ThinkBS: Basic Sciences in Engineering Education

Erasmus Plus Project, International Training Module: 10-30 May, 2021, online

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KEYWORDS

- Fractional Calculus;
- Wavelet Analysis
- Estimating of nonlinearities

Some topics in this section that will be discussed: functions approximations by polynomial interpolation, iterative methods for calculating the eigenvalues and eigenvectors of a matrix, estimation methods of probability densities functions, estimation methods for solutions of nonlinear ODEs.

Teachers

The lecturers of this course are as follows: Simona Mihaela BIBIC, Elena Corina CIPU, Mihai Rebenciuc, Carmina GEORGESCU, Emil SIMION and Antonela TOMA from the Center for Research and Training in Innovative Techniques of Applied Mathematics in Engineering "Traian Lalescu" (CiTi), University Politehnica of Bucharest.

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ESTIMATION METHODS OF PROBABILITY DENSITIES FUNCTIONS

Methods for estimating discrete distributions

Recursive formulas

Approximation of probability densities by orthogonal polynomials

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WHATdo we understand by NONLINEARITIES
and how to ESTIMATE them?

NONLINEARITIES

-what is different from linear type

-what is different from the expectance

ESTIMATIONS

-measurements, metrics of the part of the nonlinear process

First- the observer is outside the box and try to quantify the dimensions of the process

Second-the observer is inside the box and try to see the light of the outside the process

NONLINEARITIES

-what is different from linear type

a function / an operator /a sistem known is nonlinear or are known only some characteristics

-what is different from what is expected

asimptotic study, perturbation study, method of moments

ESTIMATIONS

First- interpolations, nonlinear regression, method of moments

Second- estimation parameters, perturbation study, developments, stability, make decompositions, change part of the system, numerical study, learning algorithms and neural networks

1. Methods for estimating discrete distributions.

Poisson approximation of binomial distributions

a. Let n∈N, k=1, 2, …, n and p∈(0,1), as for $X \sim Bi(p,n)$, $P(X=k)=\binom{n}{k}p^k(1-p)^{n-k}$, then

$$\sum_{i=k}^{n} P(X=i)=\sum_{i=k}^{n}\binom{n}{i}p^i(1-p)^{n-i}=n\binom{n-1}{k-1}\int_0^p t^{k-1}(1-t)^{n-k}dt. \qquad (1)$$

b. For $\lambda>0$and $n\in N_0$ we have for $\boldsymbol{X\sim Po(\lambda)}$, $P(X=k)=\frac{\lambda^k e^{-\lambda}}{k!}$,

$$\sum_{k=n+1}^{\infty} P(X=k)=\sum_{k=n+1}^{\infty} e^{-\lambda}\frac{\lambda^k}{k!}=\int_0^{\lambda}\frac{t^n e^{-t}}{n!}dt. \qquad (2)$$

c. For n=1, 2, …, p_n∈(0,1), $n\cdot p_n \underset{n\to\infty}{\to} \lambda>0$, and k=0, 1, … we have:

$$\binom{n}{k}{p_n}^k(1-p_n)^{n-k}=\frac{1}{k!}\frac{n}{n}\dots\frac{n-k+1}{n}(n\cdot p_n)^k\left(1-\frac{n\cdot p_n}{n}\right)^n\left(1-\frac{n\cdot p_n}{n}\right)^{-k}\underset{n\to\infty}{\to} e^{-\lambda}\frac{\lambda^k}{k!}. \qquad (3)$$

Poisson approximation of negative binomial distributions

For $n=1,2,\dots,p_n\in(0,1)$ and $n\cdot(1-p_n)\underset{n\to\infty}{\to}\lambda>0$, then for k=0, 1, … we have:

$$\binom{n+k-1}{k}p_n^n(1-p_n)^k=\frac{1}{k!}\frac{n}{n}\dots\frac{n+k-1}{n}\left(1-\frac{n\cdot(1-p_n)}{n}\right)^n(n\cdot(1-p_n))^k\underset{n\to\infty}{\to} e^{-\lambda}\frac{\lambda^k}{k!}. \quad (4)$$

Simulation of geometrical distributions

Next, we will show how a *geometrical distribution of random variables*, *NB* (1, p) can be obtained from *a uniform distribution*.

