INNOVATIVE MATHEMATICAL MODELING TECHNIQUES: FRACTIONAL CALCULUS

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CONTENT

1. INTRODUCTION

1.1. Fractional Calculus (FC) as mathematical field

Fractional calculus is the field of mathematical analysis which deals with the investigation and application of integrals and derivatives of arbitrary order (real or even complex). It is a generalization of the classical calculus and therefore preserves many of the basic properties. Fractional derivatives and integrals can be considered as an „interpolation"of the infinite sequence of the clasical n-fold integrals and n-fold derivatives.

1.2. Development

The fractional calculus may be considered an old and yet novel topic.
It is an old topic since, starting from some speculations of G.W. Leibniz (1695, 1697) and L. Euler (1730), it has been developed up to the present day. In fact the idea of generalizing the notion of derivative to non-integer order, in particular to the order 1/2, is contained in the correspondence of Leibniz with Bernoulli, L'Hôpital and Wallis. Euler took the first step by observing that the result of the evaluation of the derivative of the power function has a meaning for non-integer order thanks to his Gamma function. A list of mathematicians who have provided important contributions up to the middle of the 20th century includes J. Liouville (1832–1837), B. Riemann (1847), A.K. Grünwald (1867–1872), A.V. Letnikov (1868–1872), N.Ya.Sonine (1872–1884), M. Riesz (1949).
As novel topic FC has only been the subject of specialized conferences and treatises in the last 30 years – for proceedings of the First Conference on Fractional Calculus and its Applications at the University of NewHaven in June 1974 see [Ros75], [Ros77] and for the first monograph see [OldSpa74]. In addition around ten titles are explicitly devoted to FC and around twelve treatises contain a detailed analysis of some mathematical aspects and/or physical applications of FC, without referring to FC in the title (see [Gor et al20]).
The old and recent (up to 2010) history of Fractional Calculus are available on Research Gate and at the website of Fractional Calculus and Applied Analysis (FCAA) http://www.math.bas.bg/~fcaa/. The latter includes free access to full length papers from 2004-2010.
For an introductory survey on FC, respectively for engineering and physics, respectively for economics and finance see [GorMai97], [Pod99], respectively [CarMai97], [Her14], [Hil00], respectively [BaiKin96].
In recent years considerable interest in fractional calculus has been stimulated by the applications that it finds in different fields of science, including numerical analysis, economics and finance, engineering, physics, biology, etc., as well as several contributions to the fractional theory and its applications in the recent series, edited by Machado (published by De Gruyter), of 8 Handbooks on Fractional Calculus with applications, [HAND1, HAND2, HAND3, HAND4, HAND5, HAND6, HAND7, HAND8] and also in the special issue of Mathematics (MDPI) edited by Mainardi [Mai-spec18].
The regular journals devoted to fractional calculus, are Journal of FC (Descartes Press, Tokyo) and FC and Applied Analysis (De Gruyter, Berlin) - for information there is the WEB site https://www.degruyter.com/view/j/fca; a remarkable site devoted to FC is www.fracalmo.org, whose name comes from FRActional CALculusMOdelling, and the related links.

1.3. Interpretation

Integer – order derivatives and integrals have physical and geometrical interpretations. For a physical interpretation of the fractional integration in terms of two different time scales – the homogeneous, equably flowing scale and the inhomogeneous time scale see [Pod99]. Some authors [M-THam8] consider the fractional operators as linear filters and also seek the geometrical interpretation of the fractional operators in the fractal geometry of which classical geometry is a subclass.

1.4. Applications

The first application of a semi-derivative (of order ½) is done by Abel in 1823 and is in relation with the solution of an integral equation [see OldSpa9]. The last decades prove that FC is very convenient for describing properties of real materials, e.g.polymers as a tool for describing the memory and hereditary properties or solving Caputo-problems of viscoelasticity (see [Pod99]). FC also appears in the control theory, where for the description of the controlled system and the controller fractional differential equation are used.

1.5. Approaches

There are two main approaches to the FC – the continuous and the discrete approaches (see [GorMai6]). The continuous approach is based on the Riemann-Liouville fractional integral which has the Cauchy integral formula as a starting point (see [OldSpa9]. The dicrete approach is based on the Grunwald-Letnikov fractional derivative – as a limit of a fractional-order backward difference (see [GorMai6], [Pod99]).

1.6. Riemann-Liouville fractional operators vs. Caputo fractional operator

Riemann-Liouville operators play an important role in the development of FC and for its applications in pure mathematics – solution of integer-order differential equations, definition of new function classes, see[Pod99]. Caputo fractional derivative can provide initial conditions with clear physical interpretation for the differential equation of fractional order

2. SPECIAL FUNCTIONS OF THE FRACTIONAL CALCULUS

2.1. Special Functions

2.1.1. The Eulerian Functions

Definition 1.(in C)i. (Gamma function (Γ)) The Gamma function – as the Euler integral of the second kind is defined by the integral formula

$$\Gamma(z) := \int_0^\infty u^{z-1}\, e^{-u}\, du, \quad \mathrm{Re}(z) > 0. \qquad (1)$$

ii. (Beta function (B)) The Beta function – as Euler integral of the first kind is defined by the integral formula

$$B\ (p,q) = \int_0^1 u^{p-1}\,(1-u)^{q-1}\,du\,, \quad \begin{cases} \mathrm{Re}(p) > 0 \\ \mathrm{Re}(q) > 0\,. \end{cases} \qquad (2)$$

Observation 1. (domain of analycity) This integral representation (i) is the most common for Γ, even if it is valid only in the right half-plane of $\mathbb{C}$.

The analytic continuation to the left half-plane can be done in different ways. As will be shown later, the domain of analyticity D_Γ of Γ is

$D_\Gamma = \mathrm{C} \setminus Z_-.$ (3)

Lemma 1. i(the Gaussian integral representation of Γ)

$$\Gamma(z) = 2\int_0^\infty e^{-v^2} v^{2z-1} dv\,, \quad \mathrm{Re}(z) > 0 \,; \qquad (4)$$

ii(trigonometric integral reprezentation of B)

$$B(p,q) = 2\int_0^{\pi/2} (\cos\vartheta)^{2p-1} (\sin\vartheta)^{2q-1} d\vartheta, \quad \begin{cases} \mathrm{Re}(p) > 0 \\ \mathrm{Re}(q) > 0\,. \end{cases} \qquad (5)$$

Proof. From definition, by the substitution $u = v^2$ (i), respectively $u = (cosv)^2$ (ii).

Theorem 1 (essential properties) 1.(Γ) i. (recurrence formula or difference equation)

$\Gamma(z+1) = z\,\Gamma(z)$; (6)

ii. (complementary formula or the reflection) $\Gamma(z)\Gamma(1-z) = \frac{\pi}{sin\pi z}$. (7)

2. (B) i. (symmetry) $B(p,q) = B(q,p)$; (8)

ii. (relation to Gamma function) $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. (9)

Proof. 1.i. From definition using integration by parts.

ii. The simplest proof consists in proving (7) for 0 < Re(z) < 1 and extending the result by analytic continuation to C \ Z.

