

### Markov Chains and Their Applications, Problem sheet 3

Let  $G$  be a graph with eigenvalues of the adjacency matrix  $\lambda_1, \dots, \lambda_n$ .

- (1) Show that  $(\lambda_1^3 + \dots + \lambda_n^3)/6$  is the number of triangles in  $G$ .
- (2) Prove that all the  $|\lambda_i|$  are at most the maximum degree in  $G$ .
- (3) If there is equality in Problem 2, then  $G$  is regular.
- (4) Prove that the Laplacian  $L$  of  $G$  is singular.
- (5) Show that  $\dim \ker(L)$  is the number of connected components in  $G$ . In particular, 0 is an eigenvalue with multiplicity 1 if  $G$  is connected. What is the trace of the Laplacian?
- (6) Prove Cayley's theorem on the number of labelled trees on  $n$  vertices.