Let $p \in (0,1)$, X ~ U (0,1) then *N* expressed by $N := \left[\frac{ln\, X}{ln(1-p)}\right]$ is $N \sim NB(1,p)$

For k=0, 1, … we can write

$$P(N=k) = P\left(k+1 > \frac{ln\, X}{ln(1-p)} \geq k\right) =$$
$$P((1-p)^{k+1} < X \leq (1-p)^k) = F_X((1-p)^k) - F_X((1-p)^{k+1}).$$

$$\Rightarrow P(N=k) = (1-p)^k - (1-p)^{k+1} = p(1-p)^k. \qquad (5)$$

Also, *NB* (1, p) it can be constructed as a discretization of an exponential distribution as follows:

Let $Y = \frac{ln\, X}{ln(1-p)}$, N=[Y]. Y is defined in R_+, and for x>0 we have:

$$P(Y \leq x) = P(ln\, X \geq x\, ln(1-p)) = P(X \geq e^{x\, ln(1-p)}) = 1 - e^{-|ln(1-p)|x},$$

Therefore:

[Y]~NB (1, p)⇒ Y~Exp($|ln(1-p)|$), (6)

Y~Exp(a) ⇒ [Y]~NB (1, p) with p=1-exp(-a). (7)

Normal approximation and Wilson - Hilferty approximation for the Poisson distribution

Let $X^*:=(X-M(X))/\sqrt{V(X)}$ the standardized of a random variable denoted with X, with M(X) notation for expectancy of X and V(X) the variance. Then X^* is approximated using the Laplace function ϕ that explain the normal distribution $N(0,1)$ since

$$P(X^* \leq x) \approx \Phi(x) \quad \text{or} \quad P(X \leq x) = P\left(X^* \leq \frac{x-M(X)}{\sqrt{V(X)}}\right) \approx \Phi\left(\frac{x-M(X)}{\sqrt{V(X)}}\right). \quad (8)$$

For high volume selections this approximation is justified by the Central Limit Theorem.

Particular Case

$$\boldsymbol{N \sim Po_k(\lambda)}, \quad P(N \leq n) \approx \Phi(\frac{n-\lambda}{\sqrt{\lambda}}), \text{ n=1,2, …} \qquad (9)$$

$N^*:=(N-\lambda)/\sqrt{\lambda}$ has the following momentum generating function:

$$\phi_{N^*}(t) = e^{-t\sqrt{\lambda}} e^{\lambda(e^{t/\sqrt{\lambda}}-1)} = exp[\lambda(1+\frac{t}{\sqrt{\lambda}}+\frac{t^2}{2\lambda}+\ldots-1)-\sqrt{\lambda}t] = exp[\frac{t^2}{2}+\frac{t^3}{6\sqrt{\lambda}}+\ldots] \xrightarrow[\lambda\to\infty]{} exp(\frac{t^2}{2}). \qquad (10)$$

The moment-generating function for N (0,1) is $e^{t^2/2}$.

A first approximation $N=\lambda+\sqrt{\lambda}\,Z$ with $Z \approx N(0,1)$.

A better approximation could be obtained using the Wilson-Hilferty (WH) formula.

We are interested in a study of $S = \sum_{i=1}^{N} X_i$ where X_i are some random variables that could be standard Gamma distribution, $f_X(x) = \frac{x^{b-1}e^{-x}}{\Gamma(b)}, x > 0, N$ could be expressed by Poisson distribution.

For Gamma distribution, first approximation for the cumulative distribution function (cdf) is $X = b + \sqrt{b}\, Z$ with $Z \approx N(0,1)$.

Wilson-Hilferty (WH) for Gamma distribution

We look for a non-linear transformation of the type

$$X = y(Z)^p = (a + b\, Z + \psi(Z))^p$$

where X is Gamma random variable, Z is the transformed random variable, a, b, p are suitable constants to be determined such that Z is closely to a standard Gaussian random variable, $N(0,1)$.

The class of transformation include an analytic correction $\psi(z)$ with initial conditions:

$$\psi(0) = 0,\ \psi'(0) = 0,\ \psi''(0) = 0.$$

From (11) the differential of x becomes $dx = p(a + b\, z + \psi(z))^{p-1}(b\ + \psi'(z))dz$. Also

$$f_X(x,z)dx = g(z,b)dz = \frac{p}{\Gamma(b)} \exp\big(\psi(z)\big)\, dz$$

where $g(z,b)$ is the new Gamma density function expressed through z variable. Because we need to obtain gaussian approximation for $\psi(z)$ is taken a parabolic approximation $\psi(z) \cong \psi(0) - \frac{1}{2}z^2 = \varphi(z)$ that must satisfy initial conditions: $\varphi(0) = 0,\ \varphi'(0) = -1,\ \varphi''(0) = 0$. One finds X= $(a + b\, Z + \psi(Z))^3, b = \frac{1}{3\sqrt{a}}$;

or $a(\alpha) = \sqrt[3]{\alpha - \frac{1}{3}}, b(\alpha) = \frac{1}{3\sqrt[6]{\alpha - \frac{1}{3}}}$. And the following approximation $\Gamma_{WH}(x,\alpha) = \sqrt{2\pi}\left(\alpha - \frac{1}{3}\right)^{\alpha - \frac{1}{2}} e^{-\left(\alpha - \frac{1}{3}\right)}$.

