Syllabus

Remark 1.

The law of large numbers

Let be a series of independent two by two random variables $X_1, X_2, ..., X_n$, which admit finite mean values and bounded dispersions:

$$X_1, X_2, \dots, X_n \text{ a.i. } M(X_k) < \infty \forall k \in N*, D^2(X_k) \le c < \infty, \forall k \in N*, c > 0 \text{ then:}$$

$$(1) \quad \boxed{\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{almost\ sure} \frac{1}{n}\sum_{k=1}^{n} M(X_k)}, \quad \lim_{n\to\infty} P\left(\left|\frac{1}{n}\sum_{k=1}^{n} X_k - \frac{1}{n}\sum_{k=1}^{n} M(X_k)\right| < \varepsilon\right) = 1, \forall k \in N* \quad (2)$$

Particular cases:

A. If the random variables have the same mean: $M(X_k) = m, \forall k \in N*$, $\frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{almost\ sure} m$

B. If the random variables are *Bernoulli type*:

$$X_k = \begin{pmatrix} 1 & 0 \\ p & q \end{pmatrix}, p, q > 0, \ p + q = 1 \Rightarrow M(X_k) = p, \forall k \in N, \frac{1}{n}\sum_{k=1}^{n} X_k \xrightarrow{almost\ sure} p \ (3) \ \Leftrightarrow f_n = \frac{1}{n}\sum_{k=1}^{n} X_k, \ f_n \xrightarrow{a.s(n\to\infty)} p$$

Central-limit theorem

Let be $(X_n)_{n\in N^*}$ a series of independent two by two random variables, which admit finite mean values and bounded dispersions:

$$\frac{\sum_{k=1}^{n} X_k - \sum_{k=1}^{n} M(X_k)}{\sqrt{\sum_{k=1}^{n} D^2(X_k)}} \xrightarrow{repartition(n\to\infty)} N(0,1).$$

Remark 2.

$X \in N(0,1)$, is the random variable having a normalized normal distribution if the *distribution density* is defined by:

$$\varphi_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \forall x \in R.$$

Its cumulative distribution function is:

$$F(x) = P(X < x) = \int_0^x f(t)\,dt = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-t^2/2} dt = \phi(x), \forall x \in R.$$

where $\phi(x)$ is *Laplace function* having the following properties:

$$\phi(0) = 0, \text{ functie tabelata; } \phi(-x) = 1 - \phi(x);\ \phi(\infty) = \frac{1}{2}, \varphi(-\infty) = \frac{1}{2}.$$

Error function is defined as $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt, \forall x \in R$ and express the probability of a random variable, with normal distribution of mean 0 and variance 1/2 falling in the range [-x, x]. $\phi(x) = \frac{1}{2}[1 + \operatorname{erf}\left(\frac{x}{\sqrt{2}}\right)]$.

Remark 3.

From the analytical form of the normal distribution function results the impossibility of explicitly solving an equation of form F(x)=y. This leads to the development of special methods for evaluating the solution $x = F^{-1}(y)$. One of these methods is based on the Proposition:

Proposition 1

If U and V are 2 independent random variables, uniform distributed in the interval [0,1], then the random variables:

$$X = \sqrt{-2 \cdot \ln(U)} \cdot \cos(2\pi \cdot V);\ Y = \sqrt{-2 \cdot \ln(U)} \cdot \sin(2\pi \cdot V)$$

are independent and normal distributed, with mean m = 0 and mean square deviation σ^2=1.

For Z normal standard distribution, X = σ Z + m is the cumulative normal distibuion function with mean m≠ 0 and σ ≠ 1:

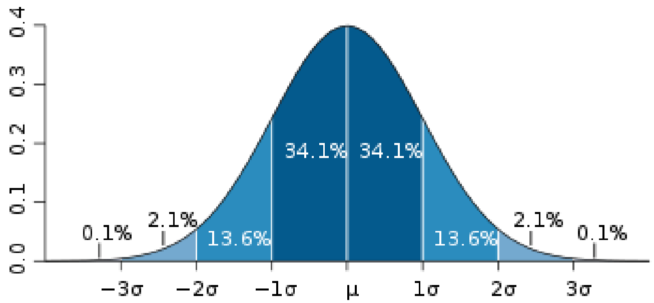
$$P(m - k\sigma < X < m + k\sigma) = P(-k < Z < k) = 2\Phi(k) - 1$$

For k=3 we have:

$$P[m-3\sigma < X < m+3\sigma] = 0.997,$$ meaning thet 99,7% values of the function Z are in the interval.

