



University POLITEHNICA of Bucharest

Applied Mathematics in Optimization Problems

Course

Chapter Linear Programming (I)



- 1. Linear Programming Problems**
- 2. Linear Programming Elements**
- 3. SIMPLEX Method**
- 4. Parametric Optimization**
- 5. Integer Programming**



1. Linear Programming Problems

1.1. Examples of problems

1.2. Forms of the problems



Deployment conditions of a certain analyzed economic activity lead to a system of relations - equations or inequalities - that contain the variables of the problem and the technological coefficients that characterize it. These relationships are the *constraints* of the problem.

The aim of the study is to optimize a certain result depending on the variables of the problem. In the formulation of mathematical programming problems, the objective appears in the form of a function, called the *objective function*, whose maximum or minimum values are searched.



The constraints of the problem and the objective function, with certain conditions imposed on the variables, constitute the *mathematical model* of the problem.

If both the objective function and the constraints are linear functions, then the model is a *linear programming problem*.



1.1. Examples of problems

Example. *Problem of production organization*

A company must realize n types of products P_j ($j = 1, \dots, n$), using m types of resources R_i ($i = 1, \dots, m$). We know: technological coefficients a_{ij} (the quantity of the resource R_i necessary to produce a unit of the product P_j and which does not depend on the intensity of the production process); available quantities b_i of resources R_i ; unit profit c_j for each product P_j .



Let us determine the optimal production program (plan) of the company so that the total realized profit is maximum .

Elaboration of the mathematical model :

Let x_j ($j = 1, \dots, n$) be the variables, i.e. the quantity of the product P_j to be produced.

The objective function is the total profit, which must be maximized :

$$[\max] f = c_1 x_1 + c_2 x_2 + \dots + c_n x_n$$

The constraints of the problem are due to limited resources. Consumption of each resource R_i ($i =$



$= 1, \dots, m$) cannot exceed the available quantity :

$$a_{i1}x_1 + a_{i2}x_2 + \dots + a_{in}x_n \leq b_i, \quad \forall i = \overline{1, m}$$

The conditions on the variables are to be non-negative : $x_j \geq 0$ ($j = 1, \dots, n$).

The mathematical model of the problem can be formulated as follows:

$$\left\{ \begin{array}{l} [\max] f = c_1x_1 + c_2x_2 + \dots + c_nx_n \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \leq b_1 \\ \dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \leq b_m \\ x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \end{array} \right.$$





Example. Problem of using the capacity of the machines

A company makes n types of products that can be manufactured on m machines, which have limited production capacities for a certain period.

We know: the percentage a_{ij} of machine capacity i ($i = 1, \dots, m$) required to manufacture a unit of the product j ($j = 1, \dots, n$); unit profit c_j of the product j .

Let us determine a manufacturing program that allows the optimal use of the available capacity of the m machines .



Elaboration of the mathematical model :

The variables are the quantities of products manufactured in the considered period . Let note x_j ($j = 1, \dots, n$) the number of products of the type j .

The objective function is the total profit obtained by the company for the n types of products; the function must be maximized :

$$[\max] f = \sum_{j=1}^n c_j x_j$$

The capacity restrictions of the m machines :

$$\sum_{j=1}^n a_{ij} x_j \leq 1, \quad i = \overline{1, m}$$



The mathematical model of the problem is as follows :

$$\left\{ \begin{array}{l} [\max] f = \sum_{j=1}^n c_j x_j \\ \sum_{j=1}^n a_{ij} x_j \leq 1, \quad i = \overline{1, m} \\ x_j \geq 0, \quad j = \overline{1, n} \end{array} \right.$$





The linear function z is the *objective function* of the problem; it must be minimized or maximized.

The other relationships constitute the system of *constraints* of the problem. Only one of the symbols " \leq ", " \geq ", " $=$ " appears in each relationship. The left hand side of each relationship is a linear function of the variables x_1, x_2, \dots, x_n .

Definition. A *linear programming (LP) problem* consists in optimization (minimizing or maximizing) a linear function, respecting some linear constraints (equalities and / or inequalities).



Definition. A LP problem has the *standard form* if all constraints are equations and all variables are subject to the non-negativity condition.

The LP problem in standard form explicitly written:

$$\left\{ \begin{array}{l} [\min] ([\max]) z = c_1x_1 + c_2x_2 + \dots + c_nx_n \\ a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \\ x_1 \geq 0, x_2 \geq 0, \dots, x_n \geq 0 \end{array} \right.$$



The LP problem in standard form restricted written:

$$\left\{ \begin{array}{l} [\min] ([\max]) z = \sum_{j=1}^n c_j x_j \\ \sum_{j=1}^n a_{ij} x_j = b_i, \quad 1 \leq i \leq m \\ x_j \geq 0, \quad 1 \leq j \leq n \end{array} \right.$$

The LP problem in matrix form written:

$$\left\{ \begin{array}{l} \min (\max) c^T x \\ Ax = b \\ x \geq 0 \end{array} \right.$$



Observation. From an algebraic point of view, the system of constraints (linear equations) $Ax = b$ can be: incompatible, unique compatible determined or indeterminate compatible. Obviously, only the last case is of practical interest, because only then is the problem of choosing the best solution.

Definition. A constraint of a LP problem is called *concordant* if it is an inequality of type " \geq " when the objective function is minimized and is of type " \leq " when the function is maximized.



Definition. A LP problem has a *mixed form* if all constraints are either equations or concordant inequalities and all variables are subject to the non-negativity condition:

$$\left\{ \begin{array}{l} \min (\max) c^T x \\ A_1 x \geq b_1 \\ A_2 x = b_2 \\ x \geq 0 \end{array} \right.$$



Any problem of LP in general form can be brought to the standard, canonical or mixed form with the following *equivalent transformations*:

a) the meaning of an inequality changes by multiplying by -1;

b) the transformation of an inequality into an equality is achieved by entering a variable, called the compensation (deviation) variable, $y \geq 0$, so that

$$a^T x \leq \alpha \Leftrightarrow \begin{cases} a^T x + y = \alpha \\ y \geq 0 \end{cases} \quad a^T x \geq \beta \Leftrightarrow \begin{cases} a^T x - y = \beta \\ y \geq 0 \end{cases}$$



c) an equality can be replaced by two inequalities of the same sign :

$$a^T x = \alpha \Leftrightarrow \begin{cases} a^T x \geq \alpha \\ -a^T x \geq -\alpha \end{cases} \quad \text{or} \quad \begin{cases} a^T x \leq \alpha \\ -a^T x \leq -\alpha \end{cases}$$

d) a non-positive variable can be replaced with a non-negative variable by substitution :

$$x \leq 0 \Leftrightarrow \begin{cases} y = -x \\ y \geq 0 \end{cases}$$

e) an arbitrary variable can be replaced by two non-negative variables by substitution :



$$x \text{ arbitrary} \Leftrightarrow \begin{cases} x = y - z \\ y, z \geq 0 \end{cases}$$

f) the transformation of the maximum operator into minimum (or vice versa) is achieved by changing the signs of the coefficients in the objective function :

$$\min \sum_{i=1}^n c_i x_i = - \max \left(- \sum_{i=1}^n c_i x_i \right)$$

Remarks :

1) The compensation variables do not appear in the expression of the objective function of the problem (their coefficients in the objective function are 0).



2) In LP problems with concrete economic significance, the compensation variables introduced receive, in their turn, precise economic interpretations; this motivates the inclusion of these variables in the set of initial variables.

Solving a LP problem consists in determining the values of the variables that satisfy the constraints of the problem and that optimize the objective function.



Example. Write the equivalent forms of the LP problem :

$$\left\{ \begin{array}{l} \min (3x_1 - 5x_2 + 2x_3) \\ x_1 + 4x_2 + 3x_3 = 15 \\ 2x_1 - x_2 \geq 1 \\ \quad 3x_2 + x_3 \leq 7 \\ x_1, x_2, x_3 \geq 0 \end{array} \right.$$

Solution.

The standard form of the problem is obtained by introducing the non-negative compensation variables x_4 and x_5 , in the last two constraints:



$$\begin{cases} \min (3x_1 - 5x_2 + 2x_3) \\ x_1 + 4x_2 + 3x_3 = 15 \\ 2x_1 - x_2 - x_4 = 1 \\ 3x_2 + x_3 + x_5 = 7 \\ x_1, \dots, x_5 \geq 0 \end{cases}$$

The canonical form of minimizing the problem is obtained by transforming the first restriction of the original problem into two inequalities of the type " \geq ":



$$\left\{ \begin{array}{l} \min (3x_1 - 5x_2 + 2x_3) \\ x_1 + 4x_2 + 3x_3 \geq 15 \\ -x_1 - 4x_2 - 3x_3 \geq -15 \\ 2x_1 - x_2 \geq 1 \\ -3x_2 - x_3 \geq -7 \\ x_1, x_2, x_3 \geq 0 \end{array} \right.$$

The canonical form of maximizing the problem is as follows :

$$\left\{ \begin{array}{l} \max (-3x_1 + 5x_2 - 2x_3) \\ x_1 + 4x_2 + 3x_3 \leq 15 \\ -x_1 - 4x_2 - 3x_3 \leq -15 \\ -2x_1 + x_2 \leq -1 \\ 3x_2 + x_3 \leq 7 \\ x_1, x_2, x_3 \geq 0 \end{array} \right.$$





2. Linear Programming Elements

2.1. Solutions of the LP problems

2.2. Geometric interpretation of a LP problem

2.3. Graphical solving method of the LP problems



2.1. Solutions of the LP problems

Let the LP problem in the standard form :

$$\begin{cases} \min [\max] z = c^T x \\ Ax = b \\ x \geq 0 \end{cases} \quad (*)$$

with $x, c \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, $A \in \mathcal{M}_{m \times n}$, $\text{rank } A = m$.