The relative error of approximation is given by $\varepsilon(\alpha) = \left|1 - \frac{\Gamma_{WH}(x,\alpha)}{\Gamma(b)}\right|$ is of order α^{-2}.

For Poisson distribution approximation is:

$$k(\lambda):=3\sqrt{\lambda+1},\ P(N\le n)\approx \Phi\left(k(\lambda)-\frac{1}{k(\lambda)}-\sqrt[3]{9\cdot\lambda\cdot k(\lambda)}\right),\ \text{n=1,2, …} \tag{11}$$

$$\boldsymbol{Po_k(\lambda)}\approx\ 1-\Phi\left(\frac{c-\mu}{\sigma}\right), c=\left(\frac{\lambda}{1+k}\right)^{\frac{1}{3}}, \mu=1-\frac{1}{9(1+k)}, \sigma=\frac{1}{3\sqrt{k+1}}, \tag{11'}$$

For discrete type distributions such as Poisson distribution, it is necessary to approximate the distribution with a continuous probability density and apply the WH-type approximation. In this case $k!=\Gamma(k+1)$ is for the denominator.

$$f_X(x,\lambda)=\frac{\lambda^k e^{-\lambda}}{\Gamma(k+1)}$$

The class of transformations is defined as:

$$k(z):=(a(\lambda)+b(\lambda)\,Z)^q+c$$

with the differential

$$\mathrm{d}k(z):=qb(\lambda)(a(\lambda)+b(\lambda)\,Z)^{q-1}dx$$

that leads to

$$f_X(x,\lambda)\mathrm{d}k=\frac{\lambda^{k(z)}e^{-\lambda}}{\Gamma(k(z)+1)}b(\lambda)(a(\lambda)+b(\lambda)\,Z)^{q-1}dx=\exp\big(\varphi(z)\big)\,dz$$

$$\varphi(z)=k(z)\ln(\lambda)-\lambda-\ln[\Gamma(k(z)+1)]+\left(1-\frac{1}{q}\right)\ln(k(z)-c)+\ln\big(b(\lambda)\big)+\ln(q).$$

Then with the approximation $\ln[\Gamma(k(z)+1)]\cong\frac{1}{2}\ln(2\pi)+\left(k(z)+\frac{1}{2}\right)\ln\left(k(z)+\frac{2}{3}\right)-\left(k(z)+\frac{2}{3}\right)$

$\varphi(z)\cong k(z)\ln k(z)+k(z)+k(z)\ln(\lambda)-\frac{1}{2}\ln k(z)+\left(1-\frac{1}{q}\right)\ln k(z)+O(1),\ \varphi(0)=0,\ \varphi'(0)=-1,\ \varphi''(0)=0.$

For $k(0) = a^q + c$, with $c \ll \ k \cong a^q$, $\ln k \cong \ln(a^q) + \frac{c}{a^q}$. For conditions imposed is choosen $a(\lambda) = \lambda^{\frac{1}{q}}$ and $c = \frac{1}{2} - \frac{1}{q}$.

Finally, optimal parameters for the transformation are

$$q = \frac{3}{2}, a(\lambda) = \lambda^{\frac{2}{3}}, b(\lambda) = \frac{2}{3}\lambda^{\frac{1}{6}}, c = -\frac{1}{6} \Rightarrow k(\lambda) = \left(\lambda^{\frac{2}{3}} + \frac{2}{3}\lambda^{\frac{1}{6}}Z\right)^{\frac{3}{2}} - \frac{1}{6}$$

2. Recursive formulas

Let N - random variables with natural values that satisfy recursive relations:

P(N=0) =p (0), $P(N = n) = P(N = n - 1)(a + \frac{b}{n})$, a, b real, n=1,2, … (12)

First, we precise that condition (12) is true only for the following repartitions:

1. $N\sim Po(\lambda)$ leads to $P(N = 0) = e^{-\lambda}$ and $\frac{P(N=n)}{P(N=n-1)} = \frac{\lambda}{n}$, n=1,2, … It results a=0 and b=λ.