Exemples of distribuions functions

1. Normal distribution with parameters m and σ^2, $X \sim N(m,\sigma^2)$ **;** $X \approx N(m,\sigma^2) \Rightarrow \frac{X-m}{\sigma} \sim N(0,1)$



2.Uniform distribution $X \sim U[a,b]$

X v.a. with *density function:* $f_X(x) = \begin{cases} \frac{1}{b-a}, x \in [a,b] \\ 0 \quad \text{în rest} \end{cases}$. X has the mean $M(X) = (a+b)/2$ and dispersion $D(X) = (b-a)^2/12$.

3. Exponential distribution $X \sim \mathrm{Exp}(\lambda,\alpha)$

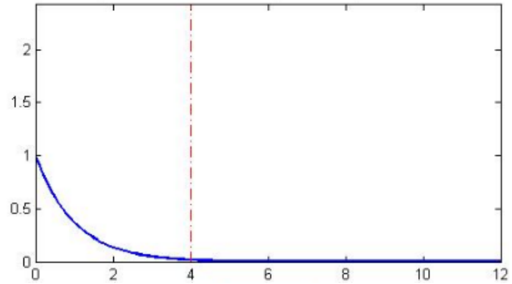
X r.d. with *density function* funcţia: $f_X(x) = \begin{cases} \frac{1}{\lambda} e^{-\frac{x-\alpha}{\lambda}}, x \geq \alpha, \lambda > 0 \\ 0 \quad \text{în rest} \end{cases}$.

X has the mean $M(X) = \lambda + \alpha$ and dispersion $D(X) = \lambda^2$. Remark: $\mathrm{Exp}(\lambda, 0) = \mathrm{Exp}(\lambda)$.

Negative exponentional distribution

X has *density function*: $f(x) = \lambda e^{-\lambda x}$, $x \in [0,\infty)$, $\lambda > 0$,

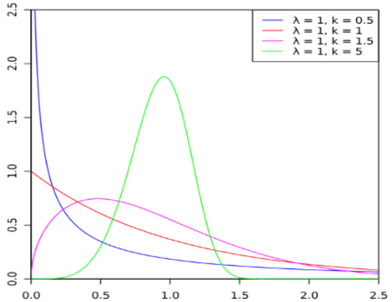
or cdf $F(x) = P[X < x] = \begin{cases} l - e^{-\lambda \cdot x}, x \geq 0 \\ 0, \text{otherwise} \end{cases}$.



Exp(1)

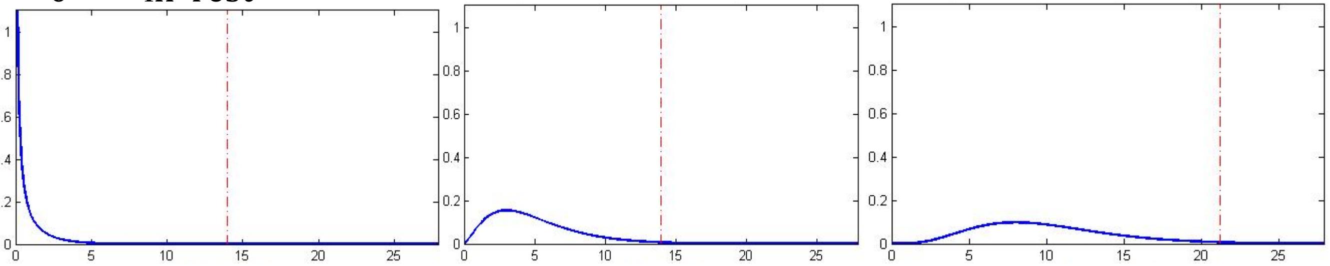
4. Weibull distribution $X \sim W(\alpha,\beta)$

X r.d. having *density function*: $f_X(x) = \alpha\beta x^{\beta-1} e^{-(\alpha x^{\beta})}$
with mean $M(X) = \lambda + \alpha$ and dispersion $D(X) = \lambda^2$.



5. chi-square distribution $X \sim \chi^2(n)$

X r.d. with *density function*: $f_X(x) = \begin{cases} \frac{x^{n/2-1}}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} e^{-x/2}, x \geq 0, \\ 0 \quad \text{în rest} \end{cases}$, and $\Gamma(p) = \int_0^\infty x^{p-1} e^{-x} dx$.