Definition. The vector $x \in \mathbf{R}^n$ whose components satisfy the given restrictions and conditions in (*) is called *possible (admissible) solution* of the LP problem.



Let's note the set of possible solutions of the problem (*) by :

$$P = \{x \in \mathbf{R}^n \mid Ax = b, x \geq 0\}$$

Note: Obviously, we are practically interested in the situation in which the set $P \neq \emptyset$.

Definition. The possible solution x^1 is *better* than the possible solution x^2 , if $z(x^1) \leq z(x^2)$ when the minimum is required for the objective function z , or if $z(x^1) \geq z(x^2)$ when the maximum is required for the objective function z .



Definition. The vector $x' \in \mathbf{R}^n$ is called *basic solution* for the system of equations $Ax = b$ if:

- a) x' is solution of the system $Ax = b$;
- b) the columns of the matrix A corresponding to non-zero components of x' form a linear independent set.

Definition. A basic solution x^* is *optimal solution* of a LP problem if $z(x^*)$ has the lowest possible value in case of a minimization problem, or the highest possible value in case of a maximization problem.



The set of possible solutions P is convex and closed. Its boundary is composed of all points whose coordinates satisfy equally, at least one of the constraints.

Definition. The vector $\bar{x} \in P$ is called *vertex (point of extremum)* of the set P if \bar{x} cannot be written as a non-trivial convex linear combination of vectors in P , meaning $\nexists x^1, x^2 \in P, x^1 \neq x^2, \nexists \lambda \in (0,1)$ such that
$$\bar{x} = \lambda x^1 + (1 - \lambda)x^2.$$

Theorem. A possible solution is basic solution if and only if it is vertex for the set P .



Definition. A group of m linearly independent columns in matrix A is called *basis* of a LP problem in standard form.

Definition. The base B that satisfies the condition $B^{-1}b \geq 0$ is called *primal admisible basis*.



2.2. Geometric interpretation of a LP problem

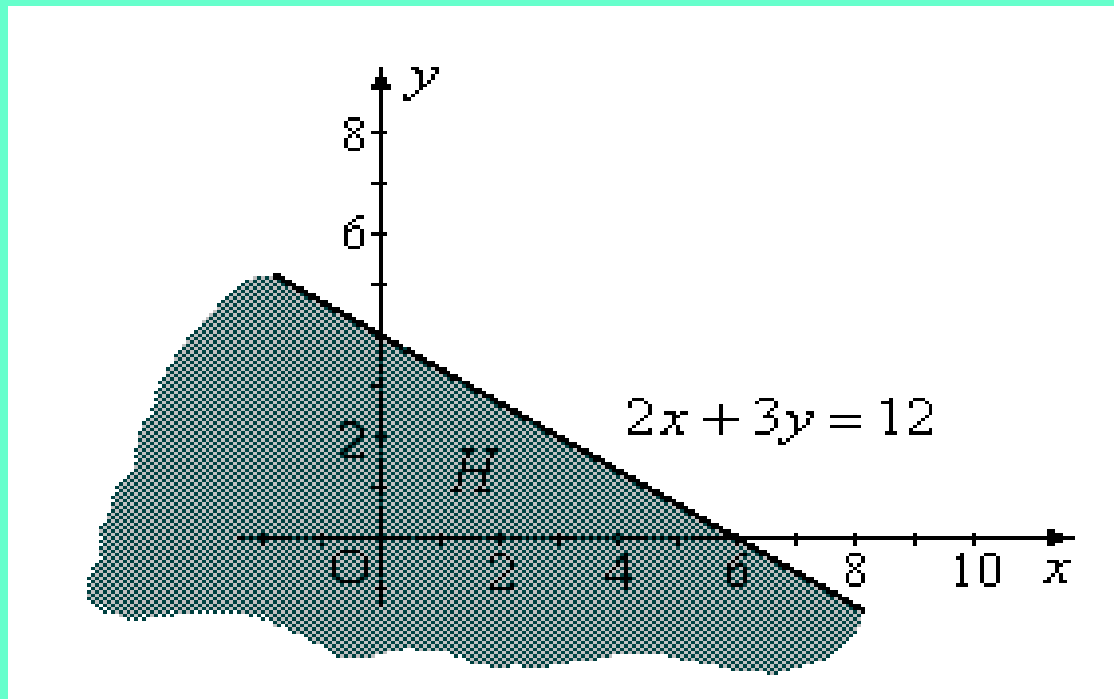
2.2.1. Geometry of the constraints

Relation $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$ represents a *hyperplane* in n -dimensional space, which divides the space into two semi-spaces analytically expressed by two inequalities.

In \mathbf{R}^2 space, the hyperplane is a *straight line* which divides the plan into two half-planes (or regions).

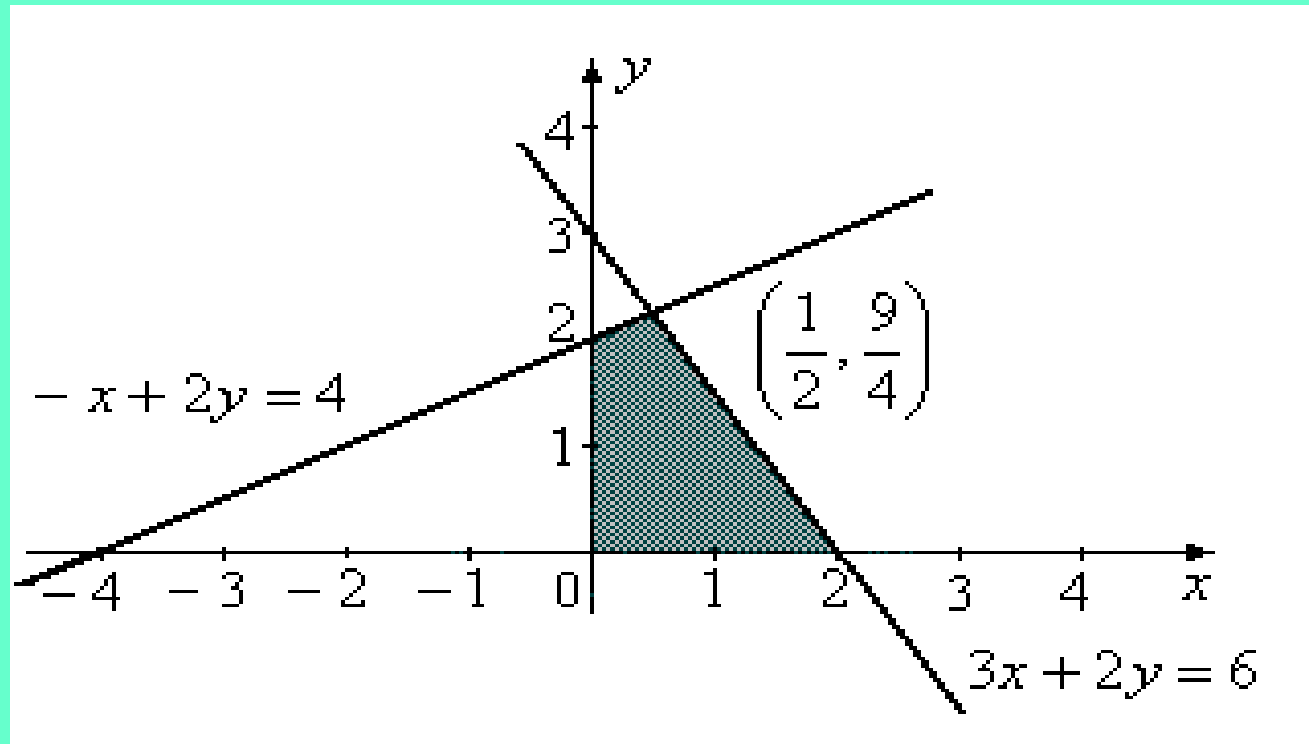
In \mathbf{R}^3 space, the hyperplane is a *plane* which divides the space into two half-spaces.