2. *N~Bi* (m, p) with $P(N = 0) = (1 - p)^m$ and $\boldsymbol{P(N = n) = P(N = n - 1) \cdot \frac{m-n+1}{n}\frac{p}{1-p}, n = 1, 2...}$,
it results $a = -p/(1 - p), b = (m + 1)p/(1 - p)$.

3. **N~** *NB* (r, p) with $P(N = 0) = p^r$ and $\frac{P(N=n)}{P(N=n-1)} = \frac{r+n-1}{n}(1 - p), n = 1,2, \ldots$, it results a=1-p, b=(r-1) (1-p).

4. Degenerate distribution with P(N=0) =1, from where a + b=0.

Counter example:

On the other hand, the relation (12) is not fulfilled in the case: N ~ Ln (p) with P(N=0) =0 and $\frac{P(N=n)}{P(N=n-1)} = p(1-\frac{1}{n})$, that is valid only for n=2,3, …, but not for n = 1.

Theorem: If the relation (12) takes place, then the corresponding distributions are only Poisson, Binomial, Negative Binomial and Logarithmic.
Proof:
i. Case a+b<0
Because of P(N=1) = (a+b) P(N=0) we find P(N=0) =0 and P(N=1) =P(N=2) =…=0 from where one concludes that is necessary to have a+b>=0.

ii. Let a+b=0, then P(N=1) =P(N=2) =…=0, so p (0) =P(N=0) =1.

iii. Now, let a+b>0 and a=0. Results that:
$P(N=n) = P(N=n-1)\frac{b}{n} = P(N=n-2)\frac{b^2}{n(n-1)} = \ldots = P(N=0)\frac{b^n}{n!}$, n=1,2, …

And because $1 = \sum_{n=0}^{\infty} P(N=n) = P(N=0)\cdot e^b$, $P(N=n) = e^{-b}\frac{b^n}{n!}$,

iv. For a<0, let $m \in N$ such that: $a + \frac{b}{m+1} = 0$, meaning $m = -\frac{a+b}{a}$ and P (N = n) =0 for n=m+1, m+2, …
Moreover, for n=1,2, …, m we have:

$$P(N=n)=P(N=n-1)(a+\frac{b}{n})=\ldots=$$
$$=P(N=0)(a+b)(a+\frac{b}{2})\ldots(a+\frac{b}{n})=P(N=0)\frac{a^n}{n!}(\frac{a+b}{a})(\frac{a+b}{a}+1)\ldots(\frac{a+b}{a}+n-1)=$$
$$=P(N=0)\frac{a^n}{n!}(-m)(-m+1)\ldots(-m+n-1)=P(N=0)(-a)^n\binom{m}{n},$$

so

$$1=\sum_{n=0}^{m}P(N=n)=P(N=0)\sum_{n=0}^{m}\binom{m}{n}(-a)^n=P(N=0)(1-a)^m$$

thus $P(N=0)=(1-a)^{-m}$ and $P(N=n)=\binom{m}{n}(\frac{-a}{1-a})^n(\frac{1}{1-a})^{m-n}$, n=0,1, …, m,

where $\frac{-a}{1-a}=1-\frac{1}{1-a}\in(0,1)$.

Further the last case, for a>0 and $r:=(a+b)/a\in R^+$, one obtains as before:

$$P(N=n)=P(N=0)(-a)^n\binom{-r}{n}=P(N=0)a^n\binom{r+n-1}{n}=(1-a)^{-r},$$
$$P(N=0)=(1-a)^r$$
$$P(N=n)=\binom{r+n-1}{n}(1-a)^r a^n, n=0,1,\ldots$$

Summary. Simple adjustments

In the applications of stochastic mathematics, the most common method of adjusting a distribution is the normal approximation:

$$P(X \leq x) \approx \Phi(\frac{x-M(X)}{\sqrt{V(X)}}). \tag{6}$$

For the variable S, $\mathrm{S} = \sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{X_i}$, and $(\mathrm{X_i})$ i.i.d. as X (6) means:

$$P(S \leq x) \approx \Phi(\frac{x-M(N(M(X))}{\sqrt{M(NV(X))+(MN)^2V(N)}}), \tag{7}$$

and for $N \sim Po(\lambda)$ one finds$P(S \leq x) \approx \Phi(\frac{x-\lambda M(X)}{\sqrt{\lambda M(X^2)}})$.

