Markov chains and Their Applications

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Week 1, University of Debrecen

Linear algebra

Basic problems

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6. If $m = n$, compute the minimal polynomial $m_A(x)$ of A.

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Runtime: $\Theta(n^3)$; not bad, but cf. the next class for much faster algorithms.

Characteristic polynomial, minimal polynomial

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$$\begin{pmatrix} 1 & r_1 & \cdots & r_1^{n-1} & r_1^n \\ 1 & r_2 & \cdots & r_2^{n-1} & r_2^n \\ & & \vdots & & \\ 1 & r_{n-1} & \cdots & r_{n-1}^{n-1} & r_{n-1}^n \\ 1 & r_n & \cdots & r_n^{n-1} & r_n^n \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \\ (-1)^n \end{pmatrix} = \begin{pmatrix} \det(A - r_0 I_n) \\ \det(A - r_1 I_n) \\ \vdots \\ \det(A - r_n I_n) \end{pmatrix}$$

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Runtime : $\Theta(n^4)$. Possible to improve to $\Theta(n^3)$ with some extra work, but not worth it : cf. the next class.

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Very roundabout, cf. the next class for a much better algorithm. The improvement is based on the Frobenius normal form, but the minimal polynomial is even simpler to calculate from the Jordan normal form.

Exercises

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1. Are the given vectors independent? In case they are, then express one of the vectors as a linear combination of the others.
   a) $(1,2,-7)^T, (-3,4,-3)^T, (-1,8,-17)^T$
   b) $(-2,2,-3)^T, (6,7,9)^T, (6,-1,11)^T$
   c) $(5,3,-10)^T, (-2,3,-9)^T, (-3,4,-5)^T$
   d) $(1,1,1)^T, (1,2,2)^T, (3,4,4)^T$

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2. Consider the four subspaces generated by the system of vectors in each subproblem of the previous exercise. Which of them contain the vector $\underline{b} = (0,10,-24)^T$?
3. Find the change of basis matrix for converting from the standard basis to the following bases.
   a) $\underline{f_1} = (2,-1,0), \underline{f_2} = (1,0,1), \underline{f_3} = (0,0,-2)$
   b) $\underline{g_1} = (1,1,1), \underline{g_2} = (1,1,0), \underline{g_3} = (2,1,1)$

   What is the change of basis matrix for converting from (f) to (g)? And that from (g) to (f)?

4. Solve the system of linear equations.

a)

$$\begin{aligned} -3x_1 + 4x_2 - x_3 + 2x_4 &= 0 \\ x_1 + 2x_2 - 4x_3 + 5x_4 &= -11 \\ 5x_1 - 5x_2 \quad + \quad 2x_4 &= -1 \\ -4x_1 - 3x_2 + 10x_3 - 9x_4 &= -3 \end{aligned}$$

b)

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 8 \\ -2x_1 - 5x_2 - 4x_3 &= -8 \\ 4x_1 + 7x_2 + x_3 &= 17 \\ 2x_1 + 9x_2 + 21x_3 &= -1 \end{aligned}$$

c)

$$\begin{aligned} -x_1 + 4x_2 - x_3 + 2x_4 &= 1 \\ x_1 + 2x_2 - 5x_3 + 4x_4 &= -13 \\ 7x_1 - 5x_2 + 6x_3 + 2x_4 &= -11 \\ 4x_1 + x_2 + 9x_3 + 2x_4 &= 4 \end{aligned}$$

d)

$$\begin{aligned} 5x_1 + 4x_2 - 6x_3 + 3x_4 &= -3 \\ 3x_1 + 2x_2 - 5x_3 + 6x_4 &= -19 \\ 2x_1 + 2x_2 - x_3 - 3x_4 &= 11 \end{aligned}$$

5. Consider the systems of linear equations in the previous problem that had at least one solution. Where there is exactly one solution and the matrix of coefficients on the left-hand side is square, compute the determinant and inverse of the square matrix. When you obtain infinitely many solutions, compute the rank of the matrix and compare it to the dimension of the affine subspace of all solutions (i.e., the number of free variables).