minimal polynomial of A are both $f(x)$. (We have to consider the monic version of the characteristic polynomial, that is, the usual notion should be multiplied by $(-1)^m$.)

Sketch of proof: $Ae_1 = e_2, Ae_2 = e_3, \ldots, Ae_{m-1} = e_m$, and
$Ae_m = -a_{m-1}e_m - \cdots - a_1 e_2 - a_0 e_1$.
Thus $f(A)e_1 = 0$, and $f(A)e_i = 0$ follows for all i, implying $f(A) = 0$.

Companion matrix

Theorem

Let $m \in \mathbb{N}$ and let $A \in M_m(K)$ be the companion matrix of the monic polynomial $f(x) \in K[x]$. Then the characteristic polynomial and minimal polynomial of A are both $f(x)$. (We have to consider the monic version of the characteristic polynomial, that is, the usual notion should be multiplied by $(-1)^m$.)

Sketch of proof: $Ae_1 = e_2, Ae_2 = e_3, \ldots, Ae_{m-1} = e_m$, and
$Ae_m = -a_{m-1}e_m - \cdots - a_1 e_2 - a_0 e_1$.
Thus $f(A)e_1 = 0$, and $f(A)e_i = 0$ follows for all i, implying $f(A) = 0$.
The above equations show that $f(x) = m_A(x)$,

Companion matrix

Theorem

Let $m \in \mathbb{N}$ and let $A \in M_m(K)$ be the companion matrix of the monic polynomial $f(x) \in K[x]$. Then the characteristic polynomial and minimal polynomial of A are both $f(x)$. (We have to consider the monic version of the characteristic polynomial, that is, the usual notion should be multiplied by $(-1)^m$.)

Sketch of proof: $Ae_1 = e_2, Ae_2 = e_3, \ldots, Ae_{m-1} = e_m$, and
$Ae_m = -a_{m-1}e_m - \cdots - a_1e_2 - a_0e_1$.
Thus $f(A)e_1 = 0$, and $f(A)e_i = 0$ follows for all i, implying $f(A) = 0$.
The above equations show that $f(x) = m_A(x)$, and then $f(x) = \chi_A(x)$ by a simple calculation. (Fill in the gaps: cf. the exercises.)

Companion matrix

The companion matrix of 0 is the one-by-one zero matrix.

Companion matrix

The companion matrix of 0 is the one-by-one zero matrix.

The companion matrix is invertible iff $a_0 \neq 0$, and then its inverse is

The companion matrix of 0 is the one-by-one zero matrix.

The companion matrix is invertible iff $a_0 \neq 0$, and then its inverse is

$$\begin{pmatrix} -a_0^{-1}a_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ -a_0^{-1}a_2 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & & & \ddots & & & \\ \vdots & & & & \ddots & & \\ -a_0^{-1}a_{m-2} & 0 & 0 & \cdots & 0 & 1 & 0 \\ -a_0^{-1}a_{m-1} & 0 & 0 & \cdots & 0 & 0 & 1 \\ -a_0^{-1} & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}$$

Frobenius normal form

Given a matrix $A \in K^{n \times n}$. Then there is a unique sequence $A_1, \ldots, A_k$ of companion matrices corresponding to some polynomials $f_1(x), \ldots, f_k(x) \in K[x]$ such that

Frobenius normal form

Given a matrix $A \in K^{n \times n}$. Then there is a unique sequence $A_1, \ldots, A_k$ of companion matrices corresponding to some polynomials $f_1(x), \ldots, f_k(x) \in K[x]$ such that

- $f_1(x) | f_2(x) | \cdots | f_k(x)$ in the ring of polynomials $K[x]$, and
- A is similar to the block matrix F with blocks $A_1, \ldots, A_k$,

Frobenius normal form

Given a matrix $A \in K^{n\times n}$. Then there is a unique sequence $A_1, \dots, A_k$ of companion matrices corresponding to some polynomials $f_1(x), \dots, f_k(x) \in K[x]$ such that

- $f_1(x) | f_2(x) | \cdots | f_k(x)$ in the ring of polynomials $K[x]$, and
- A is similar to the block matrix F with blocks $A_1, \dots, A_k$, that is, there is a regular matrix S over K such that $F = SAS^{-1}$.