2. i. This property is a simple consequence of the definition 1.ii of B.

ii. The proof of (9) can easily be obtained by writing the product Γ(p) Γ(q) as a double integral that is to be evaluated introducing polar coordinates and by lemma 1, i.e.

$$\begin{aligned}\Gamma(p)\,\Gamma(q) &= 4\int_0^\infty \int_0^\infty e^{-(u^2+v^2)} u^{2p-1} v^{2q-1} du\, dv \\ &= 4\int_0^\infty e^{-\rho^2} \rho^{2(p+q)-1} d\rho \int_0^{\pi/2} (\cos\vartheta)^{2p-1} (\sin\vartheta)^{2q-1} d\vartheta \\ &= \Gamma(p+q)\, B(p,q)\,.\end{aligned}$$

Observation 2. i. (n-recurrence formula) The recurrence formula can be iterated to n-recurrence formula:

$$\Gamma(z+n) = z\,(z+1)\,\dots\,(z+n-1)\,\Gamma(z)\,, \quad n \in \mathbb{N} \qquad (10)$$

or

$$\Gamma(z+n) = (z)_n\Gamma(z), z \in C \setminus Z, n \in N, \qquad (10')$$

where $(z)_n$ are the Pochhammer's symbols which are defined as

$(z)_0 = 1$, $(z)_n := z\,(z+1)(z+2)\ldots(z+n-1)$, $n \in N^*$. (11)

By extension to Z_- it can be defined

$(z)_{-n} := z\,(z-1)(z-2)\ldots(z-n+1)$, $n \in N$ (12)

or

$(z)_{-n} := A_z^n, n \in N$ (formal); (12')

If $z \in C \setminus Z, then$ $\Gamma(z+1) = (z)_{-n}\Gamma(z-n+1)$. (13)

ii. (the extension of the domain of analycity) A formal way to obtain

the real domain of analyticity D_Γ is to carry out the required analytical continuation via the n-recurrence formula with the help of which we can enter the left half-plane step by step. The relation (9) is of fundamental importance. Furthermore, it allows us to obtain the analytical continuation of the Beta function.

Theorem 2. (in $\mathbb{R}$) 1.(Γ)i. $\Gamma(1) = \Gamma(2) = 1, \Gamma(\frac{1}{2}) = \int_{-\infty}^{\infty} e^{-v^2} dv = 2\int_0^{\infty} e^{-v^2} dv = \sqrt{\pi} \approx$ 1.77245; (14)

ii. $\Gamma(n+1) = n!, n \in N.$ (15)

2. (B)i. $B(\frac{1}{2}, \frac{1}{2}) = \pi;$ (16)

ii. (other integral representation) Representations of $B(p, q)$ on $[0, \infty]$, respectively a representation of $B(p, q)$ on $[0, 1]$ are

$\int_0^{\infty} \frac{u(x)}{w(x)} dx, \int_0^{\infty} \frac{v(x)}{w(x)} dx, \frac{1}{2}\int_0^{\infty} \frac{u(x)+v(x)}{w(x)} dx, \int_0^{1} \frac{u(x)+v(x)}{w(x)} dx \ where\ u(x) = x^{p-1}, v(x) = x^{q-1}, w(x) = (1+x)^{p+q}.$ (17)

$Proof.$ 1.i. Usual calculation.

ii. From n-recurrence formula (10) for z=1 and according to (14).

2. i. From „relation to Gamma function" (9) and according to (14);

ii. The first representation follows from definition of B by setting $y = \frac{x}{x+1}$;

the other two are easily obtained by using the symmetry property of B(p, q) . The representation of $B(p, q)$ on $[0, 1]$ is obtained from the first integral by additivity on $[0, 1], [1, \infty]$.

Observation 3. (the graph of the Gamma function on the Real Axis) Plots of $\Gamma(x)$(continuous line) and $1/\Gamma(x)$ (dashed line) for $-4 < x \leq 4$ are shown in Fig.1 and for $0 < x \leq 3$ in Fig. 2. Hereafter we provide some analytical arguments that support the plots on the real axis. In fact, one can get an idea of the graph of the Gamma function on the real axis using the formulas

$$\Gamma(x+1) = x\Gamma(x)\,,\ \Gamma(x-1) = \frac{\Gamma(x)}{x-1}\,,$$

to be iterated starting from the interval $0 < x \leq 1$,where $\Gamma(x) \to +\infty$ as $x \to 0^+$ and $\Gamma(1) = 1.$ For $x > 0$ the integral representation (1) yields $\Gamma(x) > 0$ and $\Gamma''(x) > 0$ since

$$\Gamma(x) = \int_0^{\infty} e^{-u}\, u^{x-1}\, du\,,\ \Gamma''(x) = \int_0^{\infty} e^{-u}\, u^{x-1}\, (\log u)^2\, du\,.$$

As a consequence, on the positive real axis $\Gamma(x)$ turns out to be positive and *convex* so that it first decreases and then increases, exhibiting a minimum value. Since $\Gamma(1) = \Gamma(2) = 1$, we must have a minimum at some x_0, $1 < x_0 < 2$. It turns out that $x_0 = 1.4616...$ and $\Gamma(x_0) = \underline{0}.8856\ldots$; hence x_0 is quite close to the point $x = 1.5$ where attains the value $\sqrt{\pi}/2$=0.8862....

On the negative real axis $\Gamma(x)$ exhibits vertical asymptotes at $x = -n$ (n = 0,1,2,...); it turns out to be positive for $-2 < x < -1$, $-4 < x < -3$, ..., and negative for $-1 < x < 0$, $-3 < x < -2$, ...

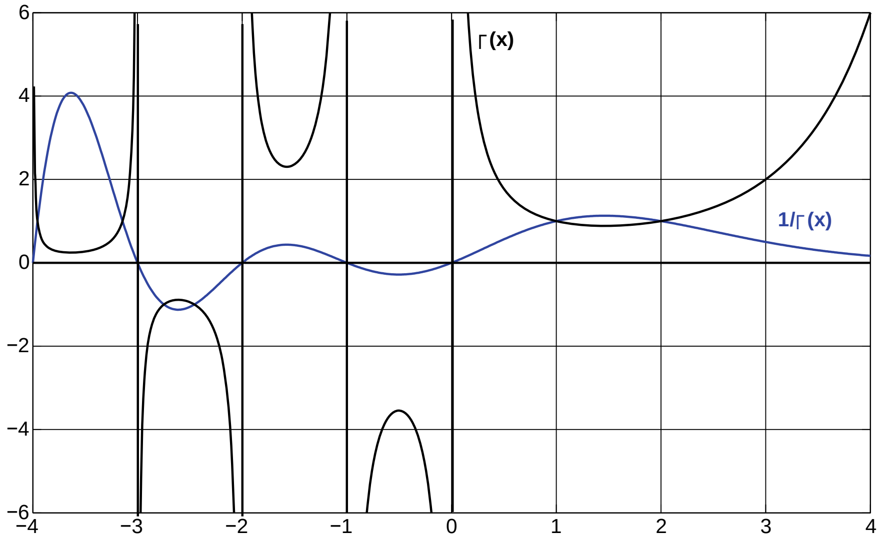

Fig.1 Plots of $\Gamma(x)$(continuous line) and $1/\Gamma(x)$ (dashed line)

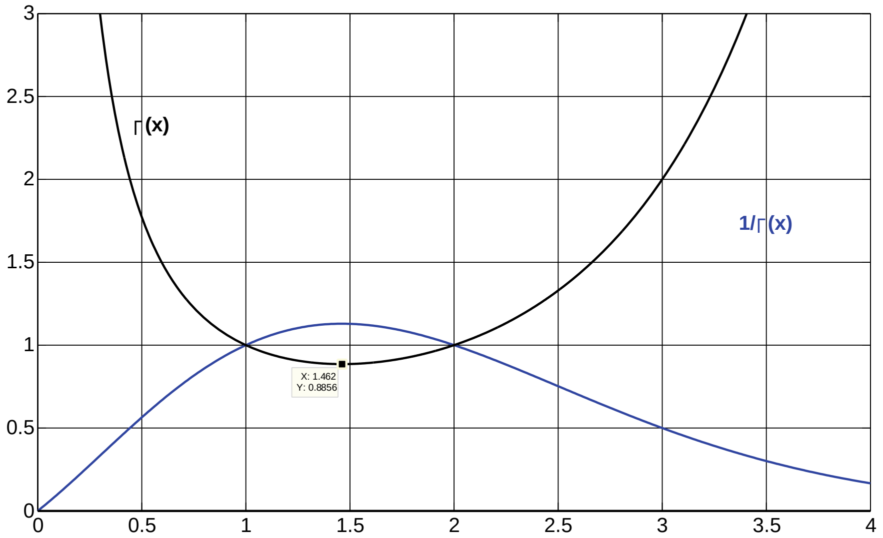

Fig. 2 Plots of $\Gamma(x)$(continuous line) and $1/\Gamma(x)$ (dashed line)

Application 1. (derivatives vs. „fractional derivatives") For power function $y(x) = x^n$ k-derivative is

$$y^{(k)}(x) = \frac{d^k y(x)}{dx^k} = A_n^k x^{n-k} = \frac{n!}{(n-k)!} x^{n-k}, k, n \in N, k \leq n \quad (18) \text{ or}$$

$$y^{(k)}(x) = \frac{\Gamma(n+1)}{\Gamma(n-k+1)} x^{n-k}; \qquad (18')$$

„ $\frac{1}{2}$ – derivative" is

$$\frac{d^{\frac{1}{2}}y(x)}{dx^{\frac{1}{2}}} = \frac{\Gamma(\mathrm{n}+1)}{\Gamma(\mathrm{n}+\frac{1}{2})}x^{n-\frac{1}{2}}. \qquad (19)$$

In particular

$$(x)^{(\frac{1}{2})} = \frac{\Gamma(2)}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}} = \frac{2}{\sqrt{\pi}}\sqrt{x},\ (x^2)^{(\frac{1}{2})} = \frac{\Gamma(3)}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}} = \frac{8}{3\sqrt{\pi}}x\sqrt{x},\ \Gamma(\frac{5}{2}) = \frac{3}{2}\Gamma(\frac{3}{2}) = \frac{3}{2}\,\frac{1}{2}\Gamma(\frac{1}{2}) = \frac{3}{4}\sqrt{\pi},$$

$$(\sqrt{x})^{(\frac{1}{2})} = \frac{\Gamma(\frac{3}{2})}{\Gamma(1)}x^0 = \frac{\sqrt{\pi}}{2} \quad (20)$$

(see (6)). More general, for power function $y(x) = x^{\alpha}$ „k-derivative" is

$$y^{(k)}(x) = A_{\alpha}^{k}x^{\alpha-k} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\mathrm{k}+1)}x^{\alpha-k}, k \in N, \alpha \in R \setminus \{-1\} \quad (21)$$

(see (12'), (13).

Exercices (verification)

1.i. $\Gamma(\mathrm{n}+\frac{1}{2}) = (\frac{1}{2})_n\Gamma(\frac{1}{2}) = \frac{(2n-1)!!}{2^n}\sqrt{\pi} = \frac{(2n)!}{2^{2n}n!}\sqrt{\pi}, n \in N;$ (22)

ii. $\Gamma(-\mathrm{n}+\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})}{(-\frac{1}{2})_{-n}} = (-1)^n\frac{2^n}{(2n-1)!!}\sqrt{\pi} = (-1)^n\frac{2^{2n}n!}{(2n)!}\sqrt{\pi}, n \in N.$ (22')

2.i. (Γ −alternative reflection formula) $\Gamma\left(\frac{1}{2}-z\right)\Gamma\left(\frac{1}{2}+z\right) = \frac{\pi}{\cos \pi z}, \quad z - \frac{1}{2} \notin \mathbb{Z}$; (23)

ii. $\Gamma(\frac{1}{2}-iy)\Gamma(\frac{1}{2}+iy) = \frac{\pi}{cosh\pi y}, y \in R.$ (23')

3.i. $\int_0^{\infty} t^{z-1}sintdt = \Gamma(\mathrm{z})\sin\frac{\pi z}{2}, Re(z) > 1;$ (24)

ii. $\int_0^{\infty} t^{z-1}costdt = \Gamma(\mathrm{z})\cos\frac{\pi z}{2}, Re(z) > 1.$ (24')

4. $I_n = \int_0^\infty \frac{dx}{x^n+1} \in \frac{\frac{\pi}{n}}{sin\frac{\pi}{n}}, n \in N \setminus \{0,1\}$. Discussion I_0, I_1, I_2, I_3, I_4.   (25)

5. $\varphi_U(u) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iux}\, e^{-\frac{x^2}{2}} dx = e^{-\frac{u^2}{2}}, \varphi_U: R \to R_+^*$ (26)

(the characteristic function associated with the normal unit distribution).

6. i. $I(p,q) = \int_0^{\frac{\pi}{2}} sin^p\, x cos^q x dx = \frac{1}{2}\mathrm{B}(\frac{p+1}{2}, \frac{q+1}{2}), p, q \in (-1, \infty)$;   (27)

ii. $I(p,-p) = \int_0^{\frac{\pi}{2}} tg^p\, xdx = \frac{\frac{\pi}{2}}{sin(p+1)\frac{\pi}{2}}, p \in (-1,1)$;   (27')

iii. $I(2m,2n) = \frac{(2m-1)!!(2n-1)!!}{(m+n)!2^{m+n}} \frac{\pi}{2} = \frac{(2m)!(2n)!}{(m+n)!m!n!4^{m+n}} \frac{\pi}{2}, m, n \in N$.   (27'')

2.1.2. (Complementary) Error function

Definition 1.i. (erf) The (Gaus-) error function $erf: C \to C$ is defined by
$erf(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\tau^2}\, d\tau$;   (1)

ii. (erfc) The complementary error function $erfc: C \to C$ is given by

erfc=1- erf.   ($\bar{1}$)

Exercise (verification) $erfc(z) = \frac{2}{\sqrt{\pi}} \int_z^\infty e^{-\tau^2}\, d\tau$.   ($\bar{1}'$)

Observation 1 (interpretations) For $X \sim N(0, \frac{1}{2})$ we have the interpretations

erf(t)=P(X falls in [-t, t]), erfc(t)=P(X resist in [-t, t]).   (2)

Proposition 1. i. (special values)

$$erfc(t) = \begin{cases} 2, & for\ t = -\infty \\ 1, & for\ t = 0 \\ 0, & for\ t = \infty \end{cases};\quad (3)$$

ii. (relations)

$$erfc(-t) = 2 - erfc(t),\ \int_0^{\infty} erfc(t)dt = \frac{1}{\sqrt{\pi}}, \int_0^{\infty} erfc^2(t)dt = \frac{2-\sqrt{2}}{\sqrt{\pi}}. \quad (4)$$

Observation 2.i. (scaled complementary function (erfcx) Is defined by
$erfcx(z) = e^{z^2} erfc(z);$ (5)

ii. (Faddeeva (Kramp)-scaled complementary function) Is defined by

$$w(z) = e^{-z^2} \cdot erfc(-iz) = erfcx(-iz) = e^{-z^2} \cdot \left(1 + \frac{2i}{\sqrt{\pi}} \cdot \int_0^z e^{\tau^2} d\tau\right), \ (6)$$

Were this function is related to the Fresnel integral, to Dawson's integral and to the Voigt function;

iii. (series representations)

$$erf(z) = \frac{2}{\sqrt{\pi}} e^{-z^2} \sum_{n=0}^{\infty} \frac{2^n}{(2n+1)!!} z^{2n+1} = \frac{1}{\sqrt{\pi}} e^{-z^2} \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(\frac{1}{2})_{n+1}}, \quad (7)$$

$$e^{-t^2} \cdot erfc(-it) = \sum_{k=0}^{\infty} \frac{(it)^k}{\Gamma\left(\frac{k}{2}+1\right)} \quad (8)$$

(see 2.1.1, formula (22)).

2.1.3. Confluent hypergeometric function

Definition 1(Confluent hypergeometric function - Kummer) The confluent hypergeometric function of the first kind or Kummer function of the first kind is defined as

$${}_1F_1(\alpha, \beta; z) = \frac{\Gamma(\alpha)}{\Gamma(\beta)} \cdot \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\beta+n)} \cdot \frac{z^n}{\Gamma(n+1)}. \qquad (1)$$

Observation 1. i. (convergence) The convergence condition of the series is fulfilled for $\alpha, \beta, z \in \mathbb{C}, -\beta \notin N$ and $|z| < \infty$;

ii. (generalization of exp) ${}_1F_1(\alpha, \alpha; z) = e^z.$ (2)

Proposition 1. i. (value) $F_1(\alpha, \beta; 0) = 1;$ (3)

ii. (relation) $\frac{d}{dz}\left({}_1F_1(\alpha, \beta; z)\right) = {}_1F_1(\alpha + 1, \beta + 1; z).$ (3)

Proof. i. (A simple calculation.

ii. Exercise 1 (verification (3)).

Observation 2 (generating differential equation) Two standard solution to Kummer's differential equation

$z\frac{d^2w}{dz^2} + (\beta - z)\frac{dw}{dz} - \alpha w = 0$ (4)

are the following

$M(\alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n n!} z^n, \beta \in N,$ (5)

$M(\alpha, \beta; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{\Gamma(\beta+n) n!} z^n.$ (6)

It is used the notation $M(\alpha, \beta; z) = \left({}_1F_1(\alpha, \beta; z)\right)$. (7)

Exercise 2 (verification) $M(\alpha, \beta;\ z) = \Gamma(\beta) M(\alpha, \beta;\ z)$. (8)

2.2. Mittag-Leffler functions

Definition 1. (Mittag-Leffler (ML) functions) i. (one-parametric) The Mittag-Leffler (ML) one-parametric function is denoted by $E_\alpha(z)$ and has the form
$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$, $Re(\alpha) > 0$, $z \in \mathbb{C}$; (1)

ii. (two-parametric-Agarwal) The Mittag-Leffler (ML) two-parametric function is denoted by $E_{\alpha,\beta}(z)$ and has the form

$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)}$, $Re(\alpha) > 0, \beta \in \mathrm{C}$, $z \in \mathbb{C}$. (2)
Observation 1. i. (generalizations of exp) We have $E_1 = exp, E_{\alpha,1} = E_\alpha$; (3)

ii. ((modern) generalizations vs. Fractional Calculus, see [Gor et al20]) in 1971 Prabhakar introduced the three-parametric Mittag-Leffler function (or generalized Mittag-Leffler function, or Prabhakar function)
$E_{\alpha,\beta}^{\rho}(z) = \sum_{n=0}^{\infty} \frac{(\rho)_n}{\Gamma(n\alpha+\beta)} z^n.$ (3)
This function appeared in the kernel of a first-order integral equation which Prabhakar treated by using Fractional Calculus. Other three-parametric Mittag-Leffler functions (also called generalized Mittag-Leffler functions or Mittag-Leffler type functions, or Kilbas–Saigo functions) were introduced by Kilbas and Saigo. These functions appeared in connection with the solution of new types of integral and differential equations and with the development of the Fractional Calculus. They are referred to as Kilbas–Saigo functions. One more generalization of the Mittag-Leffler function depending on three parameters was studied recently as
$F_{\alpha,\beta}^{(\gamma)}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(n\alpha+\beta)^\gamma}, \alpha, \beta, \gamma, z \in C;$ (4)
it plays an important role in Probability Theory and is related to the so-called Le Roy function
$R_\gamma(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!^\gamma}, z \in C.$ (4')
Generalizing the four-parametricMittag-Leffler function with $\alpha_1, \alpha_2 \in R, \alpha_1^2 + \alpha_2^2 \neq 0, \beta_1, \beta_2 \in C$ introduced by Dzherbashian (=Djrbashian) Al-Bassam and Luchko introduced the Mittag-Leffler type function with 2m parameters $\alpha_i > 0, \beta_i \in R, i \in [m]^* = \{1, \dots, m\}, z \in C$ and gave an explicit solution to a Cauchy type problem for a fractional differential equation.

In the last several decades the study of the Mittag-Leffler function has become a very important branch of Special Function Theory. Many important results have been obtained by applying integral transforms to different types of functions from the Mittag-Leffler collection. Conversely,Mittag-Leffler functions generate new kinds of integral transforms with properties making them applicable to various mathematical models. The recent notable increased interest in the study of their relevant properties is due to the close connection of the Mittag-Leffler function to the Fractional Calculus and its application to the study of Differential and Integral Equations (in particular, of fractional order). Many modern models of fractional type have recently been proposed in Probability Theory, Mechanics, Mathematical Physics, Chemistry, Biology, Mathematical Economics etc.

Exercises (verification)

1. $E_{1,n}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+n)} = \frac{1}{z^{n-1}}\left(e^z - \sum_{p=0}^{n-2} \frac{z^p}{p!}\right)$, $n \geq 2$ (5)

and in particular

$E_{1,2}(z) = \frac{1}{z}(e^z - 1),$ (5')

$E_{1,3}(z) = \frac{1}{z}[e^z - (1+z)].$ (5'')

2. $E_{2,1}(z^2) = cosh(z),$ (6)

$E_{2,2}(z^2) = \frac{1}{z} \cdot sinh(z),$ (6')

$e^{z^2} \cdot erfc(-z) = E_{\frac{1}{2},1}(z).$ (6'')

Observation 2 (order and type of entire function) A complex-valued function $F : C \to C$ is called an entire function (or integral function) if it is analytic (C-differentiable) everywhere on the complex plane. Typical examples of entire functions are the polynomials, the exponential functions and also sums, products and compositions of these functions, thus trigonometric (hyperbolic) functions. Among the special functions we point out the following entire functions: Bessel functions of the first and second, the reciprocal Gamma function $\frac{1}{\Gamma}$, the (ML) one-parametric function and its different generalizations. According to Liouville's theorem an entire function either has a singularity at infinity or it is a constant. Such a singularity can be either a pole (as is the case for a polynomial), or an essential singularity. In the latter case we speak of transcendental entire functions. All of the above-mentioned special functions are transcendental. Every entire function can be represented in the form of a power series $F(z) = \sum_{n=0}^{\infty} c_n z^n$ (7)

converging everywhere on C. Thus, according to the Cauchy.Hadamard formula, the coefficients of the series (7) satisfy the following condition (the necessary and sufficient condition for the sum of a power series to represent an entire function):

$\lim_{n} |c_n|^{\frac{1}{n}} = 0.$ (8)

The global behavior of entire functions of finite order is characterized by their order and type. Recall (see [Gor et al20]) that the order $\rho_{[F]}$ of an entire function F(z) is defined as

$$\rho_{[F]} = inf\ N_{\rho_{[F]}},\ N_{\rho_{[F]}}=\{n|M_F(r)(= max\{F(z)||z| = r\}) < e^{r^n}, r > r(n)\} \quad (9)$$

or equivalently

$$\rho_{[F]} = \limsup_{r\to\infty} \frac{loglogM_F(r)}{logr}. \quad (9')$$

The type $\sigma_{[F]}$ of an entire function F(z) of finite order $\rho_{[F]}$ is defined as

$$\sigma_{[F]} = inf\ A_{\sigma_{[F]}},\ A_{\sigma_{[F]}}=\{A > 0|M_F(r) < e^{Ar^\rho}\} \quad (10)$$

or equivalently

$$\sigma_{[F]} = \limsup_{r\to\infty} \frac{logM_F(r)}{r^\rho}. \quad (10')$$

For an entire function F(z) represented in the form of the series (7)
its order and type can by found by the following formulas

$$\rho_{[F]} = \limsup_{n} \frac{nlogn}{log\frac{1}{|c_n|}}, \quad (9'')$$

$$(\sigma_{[F]}e\rho)^{\frac{1}{\rho}} = \limsup_{n} n^{\frac{1}{\rho}}\,|c_n|^{\frac{1}{n}}. \quad (10'')$$

Proposition 1(entire function-order, type) i. (one-parametric) For each $\alpha \in C, Re(\alpha) > 0$ ($\beta \in C$) the Mittag-Leffler (ML) one (two)-parametric function $E_\alpha(z)$ ($E_{\alpha,\beta}(z)$) is an entire function of order $\rho = \frac{1}{Re(\alpha)}$ and type $\sigma = 1$.

Proof. Applying to the coefficients $c_n = \frac{1}{\Gamma(n\alpha+1)}$ (see (1), (7)) the Cauchy–Hadamard formula for the radius of convergence

$$R = \limsup_{n} \frac{|c_n|}{|c_{n+1}|} \quad (11)$$

and the asymptotic formula

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}(1 + \frac{(a-b)(a-b-1)}{2z}) + 0(\frac{1}{z^2}), z \to \infty, |arg(z)| < \pi \quad (11)$$

one can see that the series (1) converges in the whole complex plane for all $\alpha \in C, Re(\alpha) > 0$ and the MLfunction is an entire function. Moreover, it follows from the Cauchy inequality for the Taylor coefficients and simple properties of the Gamma function that there $N_{\rho_{E_\alpha}} \neq \Phi$, $E_\alpha(z)$ is an entire function of finite order. For $\alpha > 0$ by Stirling's asymptotic formula

$$\Gamma(n\alpha + 1) = \sqrt{2\pi}(n\alpha)^{n\alpha+\frac{1}{2}}e^{-n\alpha}(1 + o(1), n \to \infty \quad (12)$$

one can see that the MLfunction satisfies the relations

$$\limsup_{n} \frac{nlogn}{log\frac{1}{|c_n|}} = \lim_{n} \frac{nlogn}{log|\Gamma(n\alpha+1)|} = \frac{1}{\alpha}, \quad (13)$$

$$\limsup_{n}(n^{\frac{1}{\rho}}|c_n|^{\frac{1}{n}}) = \lim_{n}(n^{\frac{1}{\rho}}\,|\Gamma(n\alpha + 1)|^{-\frac{1}{n}}) = (\frac{e}{\alpha})^\alpha. \quad (14)$$

If $Re(\alpha) > 0,\ Im(\alpha) \neq 0$ the corresponding result is valid too. This follows from formula (11), which in particular means

$0 < A_1 < |\frac{\Gamma(n\alpha+1)}{\Gamma(n\alpha_0+1)}| < A_2 < \infty, A_1, A_2 > 0, n \in N\ (sufficiently\ large).$ (11')

For $E_{\alpha,\beta}(z)$ the reasoning is analogous.

Proposition 2. (k-derivative) For Mittag-Leffler (ML) two-parametric function, its k-th derivative is given by formula

$$E^{(k)}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{\Gamma(k+n+1)}{\Gamma(n+1)\cdot\Gamma(\alpha n+\alpha k+\beta)} \cdot z^n = k!\, E^{k+1}_{\alpha,k\alpha+\beta}(z), k \geq 1 \qquad (15)$$

where $\alpha, \beta, z \in \mathbb{C}$, $E^{(k)}_{\alpha,\beta}(z) = \frac{d^k}{dz^k}\left(E_{\alpha,\beta}(z)\right)$ and $E^{k+1}_{\alpha,k\alpha+\beta}$ is the Prabhakar function (3).

Proof. The proof is immediate by using mathematical induction.

Observation 3.i. (recurrence relation) A recurrence relation for ML two-parametric function is

$E_{\alpha,\beta}(z) = \frac{1}{\Gamma(\beta)} + zE_{\alpha,\alpha+\beta}(z);$ (16)

ii. (differential and recurrence relations) A term-by-term differentiation allows us to verify in an easy way that

$E_{\alpha,\beta}(z) = \beta E_{\alpha,\beta+1}(z) + \alpha z \frac{d}{dz} E_{\alpha,\beta+1}(z),$ (17)

$\frac{d}{dz} E_{\alpha,\beta}(z) = \frac{E_{\alpha,\beta-1}(z)+(1-\beta)E_{\alpha,\beta}(z)}{\alpha z}, z \neq 0.$ (18)

Proof. Is left for the reader as Exercise.

3. FRACTIONAL DERIVATIVES AND INTEGRALS

3.1. The Riemann-Liouville (RL) Fractional Calculus

Observation 1. (motivation) Fractional integrals are usually defined (see [Pod99], [Gor et al20]) as a generalization of the repeated integral formula

$(J^n_{a+}f)(t) = \int_a^t dt_1 \int_a^{t_1} dt_2 \ldots \int_a^{t_n} f(t_n)dt_n = \frac{1}{(n-1)!}\int_a^t f(\tau)(t-\tau)^{n-1}d\tau, n \in N^*, a, t \in R, t > a$ (1)

namely

$(J^\alpha_{a+}f)(t) = \frac{1}{\Gamma(\alpha)} \cdot \int_a^t f(\tau)(t-\tau)^{\alpha-1}d\tau,\ \alpha, a, t \in R, \alpha > 0,\ t > a.$ (1')

Definition 1. (Riemann-Liouville (RL) fractional integrals) Let be a function $f \in L_1(a,b)$ existing almost everywhere and $\alpha > 0$. The Riemann-Liouville (RL) left-sided fractional integral is given by the formula

$(J^\alpha_{a+}f)(t) = \frac{1}{\Gamma(\alpha)} \cdot \int_a^t f(\tau)(t-\tau)^{\alpha-1}d\tau$ (2)

and the (RL) right-sided fractional integral is given by the formula

$(J^\alpha_{b-}f)(t) = \frac{1}{\Gamma(\alpha)} \cdot \int_t^b f(\tau)(\tau-t)^{\alpha-1}d\tau.$ (3)

Observation 2. i. (improper case) For a function $f \in L_1(R)$ the (RL) fractional integrals are given respectively by the formulas

$(J^\alpha_{+}f)(t) = \frac{1}{\Gamma(\alpha)} \cdot \int_{-\infty}^t f(\tau)(t-\tau)^{\alpha-1}d\tau,$ (2')

$(J^\alpha_{-}f)(t) = \frac{1}{\Gamma(\alpha)} \cdot \int_t^{\infty} f(\tau)(\tau-t)^{\alpha-1}d\tau.$ (3')

ii. (commutative monoid structure) The commutative monoid structure relative to the Riemann-Liouville (RL) left-sided fractional integrals consists in the structure of commutative semigroup with identity $J_{a+}^{0} = I$ (by convention) according to the formula

$$J_{a+}^{\alpha} J_{a+}^{\beta} = J_{a+}^{\beta} J_{a+}^{\alpha} = J_{a+}^{\alpha+\beta}; \quad (4)$$

The result is also true relative to the (RL) right-sided fractional integrals.

iii. (linearity and continuity in 0) Relative to the Riemann-Liouville (RL) left-sided fractional integrals the properties of linearity and continuity in 0 are satisfied, i. e.

$$(J_{a+}^{\alpha}(\lambda f + \beta g))(t) = (\lambda J^{\alpha} f + \beta J^{\alpha} g)(t), \quad (5)$$
$$\lim_{\alpha \to 0} (J^{\alpha} f)(t) = f(t) \quad (6)$$

where f is continuas. The result is also true relative to the (RL) right-sided fractional integrals.

iv. (integration by parts) The integration by parts formula relative to the (RL) leftt-sided fractional integral and the (RL) right-sided fractional integral holds true, i.e.

$$\int_{a}^{b} g(t)(J_{a+}^{\alpha} f)(t)dt = \int_{a}^{b} f(t)(J_{b-}^{\alpha} g)(t)dt \quad (6)$$

(e.g. for $f \in L_p(a,b), q \in L_q(a,b), p,q \geq 1, \frac{1}{p} + \frac{1}{q} = \propto +1$).

Exercises 1. (verification of calculation of the (RL) fractional integrals) i. (for special functions)

$$(J_{a+}^{\propto}(\tau - a)^{\beta-1})(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(t-a)^{\alpha+\beta-1}, \quad (7)$$

$$(J_{b-}^{\propto}(b - \tau)^{\beta-1})(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}(b-t)^{\alpha+\beta-1}; \quad (7')$$

ii. (for (ML) one-parametric function)

$$\lambda(J_{a+}^{\alpha} E_{\alpha}(\lambda \tau^{\alpha}))(t) = E_{\alpha}(\lambda t^{\alpha}) - 1, \lambda \in C; \quad (8)$$

iii. (see application 2.1.1)

$$(J_{0+}^{\frac{1}{2}})(t) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \cdot \int_{0}^{t} \tau(t-\tau)^{-\frac{1}{2}} d\tau = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{0}^{t} \frac{\tau}{\sqrt{t-\tau}} d\tau, \quad \text{(ai1)}$$

$$(J_{0+}^{\frac{1}{2}})(t^2) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \cdot \int_{0}^{t} \tau^2(t-\tau)^{-\frac{1}{2}} d\tau = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{0}^{t} \frac{\tau^2}{\sqrt{t-\tau}} d\tau, \quad \text{(ai2)}$$

$$(J_{0+}^{\frac{1}{2}})(\sqrt{t}) = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \cdot \int_{0}^{t} \sqrt{\tau}(t-\tau)^{-\frac{1}{2}} d\tau = \frac{1}{\Gamma\left(\frac{1}{2}\right)} \int_{0}^{t} \sqrt{\frac{\tau}{t-\tau}} d\tau. \quad \text{(ai3)}$$

Definition 2. (Riemann-Liouville (RL) fractional derivatives) Let be $n-1 < \alpha \leq n, D^n = \frac{d^n}{dt^n}, \ t \in (a,b)$. The Riemann-Liouville (RL) left-sided fractional derivative is given by the formula

$$({}^{RL}D_{a+}^{\alpha} f)(t) = (D^n J_{a+}^{n-\alpha} f)(t) \quad (9)$$

and the (RL) right-sided fractional derivative is given by the formula

$({}^{RL}D^{\alpha}_{b-}f)(t) = (-1)^n(D^n J^{n-\alpha}_{a+}f)(t).$ (10)

Observation 3. i.(existence) The existence of the fractional derivatives is ensured by the condition

$(J^{n-\alpha}_{a+}f)(t) \in \mathcal{AC}^n_{[a,b]}$ (11)

where

$\mathcal{AC}^n_{[a,b]} = \{g \in C^{n-1}_{[a,b]} | g^{(n-1)} \in \mathcal{AC}_{[a,b]}\}$ (12)

($\mathcal{AC}$ denotes a class of absolutely continuous functions). Note that the condition (11) follows from the condition

$f \in \mathcal{AC}^n_{[a,b]}.$ (11')

In this case the Riemann–Liouville fractional derivative exists almost everywhere on the interval (a, b).

ii. (improper case) By replacing the interval (a, b) with $\mathbb{R}$ the (RL) fractional derivatives are given (as Liouville (L) fractional derivatives) respectively by the formulas

$({}^{L}D^{\alpha}_{+}f)(t) = (D^n J^{n-\alpha}_{+}f)(t),$ (9')

$({}^{L}D^{\alpha}_{-}f)(t) = (-1)^n(D^n J^{n-\alpha}_{-}f)(t).$ (10')

iii. (conditioned monoid) Conditioned by $\alpha \in \mathbb{R}, \beta < 1$ and f an analytic function the semigroup relation

$({}^{RL}D^{\alpha}_{a+}\, {}^{RL}D^{\beta}_{a+}f)(t) = ({}^{RL}D^{\alpha+\beta}_{a+}f)(t)$ (13)

holds true (note that this relation is not valid in general). In addition it is taken by convention

${}^{RL}D^{0}_{a+}f = I.$ (13')

iv. (external inverse) The (RL) fractional derivative is the left-inverse to the corresponding (RL) fractional integral, that is, the following (left-sided and analogous right-sided) relation

$({}^{RL}D^{\alpha}_{a+} J^{\alpha}_{a+}f)(t) = f(t)$ (14)

holds for any function $f \in L_1(a,b)$. The opposite is not true, i.e. the (RL) fractional derivative is not the right-inverse to the corresponding (RL) fractional integral (see Exercise 2.ii).

v. (The Leibniz rule for the Riemann–Liouville fractional derivative can be written in different forms, e.g.

$({}^{RL}D^{\alpha}_{a+}f.g)(t) = \sum_{k=0}^{\infty}\binom{\propto}{k}({}^{RL}D^{\alpha-k}_{a+}f)(t)(D^k g)(t), \propto \in \mathbb{R},$ (15)

$({}^{RL}D^{\alpha}_{a+}f.g)(t) = \sum_{k=-\infty}^{\infty}\binom{\propto}{k+\beta}({}^{RL}D^{\alpha-\beta-k}_{a+}f)(t)({}^{RL}D^{\beta+k}_{a+}g)(t), \propto, \beta \in \mathbb{R}\ (\alpha \in \mathbb{R}\setminus\mathbb{Z}\ if\ \beta \in \mathbb{Z}).$ (15')

Exercises 2. (verification) i. (see Observation 3.i)

$({}^{RL}D^{\alpha}_{a+}f)(t) = \sum_{k=0}^{n-1}(\frac{(D^k f)(a)}{\Gamma(k+1-\alpha)}(t-a)^{k-\alpha} + (J^{n-\alpha}_{a+}D^k f)(t);$ (16)

$$ii. (J_{a+}^{\alpha}\, {}^{RL}D_{a+}^{\alpha} f)(t) = f(t) - \sum_{k=0}^{n-1} \frac{(D^{n-k-1} J_{a+}^{n-\alpha} f)(t)}{\Gamma(\alpha-k)} (t-a)^{\alpha-k-1}, \quad (17)$$

where $f \in L_1(a,b)$ such that (11) works and there exists a function $g \in L_1(a,b)$ such that $f(t) = (J_{a+}^{\alpha} g)(t)$.

iii. (and calculation, see application 2.1.1 and exercise 1.iii)

$({}^{RL}D_{0+}^{\frac{1}{2}})(t) = (D\, J_{0+}^{\frac{1}{2}})(t),$ (ad1)

$({}^{RL}D_{0+}^{\frac{1}{2}})(t^2) = (D\, J_{0+}^{\frac{1}{2}})(t^2),$ (ad2)

$({}^{RL}D_{0+}^{\frac{1}{2}})(\sqrt{t}) = (D\, J_{0+}^{\frac{1}{2}})(\sqrt{t}).$ (ad3)

3.2. The Caputo (C) fractional derivatives

Definition 1. (Caputo (C) fractional derivatives) Let be $n-1 < \alpha \le n, D^n = \frac{d^n}{dt^n},\ t \in (a,b)$. The Caputo (C) left-sided fractional derivative is given by the formula

$({}^{C}D_{a+}^{\alpha} f)(t) = (J_{a+}^{n-\alpha} D^n f)(t)$ (1)

and the (RL) right-sided fractional derivative is given by the formula

$({}^{C}D_{b-}^{\alpha} f)(t) = (-1)^n (J_{b-}^{n-\alpha} D^n f)(t).$ (2)

Observation 1. i.(existence as regularization of the (RL) fractional derivatives) The existence of the (C) fractional derivatives (known also as the Caputo–Dzherbashian or Caputo–Gerasimov fractional derivatives) is ensured by the condition 3.1.11' (see observation 3.1.3.i). They are regularizations of the (RL) fractional derivatives according to the following formulas

$({}^{C}D_{a+}^{\alpha} f)(t) = ({}^{RL}D_{a+}^{\alpha}(f(\tau) - \sum_{k=0}^{n-1}(\frac{(D^k f)(a)}{k!}(\tau-a)^k)(t),$ (1')

$({}^{C}D_{b-}^{\alpha} f)(t) = ({}^{RL}D_{b-}^{\alpha}(f(\tau) - \sum_{k=0}^{n-1}(\frac{(D^k f)(b)}{k!}(b-\tau)^k)(t),$ (2')

and with an interchanging of the order of integration and differentiation for $\alpha \notin \mathbb{N}^*$.

In addition is fulfilled the following condition

$\lim_{\alpha \to n} {}_{a}^{C}D_{t}^{\alpha} f(t) = (D^n f)(t)$ (3)

which follows by applying integration by parts and taking the limit for $\alpha \to n$.

ii. (improper case) By replacing the interval (a, b) with $\mathbb{R}$ the (C) fractional derivatives are given respectively by the formulas

$({}^{C}D_{+}^{\alpha}f)(t) = (J_{+}^{n-\alpha}D^{n}f)(t),$ (1”)

$({}^{C}D_{-}^{\alpha}f)(t) = (-1)^{n}(J_{-}^{n-\alpha}D^{n}f)(t).$ (2”)

iii. (external inverse) The (C) fractional derivative is the left-inverse to the corresponding (RL) fractional integral, that is, the following (left-sided and analogous right-sided) relation

$({}^{C}D_{a+}^{\alpha}J_{a+}^{\alpha}f)(t) = f(t)$ (4)

holds for any function f which satisfies the condition 3.1.11’. The opposite is not true, i.e. the (C) fractional derivative is not the right-inverse to the corresponding (RL) fractional integral (see exercise 1.vi).

Exercises 1. (verification of calculation of the (C) fractional derivatives) i. (for special functions, see exercises 3.1.1.i) For all $\beta > n-1$,

$({}^{C}D_{a+}^{\propto}(\tau-a)^{\beta})(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha},$ (5)

$({}^{C}D_{b-}^{\propto}(b-\tau)^{\beta})(t) = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(b-t)^{\beta-\alpha};$ (6)

ii. (on the integer power monomials) For all $k \in [n-1] = \{0, 1, \dots, n-1\}$

$({}^{C}D_{a+}^{\propto}(\tau-a)^{k})(t) = 0,$ (5’)

$({}^{C}D_{b-}^{\propto}(b-\tau)^{k})(t) = 0$ (6’)

and in particular

$({}^{C}D_{a+}^{\propto}1)(t) = 0,$ (5”)

$({}^{C}D_{b-}^{\propto}1)(t) = 0$ (6”);

iii. (improper case for exponential function)

$({}^{C}D_{+}^{\alpha}e^{\lambda t})(t) = \lambda^{\alpha}e^{\lambda t},$ (7)

$({}^{C}D_{-}^{\alpha}e^{-\lambda t})(t) = \lambda^{\alpha}e^{-\lambda t}.$ (8)

iv. ((ML) one-parametric function)

$({}^{C}D_{a+}^{\propto}E_{\alpha}(\lambda(\tau-a)^{\alpha}))(t) = \lambda E_{\alpha}(\lambda(t-a)^{\alpha})).$ (9)

v. (improper case for (ML) two-parametric function)

$({}^{C}D_{-}^{\alpha}\tau^{\alpha-1}E_{\alpha}(\lambda\tau^{-\alpha}))(t) = \frac{1}{t}E_{\alpha,1-\alpha}(\lambda\tau^{-\alpha}).$ (10)

$vi.\ (J_{a+}^{\alpha}\,{}^{C}D_{a+}^{\alpha}f)(t) = f(t) - \sum_{k=0}^{n-1}\frac{(D^{k}f)(a)}{k!}(t-a)^{k}$ (11)

where $f \in \mathcal{AC}_{[a,b]}^{n}$; (3.1.11’)

vii. (see application 2.1.1 and exercises 3.1.1.iii, 3.1.2.iii)

$({}^{C}D_{0+}^{\frac{1}{2}})(t) = (J_{0+}^{\frac{1}{2}}D\ )(t),$ (ad’1)

$({}^{C}D_{0+}^{\frac{1}{2}})(t^{2}) = (J_{0+}^{\frac{1}{2}}D\ )(t^{2}),$ (ad’2)

$$({}^{C}D_{0+}^{\frac{1}{2}})(\sqrt{t}) = (J_{0+}^{\frac{1}{2}}D\ )(\sqrt{t}). \quad \text{(ad'3)}$$

Appendix A. Podlubny I., talk „Matrix-based approaches as an emerging framework for numerical solution of initial and boundary value problems for ordinary and partial differential equation of arbitrary real order”, International Symposium on Fractional PDEs- Theory, Numerics and Applications, june 3-5, 2013 (Newport, RI, USA) – selection:

A1(8). Main idea of FC: Interpolation of operators;
A2(9). From integer to non-integer;
A3(10). FC:a response to S&T needs;
A4(11). FC:136 subject areas (applications);
A5(12). The current map of the FC;
A6(15). RL-derivative;
A7(16). C-derivative;
A8(17). GL-derivative;
A9(18). Conditional equivalence;
A10(20). Left-sided „flavor”;
A11(21). Right-sided „flavor”;
A12(22). Symmetric „flavor”;
A13(24). Constant (non-integer) order (CO) „grade”;
A14(25). Variable order (VO) „grade”;
A15(26). Distributed order (DO) „grade”;
A16(27). Combinations grades-definitions-flavor;
A17(28). Intelligent fitting of date with the help of solutions of differential equations;
A18(29). The Mittag-Leffler (ML) function;
A19(33). ML-function as Queen function of FC;
A20(30). Fitting data using ML-function;
A21(34). ML-function – a complete replacement for the exponential function;
A22(49). Exact solution with C-derivative;
A23(50). Graph relative to A22;
A24(69). Exact solution for two-term ordinary FDE;
A25(67). Fractional integrals of sin;
A26(68). Fractional derivatives of sin;
A27(70). Bagley-Torvik equation;
A28(71). Variable-order fractional differentiation and integration (VO-FD, VO-FI);
A29(72). C-VO-FD;
A30(74). VO-FD of y(t)=t;
A31(75). VO-fractional relaxation equation (1);
A32(76). VO-fractional relaxation equation (3);
A33(77). DO-fractional derivatives;
A34(78). Interpretation of DO operators;
A35(84). CO-fractional relaxation equation;
A36(85). VO-fractional relaxation equation;
A37(86). DO-fractional relaxation equation;
A38(87). CO-order Bagley-Torvic equation;
A39(88). DO-order Bagley-Torvic equation:
A40(101). The Matrix Approach.

Appendix B. Table of (higher order) (C) derivatives of particular power functions (see [Ish05])

	$D_*^{\alpha} f(t)$	$D_*^{1/3} f(t)$	$D_*^{1/2} f(t)$	$D_*^{1/2} D_*^{1/2} f(t)$	$D_*^{1/2} D_*^{1/2} D_*^{1/2} f(t)$	$D_*^{1/2} D_*^{1/2} D_*^{1/2} D_*^{1/2} f(t)$
$f(t) = const$	0	0	0	0	0	0
$f(t) = t$	$\frac{1}{\Gamma(2-\alpha)} t^{1-\alpha}$	$1.1077\, t^{2/3}$	$1.1284\, t^{1/2}$	1	0	0
$f(t) = t^2$	$\frac{2}{\Gamma(3-\alpha)} t^{2-\alpha}$	$1.3293\, t^{5/3}$	$1.5045\, t^{3/2}$	$2\, t$	$2.2568\, t^{1/2}$	2
$f(t) = t^3$	$\frac{6}{\Gamma(4-\alpha)} t^{3-\alpha}$	$1.4954\, t^{8/3}$	$1.8054\, t^{5/2}$	$3\, t^2$	$4.5135\, t^{3/2}$	$6\, t$
$f(t) = t^4$	$\frac{24}{\Gamma(5-\alpha)} t^{4-\alpha}$	$1.6314\, t^{11/3}$	$2.0633\, t^{7/2}$	$4\, t^3$	$7.2216\, t^{5/2}$	$12\, t^2$
$f(t) = t^5$	$\frac{120}{\Gamma(6-\alpha)} t^{5-\alpha}$	$1.7479\, t^{14/3}$	$2.2926\, t^{9/2}$	$5\, t^4$	$10.3166\, t^{7/2}$	$20\, t^3$
$f(t) = t^{1/2}$	$\frac{\sqrt{\pi}}{2\,\Gamma(3/2-\alpha)} t^{1/2-\alpha}$	$0.9553\, t^{1/6}$	0.8862	0	0	0
$f(t) = t^{3/2}$	$\frac{3\sqrt{\pi}}{4\,\Gamma(5/2-\alpha)} t^{3/2-\alpha}$	$1.2282\, t^{7/6}$	$1.3292\, t$	$1.5\, t^{1/2}$	1.3293	0
$f(t) = e^t$	$t^{n-\alpha} E_{1,n-\alpha+1}(t)$	$t^{2/3} E_{1,5/3}(t)$	$t^{1/2} E_{1,3/2}(t)$	e^t	$t^{1/2} E_{1,3/2}(t)$	e^t

Appendix C. Table of (C) derivatives of the most used elementary functions (see [Ish05])

Function	$f(t)$	Caputo derivative $D_*^{\alpha} f(t)$
Constant function	$f(t) = c = const$	$D_*^{\alpha}\, c = 0$
Power function	$f(t) = t^p$	$D_*^{\alpha}\, t^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} t^{p-\alpha} = D^{\alpha}\, t^p\,, & n-1 < \alpha < n,\ p > n-1,\ p \in \mathbb{R}\,, \\ 0\,, & n-1 < \alpha < n,\ p \le n-1,\ p \in \mathbb{N} \end{cases}$
Exponential function	$f(t) = e^{\lambda t}$	$D_*^{\alpha}\, e^{\lambda t} = \sum_{k=0}^{\infty} \frac{\lambda^{k+n}\, t^{k+n-\alpha}}{\Gamma(k+1+n-\alpha)} = \lambda^n t^{n-\alpha}\, E_{1,n-\alpha+1}(\lambda t)$
Sine function	$f(t) = \sin \lambda t$	$D_*^{\alpha} \sin \lambda t = -\frac{1}{2} i\, (i\lambda)^n\, t^{n-\alpha}\ \left(E_{1,n-\alpha+1}(i\lambda t) - (-1)^n E_{1,n-\alpha+1}(-i\lambda t)\right)$
Cosine function	$f(t) = \cos \lambda t$	$D_*^{\alpha} \cos \lambda t = \frac{1}{2} (i\lambda)^n\, t^{n-\alpha}\ \left(E_{1,n-\alpha+1}(i\lambda t) + (-1)^n E_{1,n-\alpha+1}(-i\lambda t)\right)$

Appendix D. Table of comparison between (RL) and (C) derivatives (see [Ish05])

Property	Riemann-Liouville	Caputo
Representation	$D^{\alpha} f(t) = D^{n} J^{n-\alpha} f(t)$	$D_{*}^{\alpha} f(t) = J^{n-\alpha} D^{n} f(t)$
Interpolation	$\lim\limits_{\alpha \to n} D^{\alpha} f(t) = f^{(n)}(t)$ $\lim\limits_{\alpha \to n-1} D^{\alpha} f(t) = f^{(n-1)}(t)$	$\lim\limits_{\alpha \to n} D_{*}^{\alpha} f(t) = f^{(n)}(t)$ $\lim\limits_{\alpha \to n-1} D_{*}^{\alpha} f(t) = f^{(n-1)}(t) - f^{(n-1)}(0)$
Linearity	$D^{\alpha}(\lambda f(t) + g(t)) = \lambda D^{\alpha} f(t) + D^{\alpha} g(t)$	$D_{*}^{\alpha}(\lambda f(t) + g(t)) = \lambda D_{*}^{\alpha} f(t) + D_{*}^{\alpha} g(t)$
Non-commutation	$D^{m} D^{\alpha} f(t) = D^{\alpha+m} f(t) \neq D^{\alpha} D^{m} f(t)$	$D_{*}^{\alpha} D^{m} f(t) = D_{*}^{\alpha+m} f(t) \neq D^{m} D_{*}^{\alpha} f(t)$
Laplace transform	$L\{D^{\alpha} f(t); s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{k} \left[D^{\alpha-k-1} f(t) \right]_{t=0}$	$L\{D_{*}^{\alpha} f(t); s\} = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0)$
Leibniz rule	$D^{\alpha}(f(t)\, g(t)) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} \left(D^{\alpha-k} f(t) \right) g^{(k)}(t)$	$D_{*}^{\alpha}(f(t)\, g(t)) = \sum_{k=0}^{\infty} \begin{pmatrix} \alpha \\ k \end{pmatrix} \left(D^{\alpha-k} f(t) \right) g^{(k)}(t) - \sum_{k=0}^{n-1} \frac{t^{k-\alpha}}{\Gamma(k+1-\alpha)} \Big((f(t) g(t))^{(k)}(0) \Big)$
$f(t) = c = const$	$D^{\alpha} c = \frac{c}{\Gamma(1-\alpha)} t^{-\alpha} \neq 0, \quad c = const$	$D_{*}^{\alpha} c = 0, \quad c = const$

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