University POLITEHNICA of Bucharest

# Applied Mathematics in Optimization Problems

Course

## Chapter
## Linear Programming (II)

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# 6. Programming with piecewise linear objective function

In economic practice there are also problems whose mathematical models have linear constraints, but in which the objective function has different expressions on certain subsets of the set of admissible solutions, but each expression is linear.

The general form of a such model is :

$$\begin{cases} [\text{opt}]\, z = u_1(x_1) + u_2(x_2) + \cdots + u_n(x_n) \\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \le (\ge)(=) b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \le (\ge)(=) b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \le (\ge)(=) b_m \\ x_j \le h_{jr}, \quad j = 1, 2, \ldots, n \\ x_j \ge 0, \quad j = 1, 2, \ldots, n \end{cases}$$

where:

$$u_j(x_j)=\begin{cases} c_{j1}x_j, & x_j\in[0,h_{j1}] \\ c_{j1}h_{j1}+c_{j2}(x_j-h_{j1}), & x_j\in(h_{j1},h_{j2}] \\ c_{j1}h_{j1}+c_{j2}(h_{j2}-h_{j1})+c_{j3}(x_j-h_{j2}), & x_j\in(h_{j2},h_{j3}] \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots & \\ c_{j1}h_{j1}+c_{j2}(h_{j2}-h_{j1})+\cdots+c_{j,r-1}(h_{j,r-1}-h_{j,r-2})+ & \\ \qquad +c_{j,r}(x_j-h_{j,r-1}), & x_j\in(h_{j,r-1},h_{j,r}] \end{cases}$$

with the additional condition :

$$c_{j1}<c_{j2}<c_{j3}<\ldots<c_{j,r} \qquad (*)$$

for the case opt = min, respectively,

$$c_{j1}>c_{j2}>c_{j3}>\ldots>c_{j,r} \qquad (**)$$

for the case opt = max.

*Remarks*:

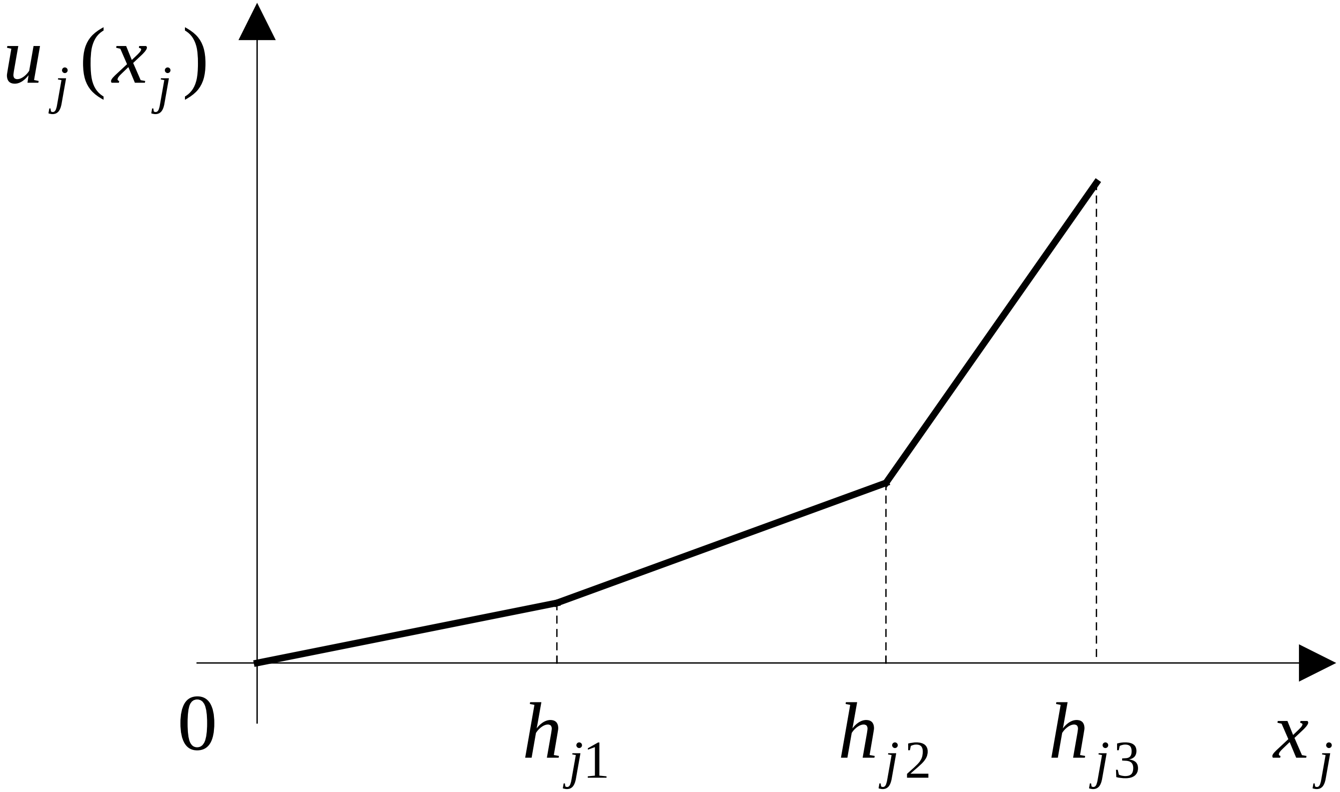
1) The index $r$ depends on the corresponding variable $x_j$, so the number of intervals used is different for the terms $u_j(x_j)$.