In cases where the distribution is not symmetrical, the normal approximation is not appropriate. In this situation, the transformation of the variable Y into Z = lnY is considered, a transformation that brings the distribution of Z closer to a normal one.

A third approach is given by introducing a Gamma distribution. This is applied as follows:

For $X \sim \Gamma(a, b)$ one has $\frac{M(X)}{V(X)} = \frac{b/a}{b/a^2} = a, \;\; \frac{(M(X))^2}{V(X)} = \frac{b^2/a^2}{b/a^2} = b.$

So, if the distribution of a random variable X is adjusted by a Gamma distribution, a suitable choice would be:

$$\Gamma(\frac{M(X)}{V(X)}, \frac{M(X)^2}{V(X)}).$$

3.Approximation of probability densities by Orthogonal Polynomials

The approximation of densities by orthogonal polynomials is based on the following idea: let $I \in R$ an interval and f a real function defined on I which will be approximate. Let w a positive continuous function on I such that exists and be finite for any polynomial $\pi$the integral $\int \pi(x)w(x)dx$. For i=0,1, … let π_i a polynomial such that $\int \pi_i(x)\pi_k(x)w(x)dx = 0$ pentru $i \neq k$ and

$$C_k := \int \pi_k^2(x)w(x)dx, \quad \text{k=0,1, …}$$

The regular function f can be developed as follows:

$$f(x) = A_0\pi_0(x)w(x) + A_1\pi_1(x)w(x) + \ldots, \qquad (8)$$

Where the coefficients $A_{0,}A_1, \ldots$ are obtained from:

$$\int \pi_k(x)f(x)dx = \int \pi_k(x)\sum A_i\pi_i(x)w(x)dx = A_k\int \pi_k^2 w(x)dx = A_kC_k. \;\; A_k = \int \pi_k(x)f(x)dx/C_k, \quad \text{k=0,1, …}$$

particularly if f is the density of a random variable X: $A_k = M(\pi_k(x))/C_k$, k=0,1, …,

From (8) choosing the first n terms for a truncated relation at step n one gets the approximation

$$f(x) \approx A_0\pi_0(x)w(x) + \ldots + A_n\pi_n(x)w(x), \qquad (9)$$

for which it is sufficient to know the moments up to the order n.

We will further consider some examples of evaluated functions and corresponding polynomial series. In all cases the evaluated functions are densities. In the limit case in which n = 0 the approximation f(x)~w(x) is obtained by the relation (9).

1. BOWERS' Gamma approximation function.

We want to approximate the density function g or the distribution function G for a positive random variable S. Instead of S we will consider the random variable $Z = \frac{M(S)}{V(S)} S$ with density function f and distribution function F.

One obtains:

$$G(x) = F(\frac{M(S)}{V(S)} x),\ g(x) = \frac{M(S)}{V(S)} f(\frac{M(S)}{V(S)} x), x \in R.$$

Let $I = R^+$ and$w(x) = \frac{1}{\Gamma(b)} x^{b-1} e^{-x}$ with $b = M(Z) = \frac{M(S)^2}{V(S)}$.

So, w has the meaning of density function for $\Gamma(1, b)$.

<u>Remark</u>: For $X \sim \Gamma(\alpha, \beta)$ we have $\alpha = \frac{M(X)}{V(X)}, \beta = \frac{M(X)^2}{V(X)}$, and with c>0, $cX \sim \Gamma(\frac{\alpha}{c}, \beta)$.

Thus $\alpha X = \frac{M(X)}{V(X)} X \sim \Gamma(1, \beta) = \Gamma(1, \frac{(EX)^2}{V(X)})$ and reversely

$$\Upsilon \sim \Gamma(1, \frac{M(X)^2}{V(X)}) \Rightarrow \frac{V(X)}{M(X)} \Upsilon \sim \Gamma(\frac{M(X)}{V(X)}, \frac{M(X)^2}{V(X)}).$$

Thus, the transition from S to Z is a kind of standardization of a Gamma random variable.
For the family of orthogonal polynomials, it is used

$$L_k(x) = (-1)^k x^{1-b} e^x \frac{d^k}{dx^k} (x^{k+b-1} e^{-x}) = \Gamma(b+k) \sum_{i=0}^{k} (-1)^{i+k} \binom{k}{i} x^i \frac{1}{\Gamma(b+i)}, \qquad k=0,1, \ldots$$