Example. Let the constraint $2x + 3y \leq 12$ and the closed half-space formed by the points that satisfy the constraint. Point $O (0, 0)$ does not belong to the line $2x + 3y = 12$, and its coordinates check for inequality. As a result, the half-space contains the origin O :

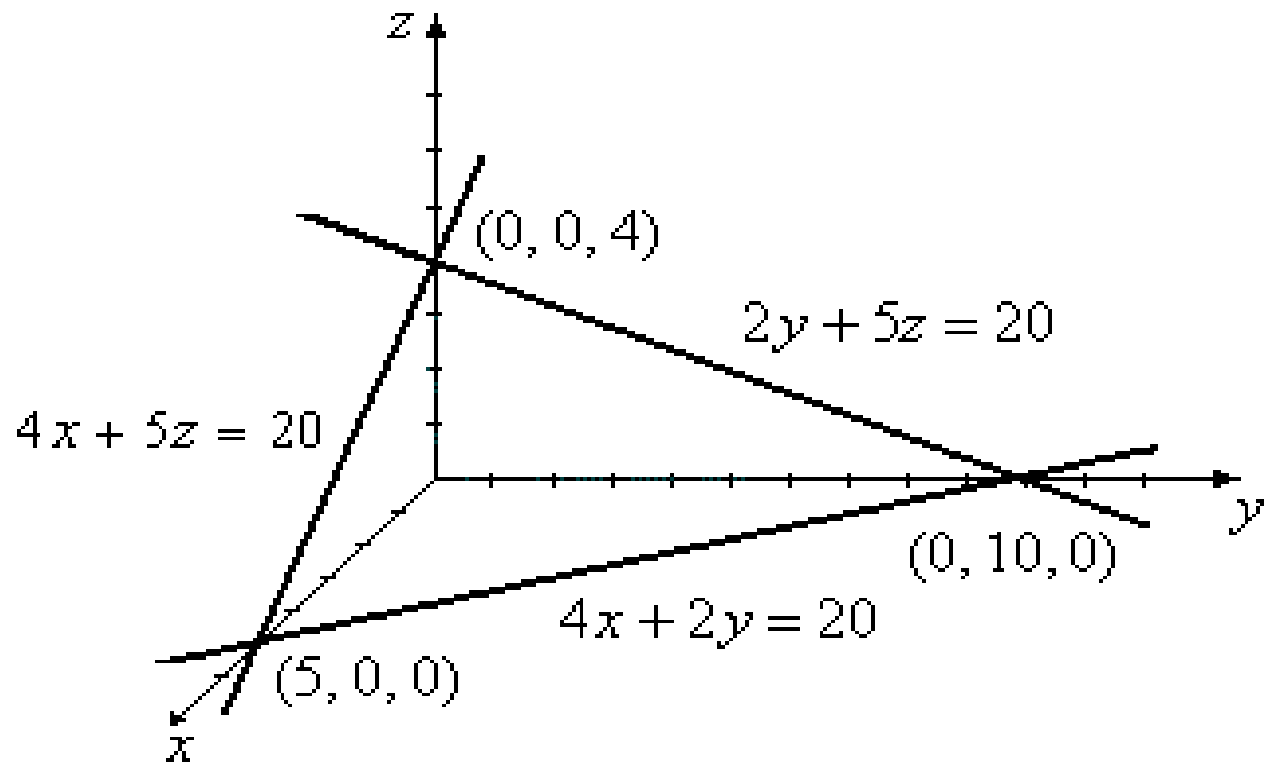


Example. The set of admissible solutions P that satisfy the constraint system:

$$\begin{cases} 3x + 2y \leq 6 \\ -x + 2y \leq 4 \\ x \geq 0 \\ y \geq 0 \end{cases} \quad \text{is the hatched domain in the figure:}$$

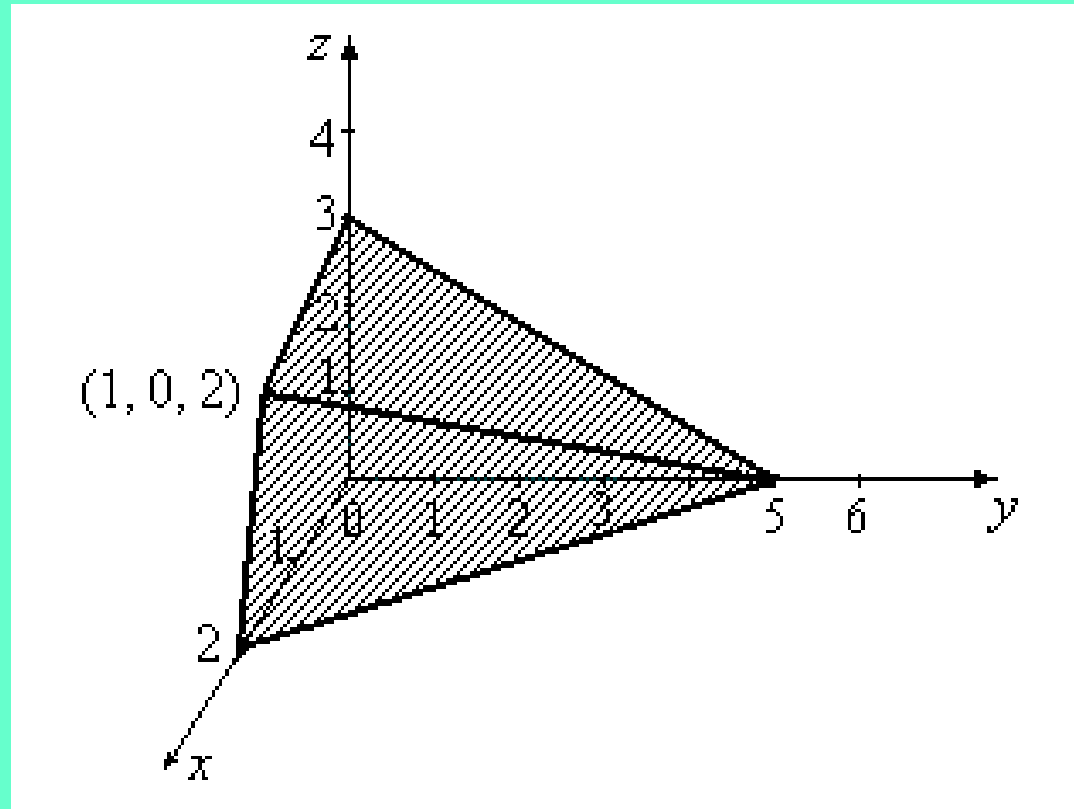


Example. The constraint $4x + 2y + 5z \leq 20$ which defines a closed half - space in \mathbf{R}^3 . It can be represented in three-dimensional space by drawing the plane $4x + 2y + 5z = 20$. Origin point O checks the constraint.



Example. The set of admissible solutions P that satisfy the system of restrictions :

$$\left\{ \begin{array}{l} 5x + 3y + 5z \leq 15 \\ 10x + 4y + 5z \leq 20 \\ x \geq 0 \\ y \geq 0 \\ z \geq 0 \end{array} \right. \text{ is the hatched domain in the figure:}$$





2.2.2. Geometry of the objective function

The objective function of a LP problem is:

$$z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

The relation below, in which k is a constant, is the equation of a hyperplane :

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n = k$$

The solution of a maximization LP problem is represented by the point (points) x in the set of admissible solutions P for which the value of k is the largest possible. From a geometric point of view, the solution is given by the hyperplane intersecting the set P , for which k is maximum.



Example.

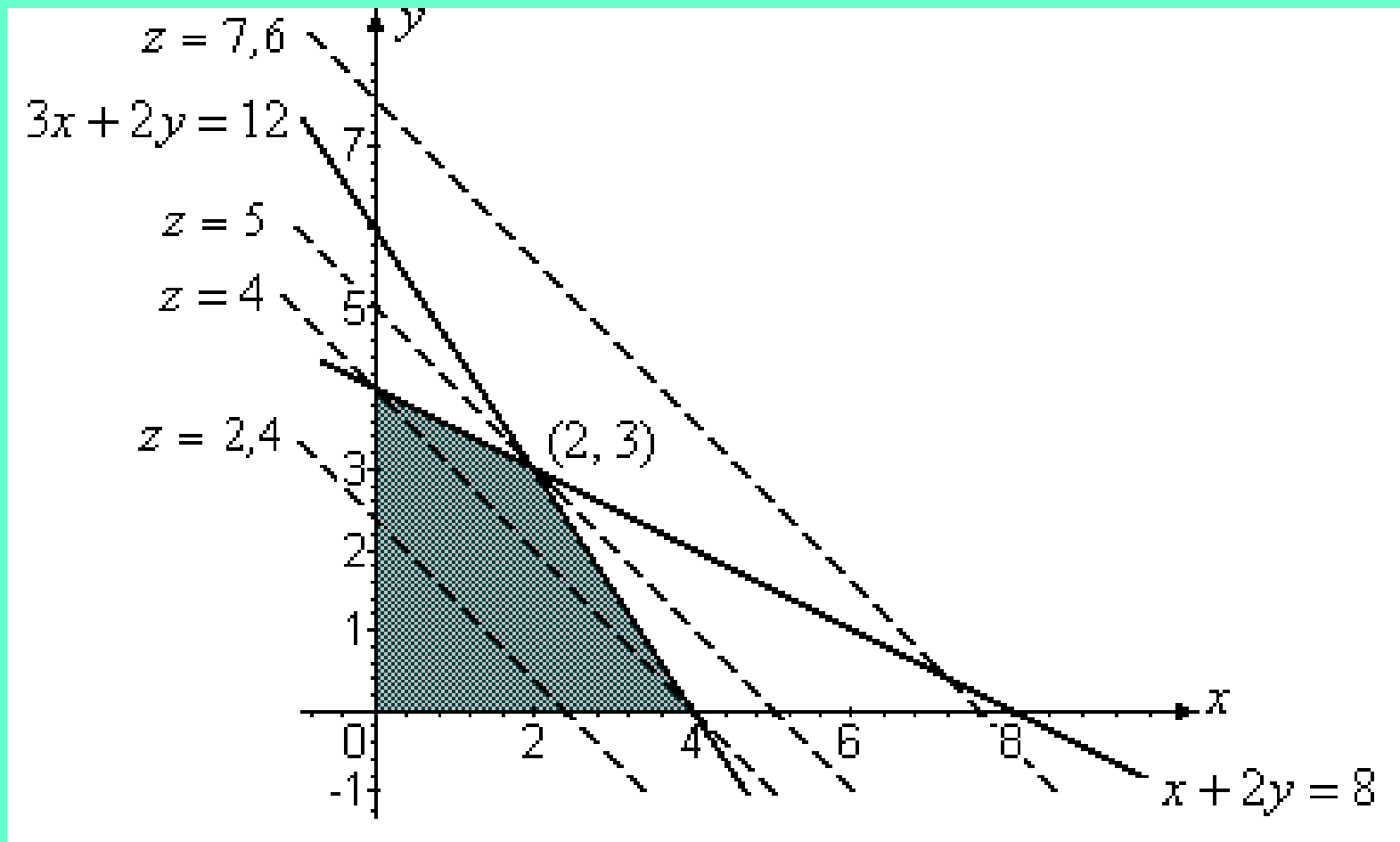
$$\begin{cases} [\max] z = x + y \\ x + 2y \leq 8 \\ 3x + 2y \leq 12 \\ x \geq 0, y \geq 0 \end{cases}$$