2) If for a certain variable $x_j$ we have $r = 1$, then the expression of the term $u_j(x_j)$ is reduced to a linear function.

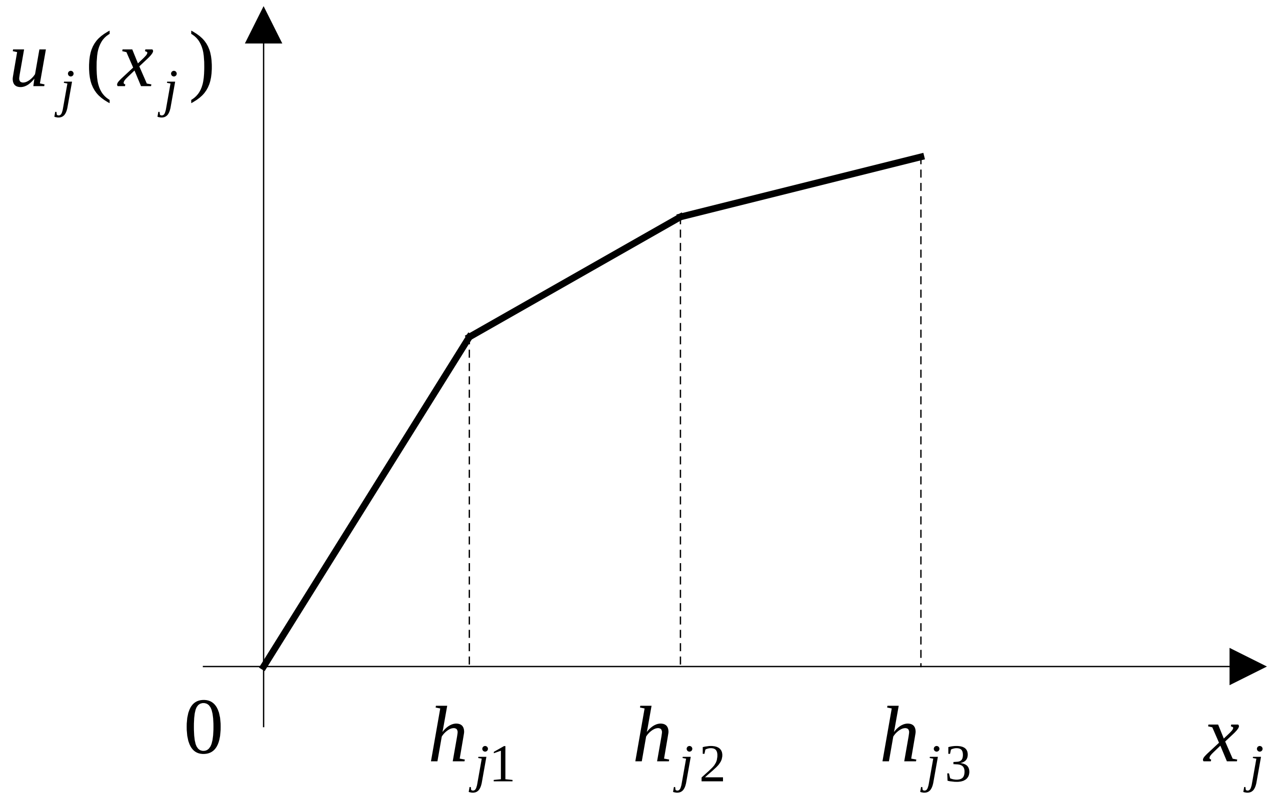
3) The last interval in the expression of the function $u_j(x_j)$ can be infinite ($h_{j,r} = +\infty$).

4) The condition (*) ensures convexity of the function $u_j(x_j)$, and the condition (**) ensures concavity of the function $u_j(x_j)$.

The below figure shows the graph of the function $u_j(x_j)$, for $j = 3$, if the minimization of the objective function is required.



The below figure shows the graph of the function $u_j(x_j)$, for $j = 3$, if the maximization of the objective function is required.



The mathematical model of these problems can be reduced to a linear model by substituting the variables $x_j$ ($j = 1,\ldots, n$) namely:

$$x_j = x_{j1} + x_{j2} + \ldots x_{j,\,r\text{-}1} + x_{j,\,r}\,,\quad j = 1,\ldots, n$$

where the new variables $x_{j1}$, $x_{j2}$ ,..., $x_{j,\,r}$ satisfy the following conditions:

$$\begin{array}{l}
0 \le x_{j1} \le h_{j1} \\
0 \le x_{j2} \le h_{j2} - h_{j1} \\
0 \le x_{j3} \le h_{j3} - h_{j2} \\
\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
0 \le x_{j,r} \le h_{j,r} - h_{j,r-1}
\end{array}$$

As a result, the general term $u_j(x_j)$ in the expression of the objective function becomes

$$u_j(x_j) = c_{j1}x_{j1} + c_{j2}x_{j2} + \ldots + c_{j,\,r}\,x_{j,\,r}\,, \quad j = 1,\ldots, n$$

The substitution is also performed in each of the $m$ constraints of the initial model :

$$a_{k1}x_1 + a_{k2}x_2 + \ldots + a_{k,\,n}\,x_n \leq (\geq)(=)\, b_k\,, \quad k = 1,\ldots, m$$

The constraints $0 \leq x_j \leq h_{j,\,r}$ ($j = 1,\ldots, n$), from the initial model, are obviously satisfied with the conditions imposed on the new variables.

