Markov Chains and Their Applications, Problem sheet 13

(1) We put a knight in a corner of a chessboard, and make random knight moves. What is the expected time of returning to the same corner square?

(2) Show that the cover time of the complete graph of n vertices is asymptotically $n \log n$.

(3) The n-lollipop graph consists of a complete graph of $n/2$ vertices with a path of length $n/2$ glued to a vertex. Show that the mean hitting time from any vertex u of the complete graph to the base of the lollipop is at least cubic. Conclude a cubic lower bound for the cover time.

(4) Prove that a connected graph is a two-sided expander iff it is not bipartite. (Hint: recall Hoffman's theorem.)

(5) Let $\mathbb{N} \cup \{0\}$ be equipped with the irreducible fair walk structure (third slide): from every positive integer we move to each neighbor with equal probability $1/2$. Show that 0 is a null-recurrent state. (Hint: show that hitting n from 0 has probability $1/n$.)

(6) Classify all states in the previous example, and also in the unfair versions. That is, stepping to the right has probability p. What if $p > 1/2$? And if $p < 1/2$?

(7) Prove the proposition about communicating recurrent states.