Frobenius normal form

Given a matrix $A \in K^{n \times n}$. Then there is a unique sequence $A_1, \dots, A_k$ of companion matrices corresponding to some polynomials $f_1(x), \dots, f_k(x) \in K[x]$ such that

- $f_1(x) | f_2(x) | \cdots | f_k(x)$ in the ring of polynomials $K[x]$, and
- A is similar to the block matrix F with blocks $A_1, \dots, A_k$, that is, there is a regular matrix S over K such that $F = SAS^{-1}$.

This unique block matrix is the Frobenius normal form (or rational canonical form) of A.

Frobenius normal form

Given a matrix $A \in K^{n\times n}$. Then there is a unique sequence $A_1, \ldots, A_k$ of companion matrices corresponding to some polynomials $f_1(x), \ldots, f_k(x) \in K[x]$ such that

- $f_1(x)|f_2(x)|\cdots|f_k(x)$ in the ring of polynomials $K[x]$, and
- A is similar to the block matrix F with blocks $A_1, \ldots, A_k$, that is, there is a regular matrix S over K such that $F = SAS^{-1}$.

This unique block matrix is the Frobenius normal form (or rational canonical form) of A.
Note that $F \in K^{n\times n}$;

Frobenius normal form

Given a matrix $A \in K^{n \times n}$. Then there is a unique sequence $A_1, \dots, A_k$ of companion matrices corresponding to some polynomials $f_1(x), \dots, f_k(x) \in K[x]$ such that

- $f_1(x) | f_2(x) | \cdots | f_k(x)$ in the ring of polynomials $K[x]$, and
- A is similar to the block matrix F with blocks $A_1, \dots, A_k$, that is, there is a regular matrix S over K such that $F = SAS^{-1}$.

This unique block matrix is the Frobenius normal form (or rational canonical form) of A.

Note that $F \in K^{n \times n}$; unlike the Jordan normal form, whose entries are in general outside F, in the algebraic closure of F.

Frobenius normal form

Given a matrix $A \in K^{n \times n}$. Then there is a unique sequence $A_1, \ldots, A_k$ of companion matrices corresponding to some polynomials $f_1(x), \ldots, f_k(x) \in K[x]$ such that

- $f_1(x) | f_2(x) | \cdots | f_k(x)$ in the ring of polynomials $K[x]$, and
- A is similar to the block matrix F with blocks $A_1, \ldots, A_k$, that is, there is a regular matrix S over K such that $F = SAS^{-1}$.

This unique block matrix is the Frobenius normal form (or rational canonical form) of A.
Note that $F \in K^{n \times n}$; unlike the Jordan normal form, whose entries are in general outside F, in the algebraic closure of F.
Remark: the 0 polynomial is allowed in the series $f_1(x), \ldots, f_k(x)$ any number of times (at the end).

Frobenius normal form

The polynomials $f_1(x), \ldots, f_k(x)$ can be read from A (in linear time).

Frobenius normal form

The polynomials $f_1(x), \ldots, f_k(x)$ can be read from A (in linear time).

Then $\chi_A(x) = f_1(x) \cdots f_k(x)$, $m_A(x) = f_k(x)$ (yielding a proof to the Cayley-Hamilton theorem).

The polynomials $f_1(x), \ldots, f_k(x)$ can be read from A (in linear time).

Then $\chi_A(x) = f_1(x) \cdots f_k(x)$, $m_A(x) = f_k(x)$ (yielding a proof to the Cayley-Hamilton theorem). In particular, $\mathrm{det}(A) = f_1(0) \cdots f_k(0)$ is the product of elements in the upper right corner of the blocks.

Frobenius normal form

The polynomials $f_1(x), \ldots, f_k(x)$ can be read from A (in linear time).

Then $\chi_A(x) = f_1(x) \cdots f_k(x)$, $m_A(x) = f_k(x)$ (yielding a proof to the Cayley-Hamilton theorem). In particular, $\det(A) = f_1(0) \cdots f_k(0)$ is the product of elements in the upper right corner of the blocks.