The equation of the hyperplane $x + y = k$ represents a line called *level line* of the objective function too.

From analytical geometry it is known that the distance d from the origin to the level line $x + y = k$ is $d = k / \sqrt{2}$. So, the maximum value of k is obtained when d has the maximum value.

Since the optimal solution satisfies both the constraints system and the level curve equation, it is ne-

cessary that the level line has a common point with the polygon. This point has the coordinates $(2, 3)$.





2.3. Graphical solving method of the LP problems

The graphical method can be used to solve only two-variable LP problems .

The set P is represented graphically. In \mathbf{R}^2 space, any constraint of the problem represents a line that divides the plane into two half-planes (regions). Draw the line and with the help of an arbitrary point chosen, but which does not belong to the line, it is decided which of the two half-spaces is the one that suits. A possible solution of the problem is a vertex for the set P .



The optimal basic solution (s) of the problem is chosen after comparing the values of the objective function calculated for the basic solutions.

Remarks:

1) The optimal solution of a LP problem is one of the vertices of the polygon representing the set of possible solutions P .

2) If the level lines are straight lines parallel to one of the sides of the polygon of the admissible solutions, the optimal solutions of the problem are in infinite number; geometrically, they are the points on that side of the polygon .



Example. Solving the LP problem:

$$\left\{ \begin{array}{l} [\max] z = 5x_1 + 4x_2 \\ 3x_1 + 2x_2 \leq 21 \\ x_1 + x_2 \leq 8 \\ x_1 \leq 6 \\ x_2 \leq 7 \\ x_1, x_2 \geq 0 \end{array} \right.$$

Solution. The non-negativity conditions of the variables involve the use of the first dial of the axis system x_1Ox_2 for the graphical representation of the straight lines (obtained from the four restrictions):



$$(d_1): 3x_1 + 2x_2 = 21$$

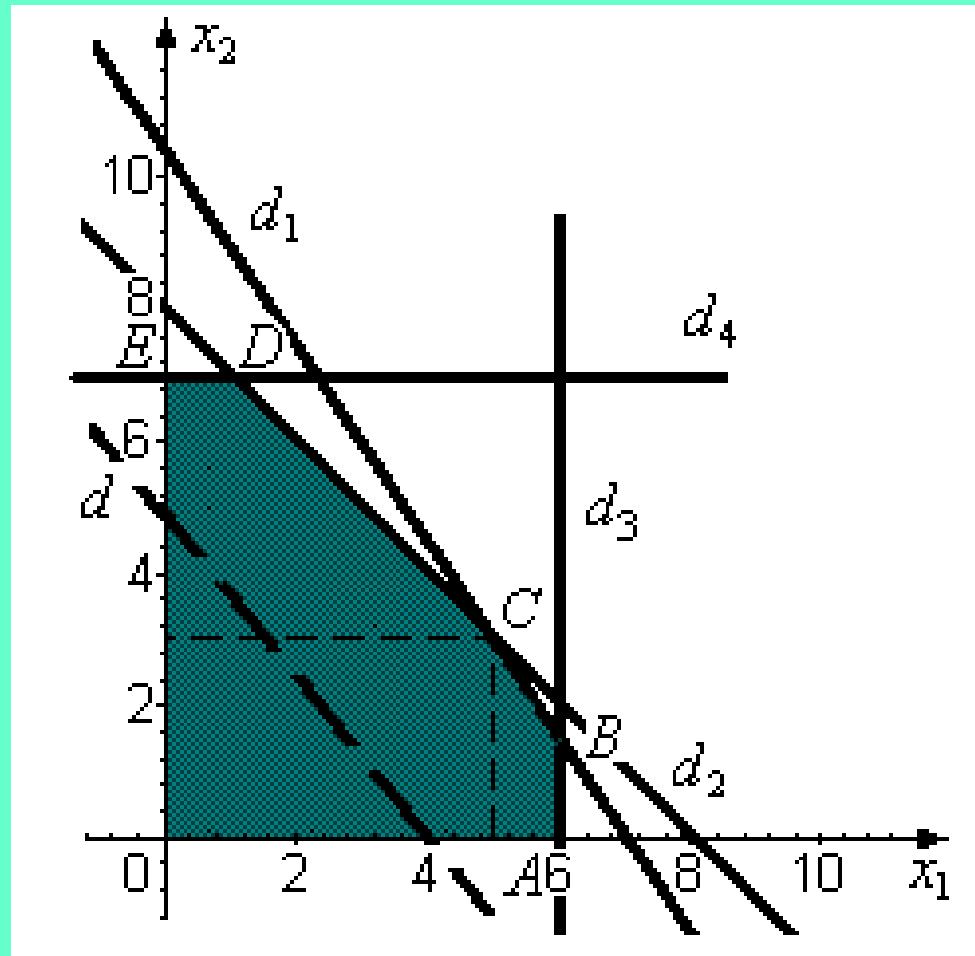
$$(d_2): x_1 + x_2 = 8$$

$$(d_3): x_1 = 6$$

$$(d_4): x_2 = 7$$

and of the level line $(d): 5x_1 + 4x_2 = 20$.

The intersection of the four half-planes is represented by the hatched set in the figure and is the set P of the admissible (possible) solutions. At least one of the vertices of this convex polygon, $O(0, 0)$, $A(6, 0)$, $B(6, 3/2)$, $C(5, 3)$, $D(1, 7)$, $E(0, 7)$ is optimal solution of the problem. In order to determine it, the coordinates of these are replaced, one by one, in the





expression of the objective function z :