L_k is called the Laguerre polynomial of order k and we get for example:

$$\begin{aligned}
&L_0(x) = 1,\\
&L_1(x) = x - b,\\
&L_2(x) = x^2 - 2(b+1)x + b(b+1),\\
&L_3(x) = x^3 - 3(b+2)x^2 + 3(b+2)(b+1)x - (b+2)(b+1)b.
\end{aligned}$$

Moreover,

$C_k = k!\frac{\Gamma(b+k)}{\Gamma(b)} = k!\,(b+k-1)\cdot(b+k-2)\cdot\ldots\cdot b$ and

$A_0 = 1, A_1 = A_2 = 0, A_3 = \frac{\Gamma(b)}{6\Gamma(b+3)}(\mu_3 - (b+2)(b+1)b)$ with $\mu_3 := M(Z^3)$.

We find the approximation:

$$f(x) \approx w(x) + A_3 L_3(x) w(x), F(x) \approx W(x) + A_3 \int_0^x L_3(t) w(t) dt, \tag{10}$$

where W is the distribution function of the density w(x). Because

$\int_0^x L_3(t)w(t)dt = -\frac{x^b e^{-x}}{\Gamma(b)}((b+2)(b+1) - 2(b+2)x + x^2)$

we finally can write:

$$F(x) \approx W(x) - x^b e^{-x}\left[\frac{1}{\Gamma(b+1)} - \frac{2x}{\Gamma(b+2)} + \frac{x^2}{\Gamma(b+3)}\right]\frac{\mu_3 - (b+2)(b+1)b}{6}. \tag{11}$$

Replacing x by $x \cdot \frac{M(S)}{V(S)}$ in the right hand of (11) an approximation of G(x) is obtained.

2. Gram-Charlier approximation

It is desired to approximate the distribution function G of the density function g for a random variable S with $\mu := M(S)$ and $\sigma^2 := V(S)$. Instead of S we consider the standardized value $Z := (S-\mu)/\sigma$ with the density function f and distribution function F. Obviously

M(Z)=0, V(Z)=1 and $G(x) = F(\frac{x-\mu}{\sigma}),\ g(x) = \frac{1}{\sigma} f(\frac{x-\mu}{\sigma}),\ \ x \in R.$

For now, we take I=R and $w = \phi$, where ϕ express the probability density function for N (0,1), and Φ is the Laplace function. For orthogonal polynomials we use

$$H_k(x) = \phi^{(k)}(x)/\phi(x)$$

$$(-1)^k \sum_{i=0}^{m} \binom{k}{2i} x^{k-2i} (-1)^i \prod_{j=0}^{i-1} (2j+1),\ m := [\tfrac{k}{2}],\ \text{k=0,1, …} \qquad (12)$$

with the recursive relations

$$H_{k+1}(x) = -xH_k(x) + H_k'(x),\ \text{k=0,1, …}$$

H_k is the Hermite polynomial of order k, for which:

$$H_0(x) = 1, H_1(x) = -x, H_2(x) = x^2 - 1, H_3(x) = -x^3 - 6x^2 + 3.$$

One finds $C_k = k!$ and

$$A_0 = 1, A_1 = A_2 = 0, A_3 = -\frac{\mu_3}{6}, A_4 = \frac{1}{24}(\mu_4 - 3)$$

with $\mu_i := M(Z^i)$, i=3,4.

We find the approximations:

$$f(x) \approx \phi(x) + A_3\phi^{(3)}(x) + A_4\phi^{(4)}(x), F(x) \approx \Phi(x) + A_3\Phi^{(3)}(x) + A_4\Phi^{(4)}(x). \qquad (13)$$

The approximation for G(x) results by replacing x with $(x-\mu)/\sigma$ in the right hand of the relation (13).

Particular Case

If we consider $S \sim CP(\lambda, Q)$, where Q has $m_k$the moment of order k, k=1,2, …, one finds:

$$\mu = \lambda m_1,\ \sigma^2 = \lambda m_2,$$
$$\mu_3 = MZ^3 = \frac{1}{\sigma^3}M(S-\mu)^3 = \frac{\lambda m_3}{\sqrt{(\lambda m_2)^3}} = \frac{m_3}{\sqrt{\lambda m_2^3}},$$
$$\mu_4 = MZ^4 = \frac{1}{\sigma^4}M(S-\mu)^4 = \frac{3\lambda^2 m_2^2 + \lambda m_4}{(\lambda m_2)^2} = \frac{m_4}{\lambda m_2^2} + 3,$$

so

$$A_3 = -\frac{m_3}{6\sqrt{\lambda m_2^3}}, A_4 = \frac{m_4}{24\lambda m_2^2}.$$

If we consider N~B (n, p) or N~NB (r, p) the values for A_3 and A_4 can be determined using recursive formulas.