The rank $\varrho(A)$ of A is the sum of the rank of the blocks.

Frobenius normal form

The polynomials $f_1(x), \dots, f_k(x)$ can be read from A (in linear time).

Then $\chi_A(x) = f_1(x) \cdots f_k(x)$, $m_A(x) = f_k(x)$ (yielding a proof to the Cayley-Hamilton theorem). In particular, $\det(A) = f_1(0) \cdots f_k(0)$ is the product of elements in the upper right corner of the blocks.

The rank $\varrho(A)$ of A is the sum of the rank of the blocks. A block has full rank iff $a_0 \neq 0$ in the corresponding polynomials.

Frobenius normal form

The polynomials $f_1(x), \ldots, f_k(x)$ can be read from A (in linear time).

Then $\chi_A(x) = f_1(x) \cdots f_k(x)$, $m_A(x) = f_k(x)$ (yielding a proof to the Cayley-Hamilton theorem). In particular, $\mathtt{det}(A) = f_1(0) \cdots f_k(0)$ is the product of elements in the upper right corner of the blocks.

The rank $\varrho(A)$ of A is the sum of the rank of the blocks. A block has full rank iff $a_0 \neq 0$ in the corresponding polynomials. Otherwise it has co-rank 1.

Frobenius normal form

The polynomials $f_1(x), \ldots, f_k(x)$ can be read from A (in linear time).

Then $\chi_A(x) = f_1(x) \cdots f_k(x)$, $m_A(x) = f_k(x)$ (yielding a proof to the Cayley-Hamilton theorem). In particular, $\det(A) = f_1(0) \cdots f_k(0)$ is the product of elements in the upper right corner of the blocks.

The rank $\varrho(A)$ of A is the sum of the rank of the blocks. A block has full rank iff $a_0 \neq 0$ in the corresponding polynomials. Otherwise it has co-rank 1. That is, $\varrho(A) = n - \ell$, where ℓ is the number of blocks with a 0 in the upper right corner (easy to determine in linear time).

Frobenius normal form

The polynomials $f_1(x), \dots, f_k(x)$ can be read from A (in linear time).

Then $\chi_A(x) = f_1(x) \cdots f_k(x)$, $m_A(x) = f_k(x)$ (yielding a proof to the Cayley-Hamilton theorem). In particular, $\det(A) = f_1(0) \cdots f_k(0)$ is the product of elements in the upper right corner of the blocks.

The rank $\varrho(A)$ of A is the sum of the rank of the blocks. A block has full rank iff $a_0 \neq 0$ in the corresponding polynomials. Otherwise it has co-rank 1. That is, $\varrho(A) = n - \ell$, where ℓ is the number of blocks with a 0 in the upper right corner (easy to determine in linear time).

The inverse matrix $A^{-1} = (SFS^{-1})^{-1} = SF^{-1}S^{-1}$ can be determined by replacing each block of F by its inverse; see an earlier slide.

Basic problems revisited

There is a randomized algorithm using matrix multiplication that computes F together with the change of basis matrix S (with high probability).

Basic problems revisited

There is a randomized algorithm using matrix multiplication that computes F together with the change of basis matrix S (with high probability).

The expected runtime is the same as that of matrix multiplication, yielding $\approx \Theta(n^{2.373})$ at the moment.

Basic problems revisited

There is a randomized algorithm using matrix multiplication that computes F together with the change of basis matrix S (with high probability).

The expected runtime is the same as that of matrix multiplication, yielding $\approx \Theta(n^{2.373})$ at the moment.

This yields a randomized algorithm of runtime $\approx \Theta(n^{2.373})$ to all basic problems in linear algebra; see the previous slide.

Basic problems revisited

There is a randomized algorithm using matrix multiplication that computes F together with the change of basis matrix S (with high probability).

The expected runtime is the same as that of matrix multiplication, yielding $\approx \Theta(n^{2.373})$ at the moment.

This yields a randomized algorithm of runtime $\approx \Theta(n^{2.373})$ to all basic problems in linear algebra; see the previous slide.

The algorithm is not yet fully de-randomized.