$$z(O) = 5 \cdot 0 + 4 \cdot 0 = 0$$

$$z(A) = 5 \cdot 6 + 4 \cdot 0 = 30$$

$$z(B) = 5 \cdot 6 + 4 \cdot 3/2 = 36$$

$$z(C) = 5 \cdot 5 + 4 \cdot 3 = 37$$

$$z(D) = 5 \cdot 1 + 4 \cdot 7 = 33$$

$$z(E) = 5 \cdot 0 + 4 \cdot 7 = 28$$

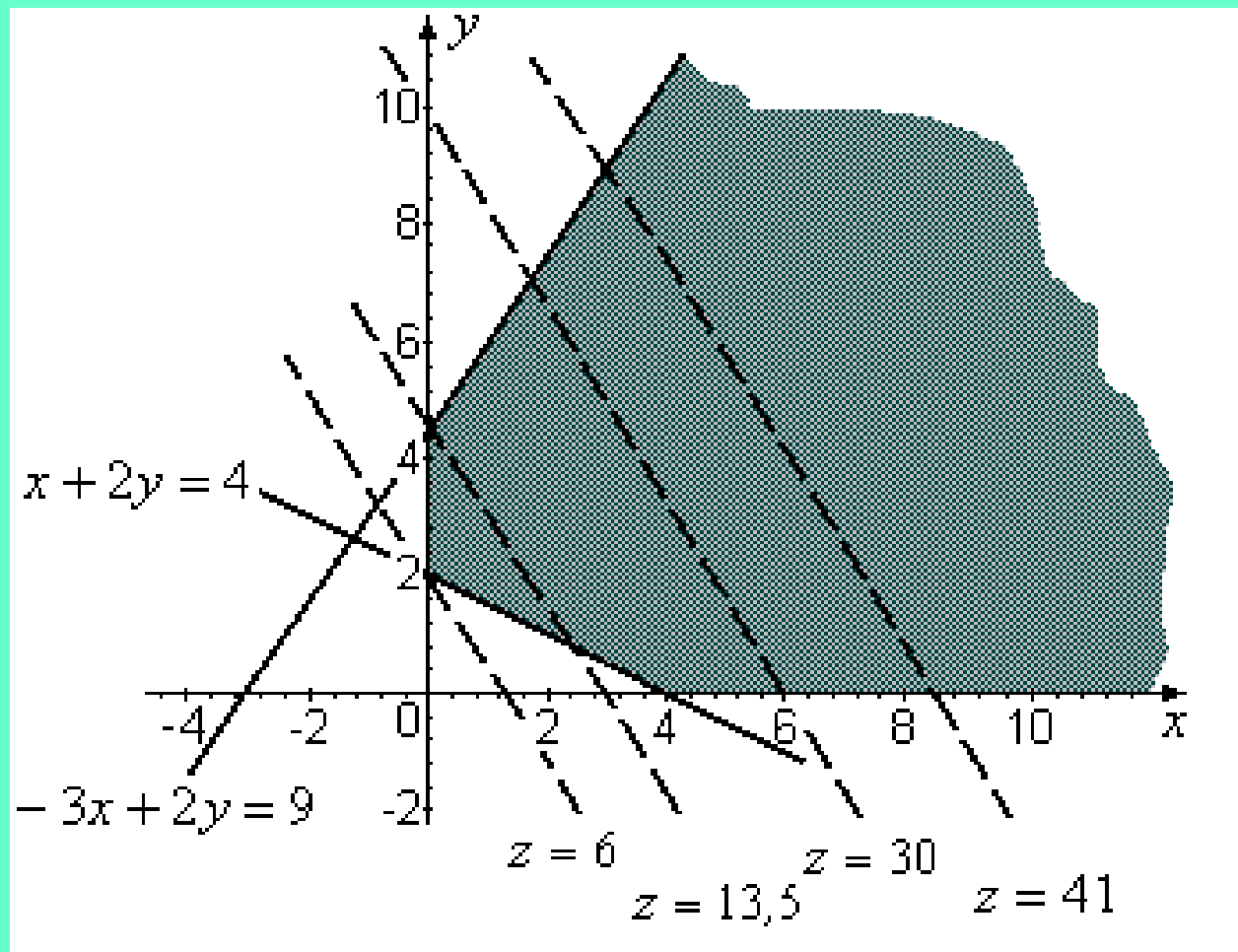
The maximum value is obtained for the point $C(5, 3)$. Therefore, the optimal solution of the problem is $\bar{x}_1^* = 5, \bar{x}_2^* = 3$. The value of the objective function $z^* = z_{\max} = 37$. \square



Example. Let the LP problems:

$$\left\{ \begin{array}{l} [\max] z = 5x + 3y \\ x + 2y \geq 4 \\ -3x + 2y \leq 9 \\ x \geq 0, y \geq 0 \end{array} \right. \quad \left\{ \begin{array}{l} [\min] z = 5x + 3y \\ x + 2y \geq 4 \\ -3x + 2y \leq 9 \\ x \geq 0, y \geq 0 \end{array} \right.$$

For the maximization problem, the level lines have in common with the section of admissible solutions a segment, no matter how large is the distance from the origin to the respective lines. The objective function can take values no matter how high, it has infinite optimal value.





For the minimization problem, the minimum value for the objective function is obtained when the distance from the origin to the contours is minimal; so the point $(0, 2)$ is the optimal solution, and the corresponding value of the objective function is $z_{\min} = 6$. \square



3. SIMPLEX Method

3.1. Theoretical foundations

3.2. The Simplex algorithm

3.3. The Big M method



3.1. Theoretical foundations

The most used method for solving LP problems is the simplex method, due to the American mathematician G. B. Dantzig (1947).

The simplex method allows the systematic investigation of the set of basic admissible solutions of a LP problem model in the standard form. Assuming known an initial solution, "better" basic achievable solutions are successively built until the optimal solution is obtained.

The method also provides a criterion for recognizing the infinite optimum.



The presentation of the simplex method will refer to the LP model in the minimization standard form :

$$\begin{cases} \min c^T x \\ Ax = b \\ x \geq 0 \end{cases}$$

where $x, c \in \mathbf{R}^n$, $b \in \mathbf{R}^m$, $A \in \mathcal{M}_{m \times n}$, $\text{rank } A = m < n$.

Let B a *primal admissible basis*, extracted from matrix A , and $x^B = B^{-1}b$ the associated basic solution.

The numerical coefficients of the problem corresponding to the base B are entered in the simplex table associated with the base B , of the form :



			C_1	...	C_k	...	C_j	...	C_n
C_B	V.B.	V.V.B.	x_1	...	x_k	...	x_j	...	x_n
C_1	x_1^B	$\frac{-B}{x_1}$	y_{11}^B	...	y_{1k}^B	...	y_{1j}^B	...	y_{1n}^B
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
C_i	x_i^B	$\frac{-B}{x_i}$	y_{i1}^B	...	y_{ik}^B	...	y_{ij}^B	...	y_{in}^B
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
C_l	x_l^B	$\frac{-B}{x_l}$	y_{l1}^B	...	y_{lk}^B	...	y_{lj}^B	...	y_{ln}^B
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots
C_m	x_m^B	$\frac{-B}{x_m}$	y_{m1}^B	...	y_{mk}^B	...	y_{mj}^B	...	y_{mn}^B
		$\frac{-B}{Z}$	$z_1^B - C_1$...	$z_k^B - C_k$...	$z_j^B - C_j$...	$z_n^B - C_n$



The simplex table contains:

- in the first column, *the basic variables* (V.B.);
- in the second column, *the values of the basic variables* (V.V.B.);
- in the following n columns, *the vectors* $y_j^B, j = 1, \dots, n$;
- on the last line, *the value of the objective function in the base* B , noted with \bar{z}^B ; *the differences* $z_j^B - c_j$.

When applying the algorithm of the simplex method, a simplex table of this type corresponds to each base B .



3.2. Simplex algorithm

Simplex algorithm for the *minimization problem*:

Step 1. It determines an initial admissible primal base B .

Step 2. It calculates $\bar{x}^B, \bar{z}^B, \underline{y}_j^B, \underline{z}_j^B - c_j, j = 1, \dots, n$.

Step 3. If $\underline{z}_j^B - c_j \leq 0, \forall j = \overline{1, n}$, then STOP: it is an optimal solution; else it determines the set

$$J_+ = \left\{ j \in J \mid \underline{z}_j^B - c_j > 0 \right\}$$

and go to step 4.

Step 4. It determines the index $k \in J_+$ with the *criterion of entry into the base*:



$$\max_{j \in J_+} (z_j^B - c_j) = z_k^B - c_k$$

If $y_k^B \leq 0$, then STOP: then the solution is unbounded; else it determines the index $l \in I$ with the *exit criterion from the base*:

$$\min_{\substack{i \in I \\ y_{ik}^B > 0}} \left(\frac{-x_i^B}{y_{ik}^B} \right) = \frac{-x_l^B}{y_{lk}^B}$$

Step 5. Let the base B' obtained from B by replacing the column $A^{(l)}$ with the column $A^{(k)}$. Go to step 2, replacing everywhere the base B with the base B' .



Simplex algorithm for the ***maximization problem***:

Step 3. If $z_j^B - c_j \geq 0, \forall j = \overline{1, n}$, then STOP: it is an optimal solution; else it determines the set

$$J_- = \left\{ j \in J \mid z_j^B - c_j < 0 \right\}$$

and go to step 4.

Step 4. It determines the index $k \in J_-$ with the *criterion of entry into the base* :

$$\min_{j \in J_-} (z_j^B - c_j) = z_k^B - c_k$$

If $y_k^B \leq 0$, then STOP: the solution is unbounded; else it determines the index $l \in I$ with the *exit criterion from the base*.



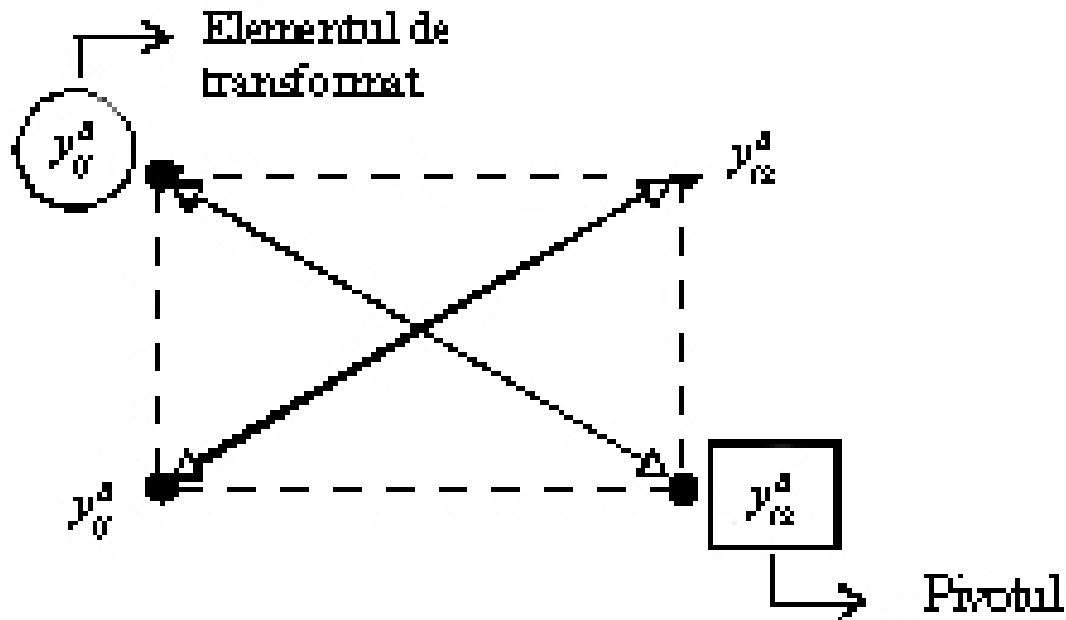
Remark: The element y_{lk}^B is called *pivot*.

