Markov Chains and Their Applications, Problem sheet 7

(1) Prove the theorem about the main recurrence time.

(2) (a) Represent coin-toss as an irreducible Markov-chain with two states.
 (b) Find the period.
 (c) Compute the mean recurrence time and compare it with the general formula.
 (d) Compute the second moment of the recurrence time. Make a general conjecture.

(3) Show that if P is irreducible, then $\frac{1}{2}(P+I)$ is regular with the same stationary distribution. (The latter one usually has a better rate of convergence for the iterative method when P has a period greater than 1.)

(4) Solve the following exercises to prove the special case of Hoffman's theorem: a graph is bipartite iff its spectrum is symmetric to the origin. (Note that the only if direction is clear.)

(5) Show that an ordered real n-tuple $\lambda_1 \geq \cdots \geq \lambda_n$ is symmetric to the origin iff for all odd $k \in \mathbb{N}$ we have $\sum_{i=1}^{n} \lambda_i^k = 0$.

(6) Observe that if A is the adjacency matrix of a graph G, then $\mathrm{Tr}(A^k)$ is the number of walks of length k with coinciding first and last vertex.

(7) Prove that $\mathrm{Tr}(A^k) = 0$ iff the main diagonal in A^k is all zero.

(8) Combine these observations to show that the spectrum of A is symmetric to the origin iff there is no walk of odd length in G with coinciding first and last vertex.