Exercises

Exercises

1. Multiply the matrices $A = \begin{pmatrix} 2 & -3 \\ 1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 6 \\ 9 & 8 \end{pmatrix}$ via the Strassen algorithm. Observe that it only requires 7 multiplications.

Exercises

1. Multiply the matrices $A = \begin{pmatrix} 2 & -3 \\ 1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 6 \\ 9 & 8 \end{pmatrix}$ via the Strassen algorithm. Observe that it only requires 7 multiplications.
2. Show that the companion matrix A of a polynomial $f(x)$ has minimal and characteristic polynomial $m_A(x) = \chi_A(x) = f(x)$.

Exercises

1. Multiply the matrices $A = \begin{pmatrix} 2 & -3 \\ 1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 6 \\ 9 & 8 \end{pmatrix}$ via the Strassen algorithm. Observe that it only requires 7 multiplications.
2. Show that the companion matrix A of a polynomial $f(x)$ has minimal and characteristic polynomial $m_A(x) = \chi_A(x) = f(x)$.
3. Using the Frobenius normal form and companion matrices, construct a matrix A with minimal polynomial $m_A(x) = x^2(x^2+1)(x-2)$ and characteristic polynomial $\chi_A(x) = x^3(x^2+1)^2(x-2)$. In particular, observe that the minimal polynomial of a matrix is not necessarily irreducible.

Exercises

1. Multiply the matrices $A = \begin{pmatrix} 2 & -3 \\ 1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 6 \\ 9 & 8 \end{pmatrix}$ via the Strassen algorithm. Observe that it only requires 7 multiplications.
2. Show that the companion matrix A of a polynomial $f(x)$ has minimal and characteristic polynomial $m_A(x) = \chi_A(x) = f(x)$.
3. Using the Frobenius normal form and companion matrices, construct a matrix A with minimal polynomial $m_A(x) = x^2(x^2+1)(x-2)$ and characteristic polynomial $\chi_A(x) = x^3(x^2+1)^2(x-2)$. In particular, observe that the minimal polynomial of a matrix is not necessarily irreducible.
4. Compute the inverse of the matrix constructed in problem 3. Show that the computation runs in linear time for a Frobenius normal form.

Exercises

1. Multiply the matrices $A = \begin{pmatrix} 2 & -3 \\ 1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 6 \\ 9 & 8 \end{pmatrix}$ via the Strassen algorithm. Observe that it only requires 7 multiplications.
2. Show that the companion matrix A of a polynomial $f(x)$ has minimal and characteristic polynomial $m_A(x) = \chi_A(x) = f(x)$.
3. Using the Frobenius normal form and companion matrices, construct a matrix A with minimal polynomial $m_A(x) = x^2(x^2+1)(x-2)$ and characteristic polynomial $\chi_A(x) = x^3(x^2+1)^2(x-2)$. In particular, observe that the minimal polynomial of a matrix is not necessarily irreducible.
4. Compute the inverse of the matrix constructed in problem 3. Show that the computation runs in linear time for a Frobenius normal form.
5. Compute the cube of the matrix constructed in problem 3.

Exercises

1. Multiply the matrices $A = \begin{pmatrix} 2 & -3 \\ 1 & 7 \end{pmatrix}$ and $B = \begin{pmatrix} -5 & 6 \\ 9 & 8 \end{pmatrix}$ via the Strassen algorithm. Observe that it only requires 7 multiplications.
2. Show that the companion matrix A of a polynomial $f(x)$ has minimal and characteristic polynomial $m_A(x) = \chi_A(x) = f(x)$.
3. Using the Frobenius normal form and companion matrices, construct a matrix A with minimal polynomial $m_A(x) = x^2(x^2+1)(x-2)$ and characteristic polynomial $\chi_A(x) = x^3(x^2+1)^2(x-2)$. In particular, observe that the minimal polynomial of a matrix is not necessarily irreducible.
4. Compute the inverse of the matrix constructed in problem 3. Show that the computation runs in linear time for a Frobenius normal form.
5. Compute the cube of the matrix constructed in problem 3.
6. What is $\dim \ker(A)$ for the matrix A constructed in problem 3?