The base change formulas are equivalent to the following transformation rules of the simplex table associated with base B :

a) the elements located on the pivot line are divided by pivot;

b) the elements located on the pivot column become 0, except for the pivot, which becomes 1;

c) the other elements of the simplex table are transformed according to *the rectangle formula*. Let an imaginary rectangle with the diagonal determined by the element to be transformed and the pivot:



The rectangle formula :

$$y_{ij}^{B'} = \frac{y_{ij}^B y_{lk}^B - y_{lj}^B y_{ik}^B}{y_{lk}^B}$$



			C_1	...	C_k	...	C_j	...	C_n
C_B	V.B.	V.V.B.	x_1	...	x_k	...	x_j	...	x_n
C_1	x_1^B	\bar{x}_1^B	y_{11}^B	...	y_{1k}^B	...	y_{1j}^B	...	y_{1n}^B
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
C_i	x_i^B	\bar{x}_i^B	y_{i1}^B	●	y_{ik}^B	...	y_{ij}^B	...	y_{in}^B
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
C_l	x_l^B	\bar{x}_l^B	y_{l1}^B	●	y_{lk}^B	...	y_{lj}^B	...	y_{ln}^B
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
C_m	x_m^B	\bar{x}_m^B	y_{m1}^B	...	y_{mk}^B	...	y_{mj}^B	...	y_{mn}^B
		\bar{z}^B	$z_1^B - C_1$...	$z_k^B - C_k$...	$z_j^B - C_j$...	$z_n^B - C_n$



Example. Solve the LP problem with the simplex algorithm :

$$\left\{ \begin{array}{l} \min (4x_1 - 3x_2 + 5x_3) \\ 3x_1 - 2x_2 + x_3 \leq 4 \\ -2x_1 + x_2 + 2x_3 \geq -1 \\ 2x_1 + 4x_2 - 3x_3 \leq 6 \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array} \right.$$

Solution. The problem is brought to the standard form by introducing the non-negative compensation variables x_4 , x_5 , x_6 in the constraints. Multiply the second constraint by -1 and we get :



$$\left\{ \begin{array}{l} \min (4x_1 - 3x_2 + 5x_3) \\ 3x_1 - 2x_2 + x_3 + x_4 = 4 \\ 2x_1 - x_2 - 2x_3 + x_5 = 1 \\ 2x_1 + 4x_2 - 3x_3 + x_6 = 6 \\ x_i \geq 0, i = \overline{1, 6} \end{array} \right.$$

The matrix of the system of the constraints is :

$$A = \begin{bmatrix} 3 & -2 & 1 & 1 & 0 & 0 \\ 2 & -1 & -2 & 0 & 1 & 0 \\ 2 & 4 & -3 & 0 & 0 & 1 \end{bmatrix}$$

with $m = 3$, $n = 6$, $\text{rank}(A) = 3 < 6$.



A base, B , can be formed with the last three columns of the matrix A , this being the unit matrix :

$$B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

In order to be able to apply the simplex algorithm, it is checked in advance whether the base B is primal admissible:

$$B^{-1} = B, \text{ and } \bar{x}^B = B^{-1}b = b = [4 \ 1 \ 6]^T > 0.$$

We build simple tables (see. next Fig.).

The optimal solution of the problem is: $x_1 = 0$;
 $x_2 = 1,5$; $x_3 = 0$; $x_4 = 7$; $x_5 = 2,5$; $x_6 = 0$; $z_{\min} = -4,5$.



			4	-3	5	0	0	0
C_B	V.B.	V.V.B.	x_1	x_2	x_3	x_4	x_5	x_6
0	x_4	4	3	-2	1	1	0	0
0	x_5	1	2	-1	-2	0	1	0
← 0	x_6	6	2	4	-3	0	0	1
		0	-4	3	-5	0	0	0

↑

			4	-3	5	0	0	0
C_B	V.B.	V.V.B.	x_1	x_2	x_3	x_4	x_5	x_6
0	x_4	7	4	0	-1/2	1	0	1/2
0	x_5	5/2	5/2	0	-11/4	0	1	1/4
-3	x_2	3/2	1/2	1	-3/4	0	0	1/4
		-9/2	-11/2	0	-11/4	0	0	-3/4



Example. Solve the LP problem:

$$\begin{cases} \max (x_1 + 3x_2 + 2x_3) \\ x_1 - x_2 + x_3 \leq 2 \\ -2x_1 + 2x_2 - x_3 \leq 4 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Solution. The problem is brought to the standard form, by introducing the compensation variables x_4, x_5 :

$$\begin{cases} \max (x_1 + 3x_2 + 2x_3) \\ x_1 - x_2 + x_3 + x_4 = 2 \\ -2x_1 + 2x_2 - x_3 + x_5 = 4 \\ x_i \geq 0, i = \overline{1, 5} \end{cases}$$



With the last two columns from the matrix of the system of constraints (equations) can be formed the base B , respectively, the unit matrix .

The base B is primal admissible because

$$B^{-1}b = b = [2 \ 4]^T > 0$$

The simplex algorithm can be applied. The simplex tables are shown in the following figures .

The last table shows that it exists $z_k - c_k < 0$; $J_- = \{1, 2\}$, so the obtained solution is not optimal. On the other hand, all the coefficients y_{i1} from the column of x_1 are not positive. So the problem has infinite optimum.



			1	3	2	0	0
C_B	V.B.	V.V.B.	x_1	x_2	x_3	x_4	x_5
0	x_4	2	1	-1	1	1	0
← 0	x_5	4	-2	2	-1	0	1
		0	-1	-3	-2	0	0

↑

			1	3	2	0	0
C_B	V.B.	V.V.B.	x_1	x_2	x_3	x_4	x_5
0	x_4	4	0	0	1/2	1	1/2
3	x_2	2	-1	1	-1/2	0	1/2
		6	-4	0	-7/2	0	3/2





3.3. The Big M method

In order to be able to start applying the simplex algorithm, two conditions must be met:

- the problem should be in standard form;
- to have a initial basic solution.

For the second condition, the random choice of a base consisting of m any vectors from the column vectors of the matrix A can lead to unfeasible basic solutions with which the application of the simplex algorithm cannot begin .

Determination of an initial basic solution can be done with *the Big M method*.



$$\left\{ \begin{array}{l}
 [\min] z = c_1x_1 + \cdots + c_nx_n + Mx_{n+m+1} + \cdots + Mx_{n+2m} \\
 a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - x_{n+1} + x_{n+m+1} = b_1 \\
 a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - x_{n+2} + x_{n+m+2} = b_2 \\
 \dots\dots\dots \\
 a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n - x_{n+m} + x_{n+2m} = b_m \\
 x_1, x_2, \dots, x_n, x_{n+1}, \dots, x_{n+m}, x_{n+m+1}, \dots, x_{n+2m} \geq 0
 \end{array} \right. \quad (2)$$

The basic initial solution of the problem (2) can be: $x_1 = x_2 = \dots = x_n = x_{n+1} = x_{n+m} = 0$, $x_{n+m+1} = -b_1$, $x_{n+m+2} = -b_2$, $x_{n+2m} = -b_m$

The solution contains artificial variables, but it is convenient for applying the simplex algorithm to the problem (2).



The problem (2) is solved, the following cases being possible:

1) The problem (2) has infinite optimum. Then the problem (1) also has infinite optimum .

2) The problem (2) has finite optimum, but in the optimal solution at least one artificial variable has a non-zero (positive) value. Then the problem (1) has no solution.

3) The problem (2) admits a finite optimum and in the optimal solution all artificial variables have zero value. Then this solution, from which the artificial variables are ignored, is the optimal solution of the problem (1).



Remarks:

1) Regardless of the type of problem (minimization or maximization), if the constraint matrix does not contain unit vectors, then the number of artificial variables entered will be equal to m .

2) If, when applying the simplex algorithm to solve the extended problem, an artificial variable leaves the base, it will never enter the base, which justifies the elimination (possible) from the subsequent calculations of the column of that variable.



Example. Solve the LP problem with the Big M method:

$$\begin{cases} \min (3x_1 + 4x_2 + 7x_3) \\ x_1 + 3x_2 - x_3 \geq 10 \\ x_1 + x_2 + x_3 \geq 6 \\ x_1, x_2, x_3 \geq 0 \end{cases}$$

Solution. The problem is first brought to the standard form by the compensation variables x_4 și x_5 .