3. Edgeworth approximation

Let S a random variable with $\mu = M(S)$, $\sigma^2 := V(S)$, and we look for approximation of the density function g or the distribution function G. As we did before, the variable is standardized by$Z = (S-\mu)/\sigma$ with density function f, and distribution function F and $\phi$the momentum generation function. Taylor's next development is supposed to take place:

$$\bar{\phi}(t) := e^{-\frac{t^2}{2}}\varphi(t) = \sum_{i=0}^{\infty} a_i t^i, \varphi(t) = \sum_{i=o}^{\infty} a_i t^i e^{\frac{t^2}{2}}.$$

By induction one observes that:

$t^i e^{\frac{t^2}{2}} = \int_{-\infty}^{\infty} e^{tu} (-1)^i \varphi^{(i)}(u)du,$ i=0,1, …, where φ express the density function for N (0,1),

thus, $\varphi(t) = \sum_{i=0}^{\infty} a_i \int_{-\infty}^{\infty} e^{tu} (-1)^i \Phi^{(i)}(u)du = \int_{-\infty}^{\infty} e^{tu}(\sum_{i=0}^{\infty} a_i (-1)^i \varphi^{(i)}(u))du.$
Consequently:

$$f(t) = \sum_{i=0}^{\infty} a_i(-1)^i \varphi^{(i)}(t), F(x) = \sum_{i=o}^{\infty} a_i(-1)^i \Phi^{(i)}(u)du, \tag{14}$$

Edgeworth approximation of the order n is obtained by truncate the above series,

$$f(x) \approx \sum_{i=0}^{n} a_i(-1)^i \varphi^{(i)}(x), \text{ or } \quad g(x) \approx \sum_{i=0}^{n} a_i(-1)^i \frac{1}{\sigma} \Phi^{(i)}(\frac{x-\mu}{\sigma}). \ (15)$$

We find equality: $a_i = \frac{1}{i!}\bar{\phi}^{(i)}(0)$, i=0,1, …. and K. Schroter gave us the following representation:

$$a_i = \frac{1}{i!}\sum_{j=0}^{i}\binom{i}{j}\mu_j H_{i-j}(0), \qquad \text{i=0,1, …}$$

where $\mu_k = M(Z^k)$ and $H_k$is the Hermite polynomial of order k, k=0,1, ….

Using the explicit representation of Hermite's polynomial, we obtain for i=0,1, …

$$a_{2i} = \sum_{j=0}^{i}(-\frac{1}{2})^{i-j}\frac{1}{(1-j)!(2j)!}\mu_{2j},$$
$$a_{2i+1} = \sum_{j=0}^{i}(-\frac{1}{2})^{i-j}\frac{1}{(i-j)!(2j+1)!}\mu_{2j+1}.$$

In particular we have:

$$a_0 = 1, a_1 = a_2 = 0, a_3 = \mu_3/6,$$
$$a_4 = (\mu_4 - 3)/24,$$
$$a_5 = (\mu_5 - 10\mu_3)/120,$$
$$a_6 = (\mu_6 - 15\mu_4 + 30)/720.$$

The moments for S and Z can be determined according to the formulas in the previous sections.

Power approximation

Power approximation is an additional method to improve Edgeworth approximation.
Let be S random variabile with cumulative density function G, $\gamma = M(S)$, $\sigma^2 = V(S)$ and random variable $Z = (S - \gamma)/\sigma$. Let be also $\gamma_1 := M(Z^3)$, $\gamma_2 := M(Z^4) - 3$. We shall study the case $S \sim CP(\lambda, Q)$.
From Edgeworth aproximation for the distribution function F of Z we have:

$$F(x) \approx e(x) := \Phi(x) - \frac{\gamma_1}{6}\Phi^{(3)}(x) + \frac{\gamma_2}{24}\Phi^{(4)}(x) + \frac{\gamma_2^1}{72}\Phi^{(6)}(x).$$

For a function p with the inverse p^{-1} such that:

$$e(p(y)) = \Phi(y). \tag{18}$$

we obtain

$$G(x) = F\left(\frac{x-\mu}{\sigma}\right) \approx e\left(\frac{x-\mu}{\sigma}\right) = \Phi\left(p^{-1}\left(\frac{x-\mu}{\sigma}\right)\right). \tag{19}$$

If $p(y) = y + \Delta y$, from (19)

$$\begin{aligned} 0 = q(\Delta y) &:= \Phi(y) - e(y + \Delta y) \\ &= \Phi(y) - \Phi(y + \Delta y) + \tfrac{1}{6}\gamma_1\Phi^{(3)}(y + \Delta y) - \tfrac{1}{24}\gamma_2\Phi^{(4)}(y + \Delta y) - \tfrac{1}{72}\gamma_1^2\Phi^{(6)}(y + \Delta y) + \ldots \end{aligned}$$

In order to obtain $\Delta y(q)$ we use newton methods of order one,

$$\Delta y \approx y_0 - \frac{q(y_0)}{q'(y_0)} - \frac{1}{2}\frac{q''(y_0)}{q'(y_0)}\left[\frac{q(y_0)}{q'(y_0)}\right]^2, \tag{20}$$

With initial value $y_0 := 0$.

Prin marirea valorii λ in cazul $CP(\lambda, Q)$ si prin omiterea termenilor care devin mult prea mici ca si valoare, se obtine din ecuatia (21):

$$\Delta y \approx -\frac{q(0)}{q'(0)} - \frac{1}{2}\frac{q''(0)}{q'(0)}\left[\frac{q(0)}{q'(0)}\right]^2$$

$$\approx -\frac{\frac{1}{6}\gamma_1\Phi^{(3)}(y)-\frac{1}{24}\gamma_2\Phi^{(4)}(y)-\frac{1}{72}\gamma_1^2\Phi^{(6)}(y)}{-\Phi'(y)+\frac{1}{6}\gamma_1\Phi^{(4)}(y)-\frac{1}{24}\gamma_2\Phi^{(5)}(y)-\frac{1}{72}\gamma_1^2\Phi^{(7)}(y)} - \frac{1}{2}\ldots$$

$$\approx \frac{\frac{1}{6}\gamma_1(y^2-1)+\frac{1}{24}\gamma_2(y^3-3y)+\frac{1}{72}\gamma_1^2(y^5-10y^3+15y)}{1+\frac{1}{6}\gamma_1(y^3-3y)} + \frac{1}{72}\gamma_1^2(y^5-2y^3+y).$$

After dividing the terms into the first fraction:

$$\frac{1}{6}\gamma_1(y^2-1) + \frac{1}{24}\gamma_2(y^3-3y) - \frac{1}{72}\gamma_1^2(y^5+2y^3-9y) + \ldots,$$

We find the following approximation:

$$\Delta y \approx \frac{1}{6}\gamma_1(y^2-1) + \frac{1}{24}\gamma_2(y^3-3y) - \frac{1}{36}\gamma_1^2(2y^3-5y)$$

And

$$p(y) = y + \frac{1}{6}\gamma_1(y^2-1) + \frac{1}{24}\gamma_2(y^3-3y) - \frac{1}{36}\gamma_1^2(2y^3-5y). \qquad (21)$$

According with (19) we must now obtain that y for which the relaion $p(y) = (x-\mu)/\sigma$ take place.

Fisrt case in (21), p(y)=y; formula (19) being the normal approximation.

Second approximation $p(y) = y + \frac{1}{6}\gamma_1(y^2 - 1)$, leads to:

$$G(x) \approx \Phi(\sqrt{\frac{9}{\gamma_1^2} + \frac{6}{\gamma_1}\frac{x-\mu}{\sigma} + 1} - \frac{3}{\gamma_1}).$$

In case $S \sim CP(\lambda, Q)$, with Q having m_k, k=1,2,…, moments:

$$G(x) \approx \Phi(\sqrt{\frac{9 \cdot \lambda \cdot m_2^3}{m_3^2} + \frac{6m_2}{m_3}(x - \lambda m_1) + 1} - \frac{3\sqrt{\lambda m_2^3}}{m_3}). \quad (22)$$

Called *normal approximation of the power two.*

In case Q=Exp(a), relation (22) becoms:

$$G(x) \approx \Phi(\sqrt{2ax+1} - \sqrt{2\lambda}).$$

In this case a better approximation is obtained if all the terms in relation (21) are taken into account. The third order approximation would be the following:

$$G(x) \approx \Phi(\sqrt{2ax + \frac{1}{2} + \frac{1}{32\lambda}} - \sqrt{2\lambda}(1 - \frac{1}{8\lambda})).$$