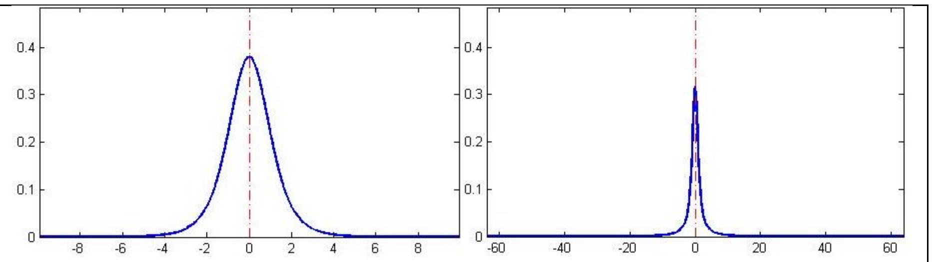
$\sigma = 1$; n=0.5 n=5 n=10

6. Student distribution $X \sim T(n)$ with n degrees of freedom

X r.d. with *density function*: $f_X(x) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{\pi \cdot n}\,\Gamma\left(\frac{n}{2}\right)} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2}$.

Proposition:
For U, V two random variables such that $U \sim N(0,1)$ and $V \sim \chi^2$ then:

$$\boldsymbol{W = U/\sqrt{V/n} \cdot \sim \mathrm{T}(n)}$$

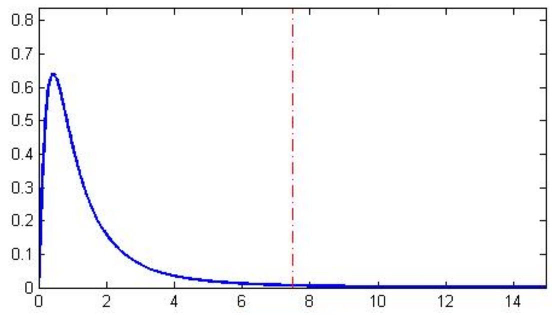


n=5 n=1

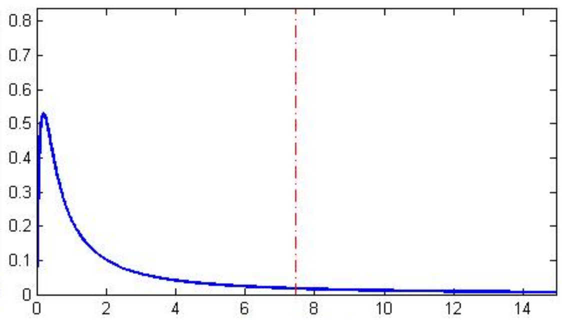
7. Fisher distribution with n, and m degrees of freedom $X \sim F(n,m)$

X r.d. with *density function*

$$f_X(x)=\frac{\Gamma\left(\frac{m+n}{2}\right)}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)}\left(\frac{n}{m}\right)^{n/2}\left(1+\frac{x^2}{n}\right)^{-(n+1)/2}$$



F(5,5)



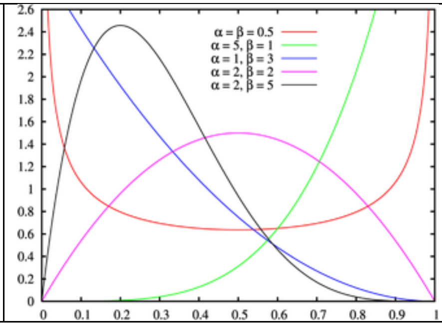
F(5,1)

8. Beta distribution $X \sim Be(\alpha,\beta)$, $\alpha>0,\beta>0$

X r.d. with *density function*:

$$f_X(x)=\frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha,\beta)}, x\in(0,1), \text{ cu } B(\alpha,\beta)=\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

M(X)= $\dfrac{a}{a+b}$, Var(X)= $\dfrac{ab}{(a+b)^2(a+b+1)}$

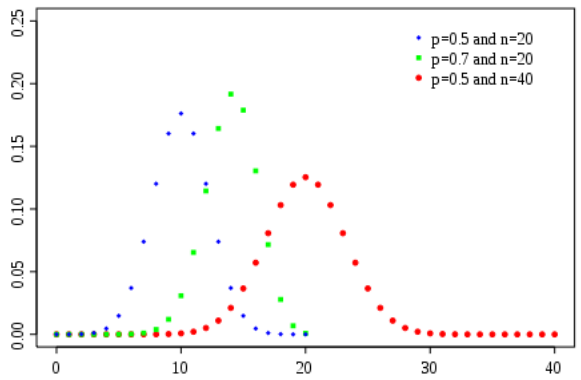
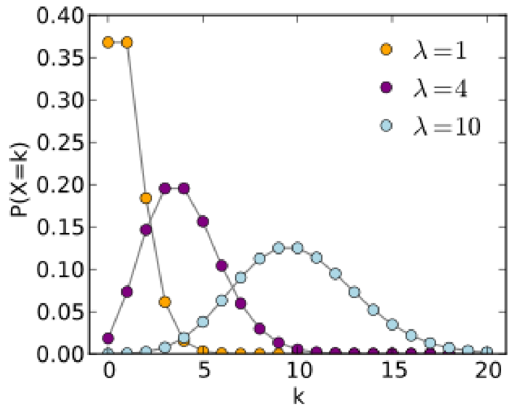


9. Gamma distribution <u>G(a,b)</u> , a>0,b>0.

X r.d. with *density function*: $f_X(x)=\dfrac{x^{b-1}e^{-ax}a^b}{\Gamma(b)}, x>0$ with M(X)=b/a ; Var(X)=b/a^2.