After solving the obtained linear model, we return to the initial variables .

**Example.** In the manufacture of three types of products $A$, $B$, $C$, three machine tools $M_1$, $M_2$, $M_3$ participate so that each type of product passes, in its processing, to all three machines. The required unit processing times of the products as well as the available times of each machine (in minutes) are indicated in the following table:

| | $A$ | $B$ | $C$ | Available time |
|---|---|---|---|---|
| $M_1$ | 2 | 3 | 4 | 320 |
| $M_2$ | 3 | 2 | 2 | 300 |
| $M_3$ | 1 | 2 | 4 | 200 |

It is possible to sell, immediately after manufacture, maximum 50 u. of each type of product, with a unit profit of 4 m.u., 6 m.u., 7 m.u., for *A*, *B*, and *C*, respectively.

The maximum manufacturing capacity for each type of product is 60 u. The quantity manufactured over 50 u. can bring a unit profit of 3 m.u., 4.5 m.u., respectively, 5 m.u., diminished due to the additional maintenance and storage expenses of the products.

Determine the manufacturing plan of the three types of products that will bring a maximum total profit.

*Solution*. Let note $x_1$, $x_2$, $x_3$ the quantities to be manufactured from products *A*, *B* and *C*, respectively.

The mathematical model of the problem has the form:

$$\begin{cases} [\max] z = u_1(x_1) + u_2(x_2) + u_3(x_3) \\ 2x_1 + 3x_2 + 4x_3 \le 320 \\ 3x_1 + 2x_2 + 2x_3 \le 300 \\ x_1 + 2x_2 + 4x_3 \le 200 \\ x_j \le 60, \quad j = 1, 2, 3 \\ x_j \ge 0, \quad j = 1, 2, 3 \end{cases}$$

where:

$$u_1(x_1) = \begin{cases} 4x_1, & x_1 \in [0, 50] \\ 200 + 3(x_1 - 50), & x_1 \in (50, 60] \end{cases}$$

$$u_2(x_2) = \begin{cases} 6x_2, & x_2 \in [0, 50] \\ 300 + 4{,}5(x_2 - 50), & x_2 \in (50, 60] \end{cases}$$

$$u_3(x_3) = \begin{cases} 7x_3, & x_3 \in [0, 50] \\ 350 + 5(x_3 - 50), & x_3 \in (50, 60] \end{cases}$$

It is found that the objective function $z$ can be written in the form of eight different expressions, obtained by combining in all possible ways the expressions attached to the terms $u_1(x_1)$, $u_2(x_2)$, $u_3(x_3)$, depending on the values of the variables $x_1$, $x_2$ şi $x_3$; all expressions are, however, linear in the three variables.

The following substitutions are made:

$x_1 = x_{11} + x_{12}$ with $x_{11} \in [0, 50]$, $x_{12} \in [0, 10]$

$x_2 = x_{21} + x_{22}$ with $x_{21} \in [0, 50]$, $x_{22} \in [0, 10]$

$x_3 = x_{31} + x_{32}$ with $x_{31} \in [0, 50]$, $x_{32} \in [0, 10]$

The associated linear model has the form:

$$\begin{cases} [\max]\, z = 4x_{11} + 3x_{12} + 6x_{21} + 4{,}5x_{22} + 7x_{31} + 5x_{32} \\ 2(x_{11} + x_{12}) + 3(x_{21} + x_{22}) + 4(x_{31} + x_{32}) \le 320 \\ 3(x_{11} + x_{12}) + 2(x_{21} + x_{22}) + 2(x_{31} + x_{32}) \le 300 \\ \quad (x_{11} + x_{12}) + 2(x_{21} + x_{22}) + 4(x_{31} + x_{32}) \le 200 \\ x_{j1} \le 50, \quad j = 1, 2, 3 \\ x_{j2} \le 10, \quad j = 1, 2, 3 \\ x_{j1}, x_{j2} \ge 0, \quad j = 1, 2, 3 \end{cases}$$

The previous model is solved and the obtained optimal solution is:

$$x_{11} = 50;\ x_{12} = 6;\ x_{21} = 50;\ x_{22} = 10;\ x_{31} = 6;\ x_{32} = 0$$

The corresponding value of the objective function is $z_{\max} = 605$.

Therefore, the optimal manufacturing program consists of 56 u. product *A*, 60 u. product *B* and 6 u. product *C*, with a maximum total profit of 605 m.u. ❑