Artificial variables x_6 și x_7 are added; they are introduced in restrictions with the coefficient 1 and in the objective function with the penalty coefficients $+M$:

$$\begin{cases} \min (3x_1 + 4x_2 + 7x_3 + Mx_6 + Mx_7) \\ x_1 + 3x_2 - x_3 - x_4 + x_6 = 10 \\ x_1 + x_2 + x_3 - x_5 + x_7 = 6 \\ x_i \geq 0, i = \overline{1, 7} \end{cases}$$

The basic initial solution is $x_1 = x_2 = x_3 = x_4 = x_5 = 0$, $x_6 = 10$, $x_7 = 6$. The first simplex table of the extended problem corresponding to this base is:

			3	4	7	0	0	M	M
c_B	V.B.	V.V.B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
M	x_6	10	1	3	-1	-1	0	1	0
M	x_7	6	1	1	1	0	-1	0	1
		$16M$	$2M-3$	$4M-4$	-7	$-M$	$-M$	0	0

↑

			3	4	7	0	0	M	M
c_B	V.B.	V.V.B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
4	x_2	10/3	1/3	1	-1/3	-1/3	0	1/3	0
M	x_7	8/3	2/3	0	4/3	1/3	-1	-1/3	1
		$8M/3 + 40/3$	$2M/3 - 5/3$	0	$4M/3 - 25/3$	$M/3 - 4/3$	$-M$	$-4M/3 + 4/3$	0

↑

			3	4	7	0	0	M	M
c_B	V.B.	V.V.B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
4	x_2	4	1/2	1	0	-1/4	-1/4	1/4	1/4
7	x_3	2	1/2	0	1	1/4	-3/4	-1/4	3/4
		30	5/2	0	0	3/4	-25/4	$-M - 3/4$	$-M + 25/4$

↑



			3	4	7	0	0	M	M
c_B	V.B.	V.V.B	x_1	x_2	x_3	x_4	x_5	x_6	x_7
4	x_2	2	0	1	-1	-1/2	1/2	1/2	-1/2
3	x_1	4	1	0	2	1/2	-3/2	-1/2	3/2
		20	0	0	-5	-1/2	-5/2	$-M+1/2$	$-M+5/2$

Because $z_k - c_k \leq 0, k = 1, \dots, 7 \Rightarrow$ the above simplex table contains the optimal solution of the extended problem, in which the artificial variables x_6 and x_7 have the value zero.

The optimal solution of the initial problem: $x_1 = 4, x_2 = 2, x_3 = 0$. The optimal value of the objective function is $z_{\text{opt}} = 20$. \square



4. Parametric Optimization



The LP problem, in which at least one of its elements - the matrix of the coefficients of the constraints A , the vector of free terms b , the vector of the coefficients of the objective function c - depends, linearly or nonlinearly, on one or more parameters, represents a *parametric LP problem*.

Parametric optimization studies, as well as postoptimization, the LP problems with variable coefficients. If the post-optimization corresponds to the discrete variation of the coefficients, the parametric optimization aims at the continuous variation of the coefficients .



Solving the parametric LP problems involves the following two steps :

1) Determining the optimal solution for a given value of the parameter or a system of values if there are several parameters.

2) The study of the sensitivity of the optimal solution determined at the variation of the parameter (parameters).

This study consists in establishing the intervals (domains) of optimality of a solution, respectively, of the set of values of the parameters for which the determined solution keeps its optimality character.



6. Integer Programming

6.1. General considerations

6.2. Examples of economic problems

6.3. Branch and bound method



6.1. General considerations

In the previously studied LP problems the variables were subjected to the condition of non-negativity, having a continuous variation.

A number of economic situations impose the requirement that variables can only take values from certain discrete sets. Such discontinuity constraints occur when the variables are expressed in indivisible units of measure (number of pieces, etc.).

Optimization problems with discontinuity constraints belong to the field of *discrete programming*.



The vast majority of discrete programming problems encountered in practice require that variables (some or all) have integer values. We are talking in these cases of *programming in integers*.

Problems in which only some variables have integer values are called *mixed* or *partially integer problems*, and problems in which all variables have integer values are called *totally integer problems*.

We will study the programming in integer problems, which does not represent a restriction of the generality because the discrete general cases can be brought to this form.



Thus, we assume that, in a problem, a variable x can take only a finite number of values : a_1, a_2, \dots, a_k . The variable x can be rewritten in the form:

$$x = \sum_{i=1}^k \delta_i a_i, \text{ with } \sum_{i=1}^k \delta_i = 1 \text{ and } \delta_i = 0 \text{ or } 1$$

which leads to a problem with integer variables.

Remark. In any discrete programming problem the objective function and the constraints, even if they are expressed by linear forms, due to the discontinuous character of the variables are no longer, from a mathematical point of view, linear functions. For this



reason, the discrete programming is treated as a special case of the linear programming.

The use of the integer variables leads to an increase in the flexibility of modeling phenomena, but also in the complexity of specific ways of solving them. .



6.2. Examples of economic problems

Example. *Loading* problem

Consider the case of loading a truck with n types of packages, with weight m_i ($i = 1, \dots, n$) and value v_i ($i = 1, \dots, n$). The truck allows a maximum load C .

Determine the optimal way to choose the load of the truck so that the total value of the transported packages is maximum.

It is denoted by x_i the number of packages of type i to be loaded into the truck.

The objective function is the total value of the



the transported packages, which must be maximized.

The mathematical model of the problem is:

$$\left\{ \begin{array}{l} [\max] z = \sum_{i=1}^n v_i x_i \\ \sum_{i=1}^n m_i x_i \leq C \\ x_i \geq 0, \quad i = \overline{1, n} \\ x_i \in Z, \quad i = \overline{1, n} \end{array} \right.$$

A totally discrete optimization problem model was obtained, in which variables can only take integer values . □



Example. *Investment* problem

A company intends to invest in retechnology. It has the opportunity to choose from n project offers. Each project will bring a profit c_i ($i = 1, \dots, n$). The cost of investing in each project a_i ($i = 1, \dots, n$) and the maximum amount available S for investments are also known.

Establish the optimal investment program that will lead to a maximum total profit.

Bivalent variables are defined :

$$x_i = \begin{cases} 1, & \text{if the project } i \text{ is chosen} \\ 0, & \text{otherwise} \end{cases}$$



The mathematical model of the problem is:

$$\left\{ \begin{array}{l} [\max] z = \sum_{i=1}^n c_i x_i \\ \sum_{i=1}^n a_i x_i \leq S \\ x_i = 0 \text{ sau } 1, \quad i = \overline{1, n} \end{array} \right.$$

The obtained model of the optimization problem is called *zero-one programming* model. \square



Example. *Cost* problem

The management of a company wants to minimize manufacturing costs for its n types of products. Each type of product j ($j = 1, \dots, n$) has a unit cost of production c_j and a preparation manufacturing cost q_j .

For each type of product j , the minimum market demand a_j and the maximum own manufacturing capacity b_j are known.

Establish the manufacturing program of the company, optimal in terms of its total expenses.



Let x_j the quantity of product j ($j = 1, \dots, n$) to be manufactured.

Bivalent variables are also defined:

$$y_j = \begin{cases} 1, & \text{if } x_j > 0 \\ 0, & \text{otherwise} \end{cases}$$

which indicates if the product j enters manufacturing or not.

The objective function is represented by the total manufacturing costs and must be minimized:

$$[\min] z = \sum_{j=1}^n (c_j x_j + q_j y_j)$$

Market demand constraints:



$$x_j \geq a_j, \quad j = \overline{1, n}$$

Production capacity constraints:

$$x_j \leq b_j, \quad j = \overline{1, n}$$

The definition of the variable y_j can be reformulated:

$$\begin{cases} y_j \geq \frac{x_j}{b_j} \\ y_j \in \{0, 1\} \end{cases}, \quad j = \overline{1, n}$$

A mixed optimization model is obtained, in which some of the variables have positive values, and the others can only take integer values (they are



bivalent variables):

$$\left\{ \begin{array}{l} [\min] z = \sum_{j=1}^n (c_j x_j + q_j y_j) \\ x_j \leq b_j y_j, \quad j = \overline{1, n} \\ x_j \geq a_j, \quad j = \overline{1, n} \\ y_j \in \{0, 1\}, \quad j = \overline{1, n} \end{array} \right.$$





6.3. Branch and bound method

The process of this method can be described in the form of a tree diagram.

The method generates the tree nodes until all paths reach the end nodes. For each node there are two branches to check. If none of the branches reaches a terminal node, then the method follows the most favorable node, the node on the other branch being considered *independent*.

Let consider the problem of mixed programming in integers :



$$\begin{cases} [\max] z = c^T x \\ Ax = b \\ x \geq 0 \\ x_j \in \mathbf{Z}, j \in T \end{cases}$$

Algorithm of the *branch and bound method*:

Step 1 (Initial solution)

Resolves the LP problem obtained from the initial integer problem, by omitting the integrity conditions. If all solution values $\bar{x}_j \in \mathbf{Z}, \forall j \in T$, then the optimal solution of the initial problem is obtained, STOP; otherwise go to step 2.