For b=1 one obtain exponentioal distribution Exp(a).

Discrete distributions

Binomial distribution $X \sim Bi(p,n)$.	**Negative Binomial distribution $X \sim NB(r,p)$**	**Poisson distribution**
$P(X=k)=C_n^k p^k (1-p)^{n-k}$ M(X)=np ; Var(X)=npq. represents the probability with which an event A is realized k times in a number n (fixed) of samples in which the probability of realization of A in each sample is p. 	$P(X=k)=C_{r+k-1}^k p^r q^k$; k=0,1,2..... M(X)=r$\frac{q}{p}$; Var(X)=r$\frac{q}{p^2}$. is the probability of having k failures before the second success in a series of Bernoulli tests (the result of each test consists in the realization of an event A with the probability p or of the event opposite to it with the probability q).	X ~ Po(λ). $P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$ M(X)=Var(X)= λ . express the number of events that result in property damage. 

The Moment generating Function (MGF)

Definition:

The moment generating function (mgf) of a random variable X is given by:

$$M_X(t) = E\left[e^{tX}\right] \quad , \quad t \in \mathbb{R}.$$

Theorem!

If the random variable X has the (mgf) $M_X(t)$ then:

$$M_X^{(r)}(0) = \frac{d^r}{dX^r}\left[M_X(t)\right]\Big|_{t=0} = E(X^r).$$

The derivative of order r of the (mgf) evaluated at $t=0$ gives the value of the r^{th} moment.

Remark:

For a discrete probability mass function $M_x(t) = \sum_{i=1}^{\infty} e^{tx_i} p_i$

For a continuous probability density function, $M_X(t) = \int_{-\infty}^{\infty} f(x) e^{tx} dx.$

Theorem 2:

Let X be a random variable with $M_X(t)$ the (mgf) and a, b constants. If $Y = a + bX$ then (mgf) of Y is given by

$$M_Y(t) = e^{at} M_X(bt)$$

Example 1.

$X \sim Po(\lambda)$ with probability mass function $p(x) = \dfrac{e^{-\lambda}\lambda^{x}}{x!}$, $x = 0, 1, \dots$

$$M_X(t) = E[e^{tX}] = \sum_{x=0}^{\infty} e^{tx}\frac{e^{-\lambda}\lambda^{x}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^{t}\lambda)^{x}}{x!}$$

$$M_X(t) = e^{-\lambda}e^{e^{t}\lambda} = e^{\lambda(e^{t}-1)}$$

Thus: $M_X'(t) = \dfrac{d}{dt}\left[e^{\lambda(e^{t}-1)}\right] = \lambda e^{t} e^{\lambda(e^{t}-1)}$

$$M_X''(t) = \frac{d^2}{dt^2}\left[e^{\lambda(e^{t}-1)}\right] = \lambda e^{t}e^{\lambda(e^{t}-1)} + \lambda^2 e^{2t}e^{\lambda(e^{t}-1)}$$

$E[X] = M_X'(0) = \lambda$

$E[X^2] = M_X''(0) = \lambda + \lambda^2$

$VAR(X) = E[X^2] - (E[X])^2 = \lambda$

Example 2

Let $X \sim$ Bernoulli (p) distribution with probability mass function $p(x) = \begin{cases} 1-p, & \text{if } x=0 \\ p, & \text{if } x=1 \end{cases}$.

Then $M_X(t) = E[e^{tX}] = e^{tX}\Big|_{=0}(1-p) + e^{tX}\Big|_{=1} p = 1-p+e^t p.$

Example 3

Let $X \sim$ Binomial (p) distribution.

$M_X(t) = (1-p+e^t p)^n$

Compund distribution

Let be $X_1, X_2, \ldots, X_n$ random variables i.i.d. that describe the same distribution function X and n becomes an integer positive and/or N random variable

For $S = \sum_{i=1}^{N} X_i$, one find: $M(S) = M(X)\,M(N)$

$$D^2(S) = M^2(X) \cdot D^2(N) + D^2(X)\,M(N)$$

If $N \sim Po(\lambda) \Rightarrow M(N) = D^2(N) = \lambda$; $M(N^2) = \lambda^2 + \lambda$, then:

$M(S) = \lambda M(X)$, $D^2(S) = \lambda M(X^2)$.