Step 2 (Selection of branching variables)

Choose from the last simplex table from step 1, from those variables x_j , $j \in T$, that do not have integer values, the variable x_l having the fractional part, f_l , the largest, for generating branching constraints.

Because the variable x_l must have integer values, and $\bar{x}_l = [\bar{x}_l] + f_l$ it must satisfy either condition:

$$\bar{x}_l \leq [\bar{x}_l] \quad (*)$$

either

$$\bar{x}_l \geq [\bar{x}_l] + 1 \quad (**)$$



Step 3 (Generating new nodes)

Two new integer programming problems are formed, relative to the node considered in step 2. A problem is formed by adding the constraint (*), and the other by adding the constraint (**).

Solve each problem with the simplex algorithm.

Step 4 (Terminal nodes testing)

Each of the nodes generated in step 3 can be a terminal node if: the problem represented by that node does not have admissible solutions or all the variables $x_j, j \in T$, have integer values.



In the first case, the nodes are labeled as terminal nodes and go to step 5.

In the second case, the values of the objective function are first compared with the best current value. If the value of the objective function for the new node is better, it becomes the best value; go to step 5.

Step 5 (Nodes selection)

a) If both nodes in step 4 are terminal, then the next node considered is the next in the list of independent nodes. If the independent node has a value of the objective function greater than the best current value,



then, with this node, go to step 2; else the next node in the independent list is checked. When the list of independent nodes is exhausted, the best current value is the optimal solution, STOP.

b) If only one node in step 4 is terminal, then use the non-terminal node and go to step 2.

c) If both nodes in step 4 are non-terminal, then the node with the highest objective function value is chosen. The other node is added to the list of independent nodes.



Example. Solve the programming problem in integers:

$$\left\{ \begin{array}{l} \max (7x_1 + 3x_2) \\ 2x_1 + 5x_2 \leq 30 \\ 8x_1 + 3x_2 \leq 48 \\ x_1, x_2 \geq 0 \\ x_1, x_2 \in \mathbb{Z} \end{array} \right.$$

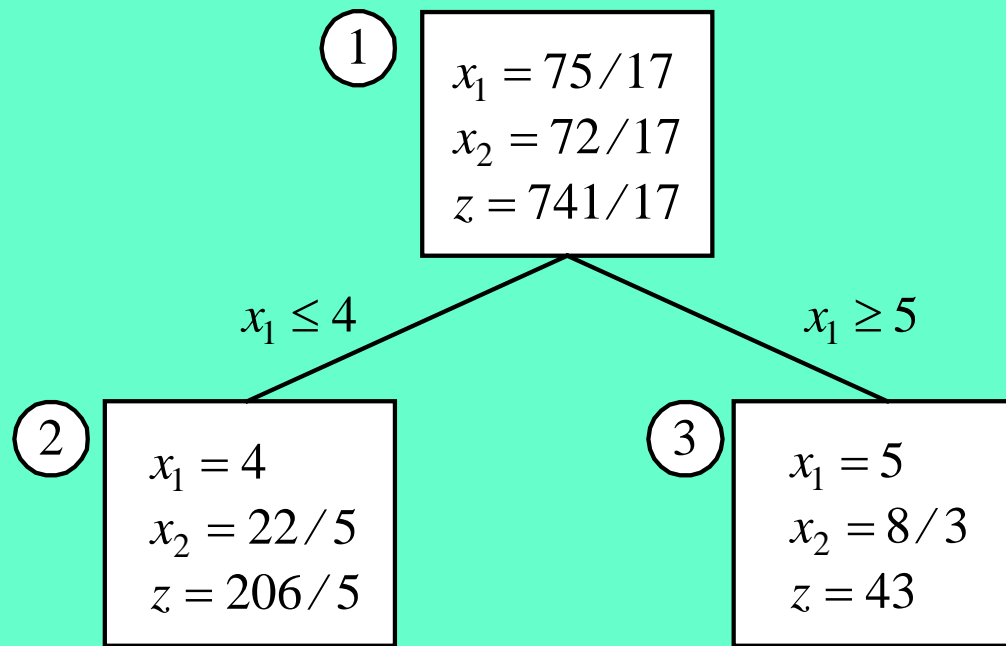
Solution. The LP problem, obtained from the initial by giving up the integrity conditions, is solved with the simplex algorithm and the final simplex table is:



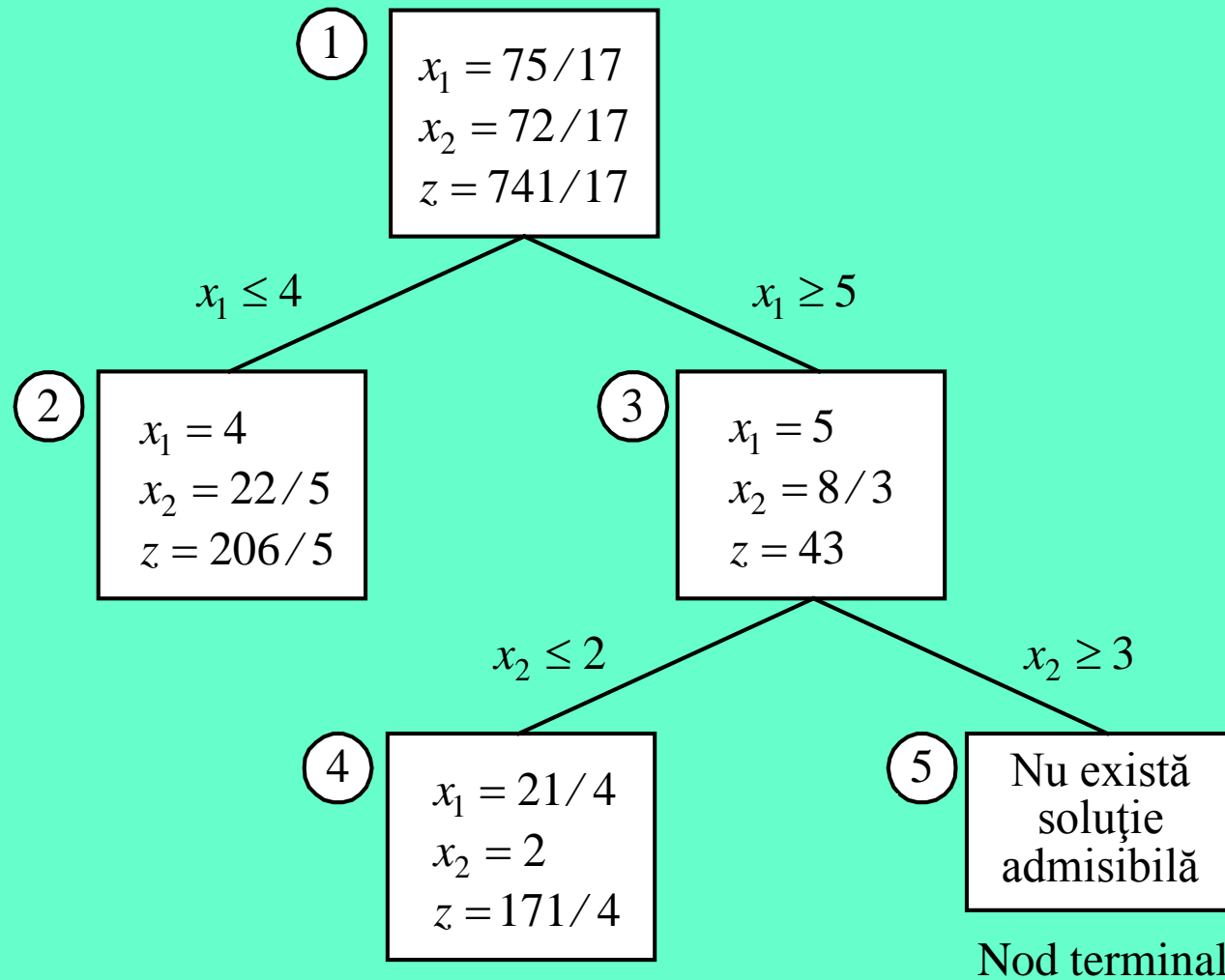
			7	3	0	0	
C_B	V.B.	V.V.B.	x_1	x_2	x_3	x_4	
3	x_2	$72/17$	0	1	$4/17$	$-1/17$	
7	x_1	$75/17$	1	0	$-3/34$	$5/34$	
			$741/17$	0	0	$3/34$	$29/34$

It is found that $\bar{x}_1 \notin \mathbf{Z}$, $\bar{x}_2 \notin \mathbf{Z}$. The branch and bound method is applied.

The fractional parts are: $f_1 = 7/17$, $f_2 = 4/17$. Because $f_1 > f_2$, x_1 is chosen as a branching variable, and the constraints are: $x_1 \leq 4$ and $x_1 \geq 5$.



The value of the objective function in node 3 is higher than that in node 2 ($43 > 206/5$). As a result, node 2 is entered in the list of independent nodes. In node 3 the conditions are: $x_2 \leq 2$ and $x_2 \geq 3$.





In node 3, x_1 has an integer value, but in node 4, x_1 has no integer value. In node 4 the conditions are: $x_1 \leq 5$ and $x_1 \geq 6$; it is obtained:

