**Csaba Gábor Kézi and Gergő Nagy**

# INTRODUCTION TO REAL AND COMPLEX ANALYSIS

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# Chapter 1.

# Introduction to real analysis

# 1. Real valued functions and their applications in real life

## Theoretical summary

**Definition 1.1.1.** If $f$ and $g$ are real valued functions defined on $\mathbb{R}$, then

$$f \circ g(x) = f\big(g(x)\big).$$

**Definition 1.1.2.** Let $C(x)$ be the total cost of production, where $x$ is the amount of units.

Let $R(x)$ be the total revenue of production, where $x$ is the amount of units.

Let $\Pi(x)$ be the total profit of production, where $x$ is the amount of units.

**Definition 1.1.3.** A firm will choose to implement a shutdown of production when the revenue received from the sale of the goods or services produced cannot cover even the variable costs of production. In that situation, which is called a shutdown point, the firm will experience a higher loss when it produces, compared to not producing at all.

**Remark 1.1.4.** Technically, a shutdown point occurs if the average revenue is below the average variable costs at the profit-maximizing positive level of output.

**Definition 1.1.5.** If the unit price of the product is $p$ and $p \mapsto D(p)$ is a demand function and $p \mapsto S(p)$ is a supply function, then the solution of the equation $D(p) = S(p)$ is called *equilibrium price*. If $p_0$ is the equilibrium price, then $D(p_0)$ or equivalently $S(p_0)$ is called *equilibrium quantity*. The point $\big(p_0, D(p_0)\big)$ is called *equilibrium point*.

## Solved exercises

**Exercise 1.** Let $f(x) = \sin x$ and $g(x) = x^2 + 7x - 3$ be real valued functions!
a) Calculate the value $f(\pi)$!
b) Calculate the function $f \circ g$!
c) Calculate the function $g \circ f$!

**Solution:**

a) The value $f(\pi)$ is $\sin \pi = 0$.

b) The function $f \circ g$ is
$$f \circ g(x) = f\big(g(x)\big) = f(x^2 + 7x - 3) = \sin(x^2 + 7x - 3).$$

c) The function $g \circ f$ is
$$\begin{aligned} g \circ f(x) = g\big(f(x)\big) = g(\sin x) &= (\sin x)^2 + 7 \cdot \sin x - 3 \\ &= \sin^2 x + 7 \cdot \sin x - 3. \end{aligned}$$

**Exercise 2.** Let $f(x) = \sqrt{x}$ and $g(x) = x + 7$ be real valued functions!
a) Calculate the function $f \circ g$!
b) Calculate the function $g \circ f$!
c) Solve the equation $f \circ g(x) = g \circ f(x)$!

**Solution:**

a) The function $f \circ g$ is:
$$f \circ g(x) = f\big(g(x)\big) = f(x + 7) = \sqrt{x + 7}.$$

b) The function $g \circ f$ is:
$$g \circ f(x) = g\big(f(x)\big) = g(\sqrt{x}) = \sqrt{x} + 7.$$

c) We have to solve the equation
$$\sqrt{x + 7} = \sqrt{x} + 7.$$

We get that
$$\begin{aligned} x + 7 &= (\sqrt{x} + 7)^2 \\ x + 7 &= x + 14 \cdot \sqrt{x} + 49 \\ -42 &= 14 \cdot \sqrt{x}. \end{aligned}$$

It means, there is a contradiction and so the equation has no solutions.

**Exercise 3.** Let $f(p) = 100 - 10p$ be a demand function, where $p$ is the unit price of the product in euros and $f(p)$ is number of pieces measured in thousand pieces.

a) What is the revenue function?

b) Plot the revenue function!

c) Determine the price which maximizes the revenue!

**Solution:**

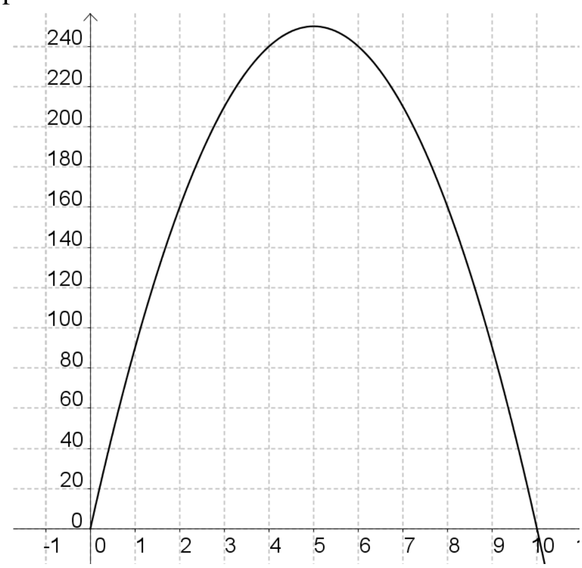
a) The revenue function is:

$$R(p) = p \cdot f(p) = p \cdot (100 - 10p) = 100p - 10p^2.$$

b) If we apply some algebraic manipulations, we get that

$$R(p) = -10 \cdot (p^2 - 10p) = -10 \cdot [(p-5)^2 - 25] =$$
$$= -10 \cdot (p-5)^2 + 250.$$

Thus the graph of the revenue function is as follows:




c) The revenue function has a maximum value at $p = 5$. It means that it is attained when the unit price is 5 euros.

In this situation:

$$q = f(5) = 100 - 10 \cdot 5 = 50,$$

thus we have to produce 50 products to obtain the maximum revenue. It is

$$R(5) = 5 \cdot f(5) = p \cdot q = 250.$$

**Exercise 4.** The company has a cost function

$$C(q) = q^3 - 4q^2 + 10q + 10,$$

where $q$ is the quantity in thousand pieces and the cost is $C(q)$ thousand euros.

a) Give the fixed cost!

b) Determine the variable cost function!

c) Calculate the average cost function!

d) Determine the average fixed cost function!

e) Calculate the average variable cost function!

f) Give the shutdown point!

**Solution:**

a) Because

$$C(0) = 0^3 - 4 \cdot 0^2 + 10 \cdot 0 + 10 = 10,$$

we get that the fixed cost is 10 000 euros.

b) The variable cost function is as follows:

$$VC(q) = q^3 - 4q^2 + 10q.$$

c) The average cost function is as follows:

$$AC(q) = \frac{C(q)}{q} = q^2 - 4q + 10 + \frac{10}{q}.$$

d) The average fixed cost function is as follows:

$$AFC(q) = \frac{10}{q},$$

e) The average variable cost function is as follows:

$$AVC(q) = q^2 - 4q + 10.$$

f) We have to find the minimum of the $AVC$ function. Because

$$AVC(q) = q^2 - 4q + 10 = (q - 2)^2 + 6,$$

we get that the minimum of the function is $q = 2$, thus we have to produce 2 000 products to reach the shutdown point.

**Exercise 5.** The demand function of the product is

$$D(p) = 150 - 3p.$$

The supply function is

$$S(p) = 2p - 20.$$

The unit price is measured in euros. Find the equilibrium point.

**Solution:**

The equilibrium price is the solution of the equation:

$$D(p) = S(p).$$

If we substitute the data and solve the equation, we get that

$$150 - 3p = 2p - 20$$
$$170 = 5p.$$

It means that the equilibrium price is $p = 34$ euro. The equilibrium quantity:

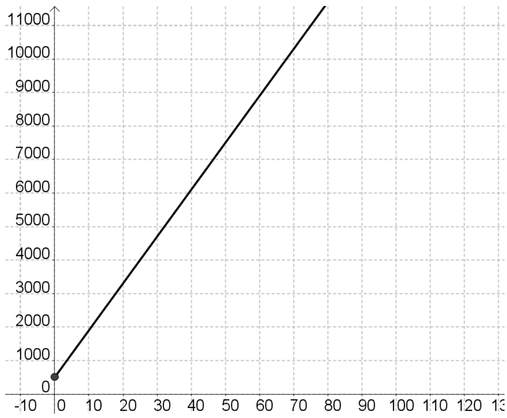
$$q = D(34) = S(34) = 150 - 3 \cdot 34 = 48.$$

**Exercise 6.** The cost function is $C(q) = 500 + 140q$ euros. The revenue function is $R(q) = 200q - q^2$ euros.

a) Sketch the graph of the cost function!
b) Calculate the value $C(2)$!
c) If $C(q) = 1\,900$ what is the value of $q$?
d) Find the fixed cost!
e) Calculate the variable cost!
f) Calculate the average cost!
g) Determine the average fixed cost!
h) Determine the average variable cost!
i) Sketch the graph of the revenue function in the same coordinatesystem which you drew in part a)!
j) What is the revenue if $q = 2$?
k) Determine the profit function!
l) Calculate the maximum of the profit!
m) Sketch the graph of the profit function!

**Solution:**

a) The graph of the function $C$ is as follows:




b) The value of the cost function at $q = 2$ is as follows:

$$C(2) = 500 + 140 \cdot 2 = 780.$$

c) We have to solve the equation

$$1\,900 = 500 + 140q.$$

We get that

$$1\,900 = 500 + 140q \quad \Rightarrow \quad 1\,400 = 140q \quad \Rightarrow \quad q = 10.$$

d) The fixed cost is: $FC = 500$.

e) The variable cost is:

$$VC(q) = 140q.$$

f) The average cost function:

$$AC(q) = \frac{C(q)}{q} = 140 + \frac{500}{q}.$$

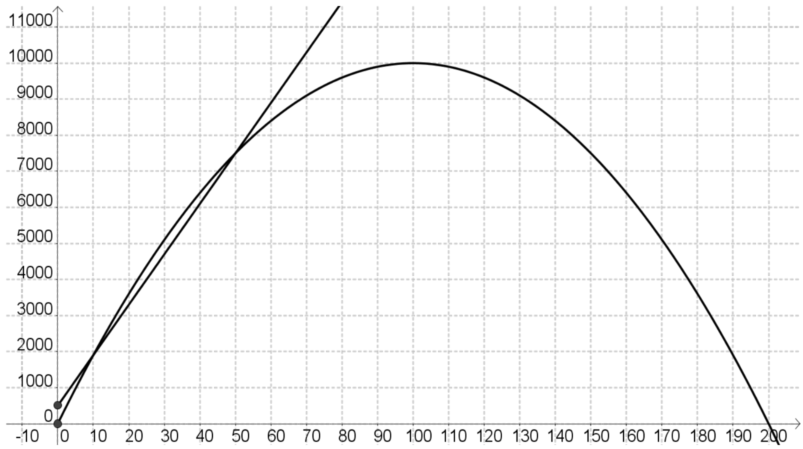
g) The average fixed cost function:

$$AFC = \frac{500}{q}.$$

h) The average variable cost function:

$$AVC(q) = 140.$$

i) The graphs of the cost function and the revenue function are:




j) The revenue at $q = 2$:

$$R(2) = 200 \cdot 2 - 2^2 = 396,$$

k) The profit function is as follows:

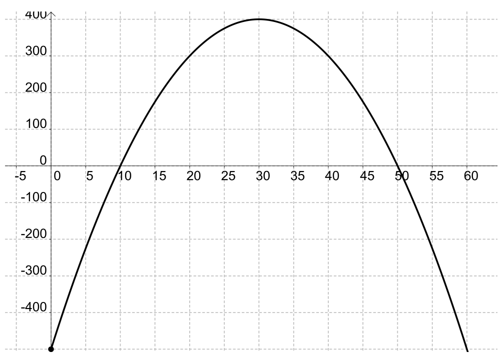
$$\Pi(q) = R(q) - C(q) = (200q - q^2) - (500 + 140q) =$$
$$= -q^2 + 60q - 500.$$

l) By applying some algebraic manipulations, we get that

$$\Pi(q) = -(q^2 - 60q) - 500 = -\big((q-30)^2 - 900\big) - 500 =$$
$$= -(q-30)^2 + 900 - 500 = -(q-30)^2 + 400.$$

The maximum of the profit function is attained at $q = 30$.

m) The graph of the profit function is as follows:



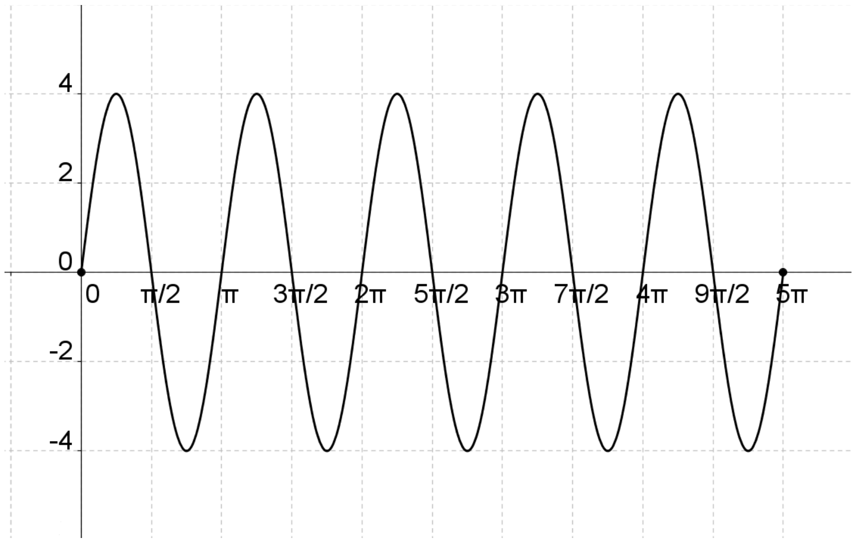
**Exercise 7.** The motion equation of the simple harmonic motion is

$$y(t) = 4 \cdot \sin(2t).$$

Sketch the graph of the function describing the motion on the interval $[0; 5\pi]$.

**Solution:**

The graph of the function is as follows:



# 2. Sequences, series and their applications

## Theoretical summary

**Definition 1.2.1.** A function

$$a\colon \mathbb{N} \to \mathbb{R}$$

is called a *(real) sequence*. We denote the sequence $a$ by $a_n$.

**Definition 1.2.2.** The sequence $a_n$ is

- *monotonically increasing* if $a_n \le a_{n+1}$ for all $n \in \mathbb{N}$;
- *monotonically decreasing* if $a_n \ge a_{n+1}$ for all $n \in \mathbb{N}$;
- *stricly monotonically increasing* if $a_n < a_{n+1}$ for all $n \in \mathbb{N}$;
- *stricly monotonically decreasing* if $a_n > a_{n+1}$ for all $n \in \mathbb{N}$.

**Example 1.2.3.** Consider the sequence $a_n = \dfrac{1}{n}$. Since

$$a_{n+1} - a_n = \frac{1}{n+1} - \frac{1}{n} = \frac{n-(n+1)}{n\cdot(n+1)} = \frac{-1}{n\cdot(n+1)} < 0$$

for all $n \in \mathbb{N}$, we get that $a_{n+1} - a_n < 0$, thus $a_{n+1} < a_n$. It means that the sequence is stricly monotonically decreasing.

**Definition 1.2.4.** The sequence $a_n$ is

- *bounded from above*, if there exists a real number $K$ such that $a_n \le K$ for all $n \in \mathbb{N}$. In this case, the range of $a_n$ is bounded from above and nonempty, so it has a supremum, which is called the *supremum* of $a_n$.
- *bounded from below*, if there exists a real number $k$ such that $a_n \ge k$ for all $n \in \mathbb{N}$. In this case, the range of $a_n$ is bounded from below and nonempty, so it has an infimum, which is called the *infimum* of $a_n$.
- *bounded*, if it is bounded from above and bounded from below.

**Definition 1.2.5.** A sequence of real numbers $a_n$ converges to a real number $a$ if for all $\varepsilon > 0$, there exists a number $n_0 \in \mathbb{N}$ such that in case $n \ge n_0$, we have $|a_n - a| < \varepsilon$. Notation: $a_n \to a$ or $\lim\limits_{n\to\infty} a_n = a$.

A sequence is *divergent* if it is not convergent.

**Example 1.2.6.** The sequence $a_n = \frac{1}{n}$ is convergent. The limit is $0$.

**Theorem 1.2.7.** (connection between operations and limit)
If the sequences $a_n$ and $b_n$ are convergent and the limit of $a_n$ is $a$, the limit of $b_n$ is $b$ and $\lambda \in \mathbb{R}$, then

- $a_n + b_n$ is convergent and its limit is $a + b$;
- $a_n \cdot b_n$ is convergent and its limit is $a \cdot b$;
- $\lambda \cdot a_n$ is convergent and its limit is $\lambda \cdot a$;
- if $b_n \neq 0$ (for $n \in \mathbb{N}$) and $b \neq 0$, then $\dfrac{a_n}{b_n}$ is convergent and its limit is $\dfrac{a}{b}$;
- if $a_n, a > 0$, then $a_n^{b_n}$ is convergent and its limit is $a^b$.

**Example 1.2.8.** The limit of the sequence $a_n = \frac{2}{n} + 1$ is $2 \cdot 0 + 1 = 1$.

**Theorem 1.2.9.** The limit of convergent monotonically increasing (decreasing) sequences (which are known to be bounded from above (below)) is their supremum (infimum).

**Theorem 1.2.10.** If $a_n$, $b_n$ and $c_n$ are sequences such that

$$a_n \leq b_n \leq c_n$$

and $a_n$ is convergent and its limit is $a$ and $c_n$ is convergent and its limits is also $a$, then the sequence $b_n$ is convergent and its limit is $a$.

**Example 1.2.11.** Since

$$-\frac{1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n},$$

and $-\frac{1}{n} \to 0$ and $\frac{1}{n} \to 0$, we get that the sequence $\frac{\sin n}{n}$ is convergent and its limit is 0.

**Theorem 1.2.12.**

- The limit of the sequence $a_n = \sqrt[n]{a}$ $(a > 0)$ is 1.
- The limit of the sequence $a_n = \sqrt[n]{n}$ is 1.
- The limit of the sequence $a_n = q^n$ $(-1 < q < 1)$ is 0.
- The limit of the sequence $a_n = q^n$ $(q > 1)$ is infinity.
- The limit of the sequence $a_n = \left(1 + \dfrac{1}{n}\right)^n$ is the number e (e $\approx$ 2.718).
- The limit of the equence $a_n = \left(1 + \dfrac{a}{n}\right)^n$ $(a \in \mathbb{R})$ is $\mathrm{e}^a$.

- For all unbounded monotonically increasing or decreasing sequences $b_n$, the limit of the sequence $a_n = \left(1 + \frac{1}{b_n}\right)^{b_n}$ is e.
- The limit of the sequence $a_n = \frac{a^n}{n!}$ $(a \in \mathbb{R})$ is $0$.

**Definition 1.2.13.** A *series* is the sum of the terms of a sequence. Let $a_n$ be a real sequence. The sum

$$a_1 + a_2 + \ldots + a_n + \ldots$$

is a *series*. The notation for this expression is

$$\sum_{n=1}^{\infty} a_n$$

or simply $\sum a_n$. The $n$-th *partial sum* of the latter series is

$$S_n = a_1 + a_2 + \ldots + a_n.$$

**Definition 1.2.14.** The series $\sum a_n$ is *convergent*, if the sequence of its $n$-th partial sums is convergent.

**Theorem 1.2.15.** If the series $\sum a_n$ is convergent then

$$\lim_{n\to\infty} a_n = 0.$$

**Example 1.2.16.** The series

$$\sum_{n=1}^{\infty} 2 + n$$

is not convergent because the limit of $a_n = 2 + n$ is not zero.

**Theorem 1.2.17.** The sum

$$\sum_{n=1}^{\infty} \frac{1}{n^k}$$

is convergent if $k > 1$, not convergent if $k \leq 1$.

**Theorem 1.2.18.** Let $\sum a_n$ be a series.

- If $\sqrt[n]{|a_n|}$ is convergent and $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, then $\sum a_n$ is convergent;
- If $\sqrt[n]{|a_n|}$ is convergent and $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum a_n$ is not convergent.

**Theorem 1.2.19.** If $q$ is a real number such that $|q| < 1$ then the series $\sum q^n$ is convergent and

$$\sum a_n = a \cdot \frac{1}{1-q},$$

where $a$ is the first term of the series.

**Example 1.2.20.**

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{2} \cdot \frac{1}{1-\frac{1}{2}} = 1.$$

**Solved exercises**

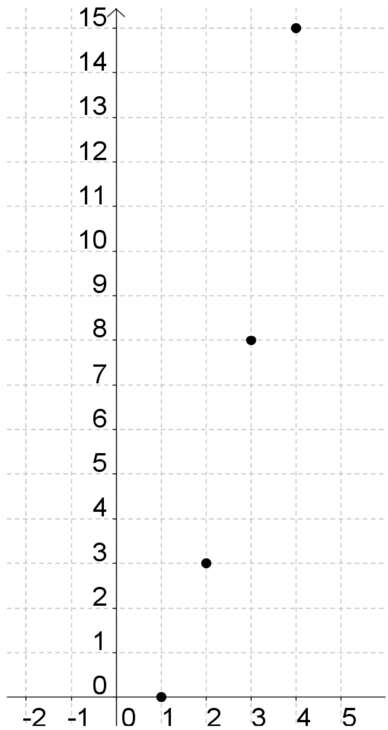
**Exercise 8.** Let's represent the first four terms of the sequence $a_n = n^2 - 1$ in a coordinate system in the plane!

**Solution:**

The fist four terms are:

$$a_1 = 0, \quad a_2 = 3, \quad a_3 = 8, \quad a_4 = 15.$$

In a coordinate sytem:




**Exercise 9.** Let $a_n = 2 + \frac{6}{n}$! What are the first four terms of the sequence? Describe the monotonicity, boundedness and limit properties of $a_n$!

**Solution:**

The first four elements are:

$$a_1 = 8, \quad a_2 = 5, \quad a_3 = 4, \quad a_4 = 3{,}5.$$

The sequence

- is stricly monotonically decreasing;
- is convergent, the limit is 2;
- has an infimum, 2; and a supremum, 8;
- is bounded;
- has no minimum value; has a maximum value, 8.

**Exercise 10.** Let $a_n = \frac{n}{n+1}$!

a) Prove that the sequence is stricly monotonically increasing!

b) Determine the limit if exists!

c) Prove that the sequence is bounded!

d) What is the minimum value and maximum value of the sequence if exists?

**Solution:**

a) Since

$$a_{n+1} = \frac{n+1}{(n+1)+1} = \frac{n+1}{n+2},$$

we get that

$$a_{n+1} - a_n = \frac{n+1}{n+2} - \frac{n}{n+1} = \frac{(n+1)^2 - n\cdot(n+2)}{(n+1)\cdot(n+2)} = $$
$$= \frac{n^2+2n+1-n^2-2n}{(n+1)\cdot(n+2)} = \frac{1}{(n+1)\cdot(n+2)} > 0.$$

The difference $a_{n+1} - a_n$ is positive for all $n \in \mathbb{N}$, hence we get that

$$a_{n+1} - a_n > 0 \qquad \Rightarrow \qquad a_{n+1} > a_n.$$

It means that the sequence is stricly monotonically increasing.

b) Since

$$a_n = \frac{n}{n+1} = \frac{n+1-1}{n+1} = 1 - \frac{1}{n+1},$$

the limit of the sequence is

$$\lim_{n\to\infty} a_n = 1.$$

c) Since $a_n$ is monotonically increasing and convergent, it is bounded.

d) The minimum value is

$$a_1 = \frac{1}{1+1} = \frac{1}{2}.$$

The maximum value would be the supremum, i.e. the limit which is 1. However

$$a_n = 1 - \frac{1}{n+1} < 1,$$

so 1 is not attained by $a_n$, therefore it has no maximum value.

**Exercise 11.** Calculate the limit of the sequence

$$a_n = \frac{n^2+3n+5}{4n^2-7n+6}.$$

**Solution:**

If we apply some algebraic manipulations, we get that

$$a_n = \frac{n^2+3n+5}{4n^2-7n+6} = \frac{\frac{n^2}{n^2}+\frac{3n}{n^2}+\frac{5}{n^2}}{\frac{4n^2}{n^2}-\frac{7n}{n^2}+\frac{6}{n^2}} = \frac{1+\frac{3}{n}+\frac{5}{n^2}}{4-\frac{7}{n}+\frac{6}{n^2}}.$$

It means that the limit of the sequence is

$$\lim_{n\to\infty} a_n = \frac{1+0+0}{4-0+0} = \frac{1}{4}.$$

**Exercise 12.** Calculate the limit of the sequence

$$a_n = \frac{(n+3)^3-(n+1)^2}{2n^3+n-1}.$$

**Solution:**

Apply the identity

$$(n+3)^3 = n^3+9n^2+27n+27,$$

and the identity

$$(n+1)^2 = n^2+2n+1$$

to get that

$$a_n = \frac{n^3+9n^2+27n+27-n^2-2n-1}{2n^3+n-1} = \frac{n^3+8n^2+25n+26}{2n^3+n-1} =$$

$$= \frac{1+\frac{8}{n}+\frac{25}{n^2}+\frac{26}{n^3}}{2+\frac{1}{n^2}-\frac{1}{n^3}} \to \frac{1}{2}.$$

**Exercise 13.** Calculate the limit of the sequence $a_n = \sqrt{9n^2+7n+3} - 3n$.

**Solution:**

Since

$$a_n = \sqrt{9n^2+7n+3} - 3n =$$
$$= \left(\sqrt{9n^2+7n+3} - 3n\right) \cdot \frac{\sqrt{9n^2+7n+3}+3n}{\sqrt{9n^2+7n+3}+3n},$$

we get that

$$a_n = \frac{9n^2+7n+3-9n^2}{\sqrt{9n^2+7n+3}+3n} = \frac{7n+3}{\sqrt{9n^2+7n+3}+3n}.$$

So

$$a_n = \frac{7+\frac{3}{n}}{\sqrt{9+\frac{7}{n}+\frac{3}{n^2}}+3} \to \frac{7+0}{\sqrt{9+0+0}+3} = \frac{7}{6}.$$

**Exercise 14.** Calculate the limit of the sequence

$$a_n = \left(\frac{2n+1}{2n+5}\right)^{3n+2}.$$

**Solution:**

Since

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \mathrm{e},$$

we get that

$$a_n = \left(\frac{2n+1}{2n+5}\right)^{3n+2} = \left(\frac{2n+5-4}{2n+5}\right)^{3n+2} = \left(1-\frac{4}{2n+5}\right)^{3n+2} =$$
$$= \left(1+\frac{-4}{2n+5}\right)^{3n+2} = \left[\left(1+\frac{1}{\frac{2n+5}{-4}}\right)^{\frac{2n+5}{-4}}\right]^{\frac{-4}{2n+5}\cdot(3n+2)} \to e^{-6} = \frac{1}{\mathrm{e}^6}.$$

**Exercise 15.** Calculate the sum

$$\sum_{n=0}^{\infty}\frac{2^n+3^n}{6^n}.$$

**Solution:**

We write another form of the series:

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{6^n} = \sum_{n=0}^{\infty} \frac{2^n}{6^n} + \frac{3^n}{6^n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n + \left(\frac{1}{2}\right)^n.$$

If we apply the identity

$$\sum_{n=0}^{\infty} aq^n = a \cdot \frac{1}{1-q} \qquad (a, q \in \mathbb{R};\ |q| < 1),$$

we get that

$$\sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{1}{1-\frac{1}{3}} = \frac{3}{2},$$

and

$$\sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n = \frac{1}{1-\frac{1}{2}} = 2,$$

thus the sum is

$$\sum_{n=0}^{\infty} \frac{2^n + 3^n}{6^n} = \frac{3}{2} + 2 = \frac{7}{2}.$$

**Exercise 16.** Calculate the sum

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)}.$$

**Solution:**

By definition, we get that

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)} = \lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k \cdot (k+1)}.$$

By decomoposing into partial fractions,

$$\frac{1}{k \cdot (k+1)} = \frac{A}{k} + \frac{B}{k+1}$$

we get that

$$1 = A \cdot (k+1) + B \cdot k.$$

It means that $1 = (A+B) \cdot k + A$, thus $A = 1$ and $A + B = 0$, that is $B = -1$. We get that

$$\frac{1}{k \cdot (k+1)} = \frac{1}{k} - \frac{1}{k+1},$$

so

$$\sum_{n=1}^{\infty} \frac{1}{n \cdot (n+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k \cdot (k+1)} = \lim_{n\to\infty} \sum_{k=1}^{n} \left( \frac{1}{k} - \frac{1}{k+1} \right) =$$
$$= \lim_{n\to\infty} \left( 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \ldots + \frac{1}{n} - \frac{1}{n+1} \right) =$$
$$= \lim_{n\to\infty} \left( 1 - \frac{1}{n+1} \right) = 1.$$

**Exercise 17.** Consider the region bounded by the midsegments of a regular triangle with side length 2 and the one obtained by applying the same procedure to the latter region and so on. Performing the procedure infinitely many times, determine the sum of the perimeters, resp. areas of the obtained triangles.

**Solution:**

In the case of the perimeter, that of the original triangle is 6 units, the perimeter of the first inscribed triangle is 3 units, that of the next inscribed triangle is $1{,}5$ units and so on. So the sum of the perimeters of the obtained triangles:

$$P = 6 + 3 + 1{,}5 + \ldots = 6 \left( 1 + \frac{1}{2} + \frac{1}{4} + \ldots \right) = 6 \cdot \frac{1}{1 - \frac{1}{2}} = 6 \cdot 2 = 12.$$

In the case of the area, that of the original triangle:

$$2^2 \cdot \frac{\sqrt{3}}{4} = \sqrt{3}.$$

The area of the first inscribed triangle is $\frac{\sqrt{3}}{4}$, that of the next inscribed triangle is $\frac{\sqrt{3}}{16}$, and so on. So the sum of the areas of the obtained triangles:

$$A = \sqrt{3} \left( 1 + \frac{1}{4} + \frac{1}{16} + \ldots \right) = \sqrt{3} \cdot \frac{1}{1 - \frac{1}{4}} = \sqrt{3} \cdot \frac{4}{3} = \frac{4\sqrt{3}}{3}.$$

# Chapter 2.

# Differential calculus and its applications

# 3. Limits of functions, introduction to differential calculus, description of motions

## Theoretical summary

**Definition 2.1.1.** Let $(X; d)$ and $(Y; \rho)$ be metric spaces, $D \subset X$ be a set, $x_0$ be a limit point of $D$ and $f\colon D \to Y$ be a function. We say that the *limit* of $f$, as $x$ approaches $x_0$, is $L \in Y$ if for each number $\varepsilon > 0$, there is a number $\delta > 0$ such that $\rho(f(x), L) < \varepsilon$ for all $x \in D$ with $0 < d(x, x_0) < \delta$. In this case, we define $\lim\limits_{x \to x_0} f(x) = L$.

In the case of real functions, this definition concerns their limits at finite numbers. However, we can define their limits also at infinity as follows.

**Definition 2.1.2.** Let $D \subset \mathbb{R}$ be a set which is not bounded from above and $f\colon D \to \mathbb{R}$ be a function. We say that the *limit* of $f$, as $x$ approaches $\infty$ is $L \in \mathbb{R}$ if for each number $\varepsilon > 0$, there is a number $T \in \mathbb{R}$ such that $|f(x) - L| < \varepsilon$ for all $x \in D$ with $x > T$. In this case, we define $\lim\limits_{x \to \infty} f(x) = L$. Limits of real functions at $-\infty$ can be defined similarly.

**Remark 2.1.3.** In the rest of this section $I$ is an open interval.

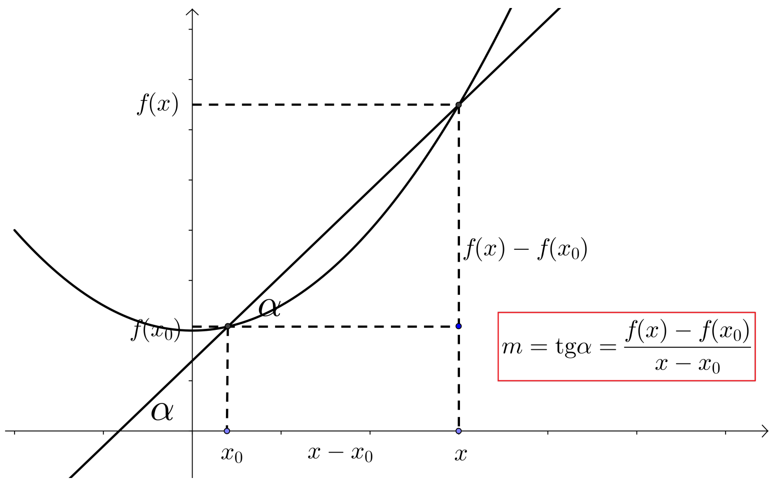
**Definition 2.1.4.** Let $f\colon I \to \mathbb{R}$ be a function and $x_1$ and $x_2$ be real numbers such that $x_1, x_2 \in I$ and $x_1 \neq x_2$. The *difference quotient* of the function $f$ on the points $x_1$ and $x_2$ is

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

**Example 2.1.5.** Let $f(x) = x^2$ be a real-valued function, $x_1 = 2$ and $x_2 = 4$. In this case

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{4^2 - 2^2}{4 - 2} = \frac{16 - 4}{2} = 6.$$

**Remark 2.1.6.** The geometric representation of the difference quotient of the function $f$ on the points $x_0$ and $x$ is the slope of a secant line.




**Example 2.1.7.** The slope of a secant line of the function $f(x) = x^2$ on points $x_1 = 2$ and $x_2 = 4$ is 6.

**Definition 2.1.8.** A function $f\colon I \to \mathbb{R}$ is *differentiable* at the point $x_0 \in I$ if the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

of difference quotients exists. The value of the limit is the derivative or differential quotient of the function at the point $x_0$. Notation: $f'(x_0),\ \dfrac{df}{dx}(x_0),$ $\dot{f}(x_0),\ \dfrac{d}{dx} f(x_0)$. The function $f'\colon x \mapsto f'(x)$ $(x \in I,$ f is differentiable at $x)$ is called the *derivative* of $f$. If $f$ is differentiable at each point of $I$, then we say that it is *differentiable*.

**Example 2.1.9.** The function $f(x) = x^2$ is a differentiable at $x_0 \in \mathbb{R}$, and the derivative function at $x_0$ is $2x_0$, because

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{(x - x_0) \cdot (x + x_0)}{x - x_0} = $$
$$= \lim_{x \to x_0} x + x_0 = 2x_0.$$

**Definition 2.1.10.** If $f\colon I \to \mathbb{R}$ is a function which is differentiable at the point $x_0 \in I$ then the *tangent line* of the function $f$ is

$$y = f(x_0) + f'(x_0) \cdot (x - x_0).$$

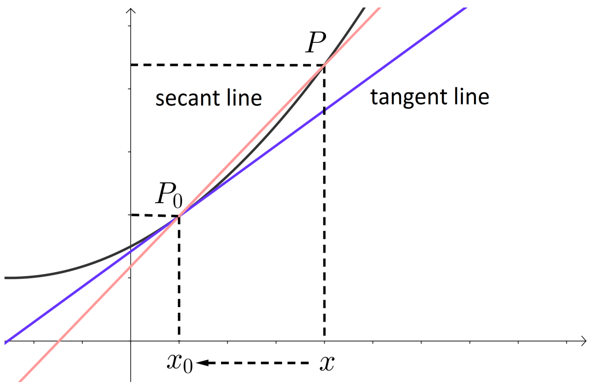
**Example 2.1.11.** If $f(x) = x^2$ and $x_0 = 2$, the the tangent line of $f$ at $x_0$ is

$$y = 4 + 4 \cdot (x - 2),$$

thus

$$y = 4x - 4.$$

**Remark 2.1.12.** The geometric representation of the differential quotient of the function $f$ at the point $x_0$ is the slope of a tangent line.




**Remark 2.1.13.**

| | difference quotient | differential quotient |
|---|---|---|
| definition 1 | $\frac{f(x)-f(x_0)}{x-x_0}$ | $\lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ |
| definition 2 | $\frac{f(x+h)-f(x)}{h}$ | $\lim\limits_{h\to 0} \frac{f(x+h)-f(x)}{h}$ |
| geometric interpretation | slope of the secant line | slope of the tangent line |
| physical interpretation | average change | instantaneous value |

Let $r \in \mathbb{Q}$, $c \in \mathbb{R}$, $0 < a \neq 1$. The derivatives of the elementary functions are in the next table:

| $f(x)$ | $D_f$ | $f'(x)$ | $D_{f'}$ |
|---|---|---|---|
| $c$ | $\mathbb{R}$ | $0$ | $\mathbb{R}$ |
| $x$ | $\mathbb{R}$ | $1$ | $\mathbb{R}$ |
| $x^r$ | $]0;\infty[$ | $r \cdot x^{r-1}$ | $]0;\infty[$ |
| $\sin x$ | $\mathbb{R}$ | $\cos x$ | $\mathbb{R}$ |
| $\cos x$ | $\mathbb{R}$ | $-\sin x$ | $\mathbb{R}$ |
| $\tan x$ | $\mathbb{R} \setminus \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$ | $\frac{1}{\cos^2 x}$ | $\mathbb{R} \setminus \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$ |
| $\cot x$ | $\mathbb{R} \setminus \{\pi + k \cdot \pi \mid k \in \mathbb{Z}\}$ | $-\frac{1}{\sin^2 x}$ | $\mathbb{R} \setminus \{\pi + k \cdot \pi \mid k \in \mathbb{Z}\}$ |
| $\mathrm{e}^x$ | $\mathbb{R}$ | $\mathrm{e}^x$ | $\mathbb{R}$ |
| $a^x$ | $\mathbb{R}$ | $a^x \cdot \ln a$ | $\mathbb{R}$ |
| $\ln x$ | $]0;\infty[$ | $\frac{1}{x}$ | $]0;\infty[$ |
| $\log_a x$ | $]0;\infty[$ | $\frac{1}{x \cdot \ln a}$ | $]0;\infty[$ |
| $\arcsin x$ | $[-1;1]$ | $\frac{1}{\sqrt{1-x^2}}$ | $]-1;1[$ |
| $\arccos x$ | $[-1;1]$ | $-\frac{1}{\sqrt{1-x^2}}$ | $]-1;1[$ |

| | | | |
|---|---|---|---|
| $\arctan x$ | $\mathbb{R}$ | $\frac{1}{1+x^2}$ | $\mathbb{R}$ |
| $\operatorname{arccot} x$ | $\mathbb{R}$ | $-\frac{1}{1+x^2}$ | $\mathbb{R}$ |
| $\sinh x$ | $\mathbb{R}$ | $\cosh x$ | $\mathbb{R}$ |
| $\cosh x$ | $\mathbb{R}$ | $\sinh x$ | $\mathbb{R}$ |
| $\tanh x$ | $\mathbb{R}$ | $\frac{1}{\cosh^2 x}$ | $\mathbb{R}$ |
| $\coth x$ | $\mathbb{R} \setminus \{0\}$ | $-\frac{1}{\sinh^2 x}$ | $\mathbb{R} \setminus \{0\}$ |
| $\operatorname{arsinh} x$ | $\mathbb{R}$ | $\frac{1}{\sqrt{1+x^2}}$ | $\mathbb{R}$ |
| $\operatorname{arcosh} x$ | $[1; \infty[$ | $\frac{1}{\sqrt{x^2-1}}$ | $]1; \infty[$ |
| $\operatorname{artanh} x$ | $]-1; 1[$ | $\frac{1}{1-x^2}$ | $]-1; 1[$ |
| $\operatorname{arcoth} x$ | $]-\infty; -1[\cup]1; \infty[$ | $-\frac{1}{1-x^2}$ | $]-\infty; -1[\cup]1; \infty[$ |

**Theorem 2.1.14.** If the functions $f, g\colon I \to \mathbb{R}$ are differentiable at the point $x_0 \in I$ then $f + g$ is also differentiable at $x_0$ and

$$(f + g)'(x_0) = f'(x_0) + g'(x_0).$$

**Example 2.1.15.** If $u(x) = x^2 + 3$, then

$$u'(x) = (x^2 + 3)' = (x^2)' + 3' = 2x + 0 = 2x.$$

**Theorem 2.1.16.** If the function $f\colon I \to \mathbb{R}$ is differentiable at the point $x_0 \in I$ and $c \in \mathbb{R}$, then $c \cdot f$ is also differentiable at $x_0$ and

$$(c \cdot f)'(x_0) = c \cdot f'(x_0).$$

**Example 2.1.17.** If $u(x) = 5 \sin x$, then

$$u'(x) = (5 \sin x)' = 5 \cdot (\sin x)' = 5 \cos x.$$

**Theorem 2.1.18.** If the functions $f, g\colon I \to \mathbb{R}$ are differentiable at the point $x_0 \in I$ then $f \cdot g$ is also differentiable at $x_0$ and

$$(f \cdot g)'(x_0) = f'(x_0) \cdot g(x_0) + f(x_0) \cdot g'(x_0).$$

**Example 2.1.19.** If $u(x) = x^2 \cdot \sin x$, then

$$u'(x) = (x^2)' \cdot \sin x + x^2 \cdot (\sin x)' = 2x \cdot \sin x + x^2 \cdot \cos x.$$

**Corollary 2.1.20.** If the functions $f, g, h\colon I \to \mathbb{R}$ are differentiable at the point $x_0 \in I$ then $f \cdot g \cdot h$ is also differentiable at $x_0$ and

$$(f \cdot g \cdot h)'(x_0) = f'(x_0) \cdot g(x_0) \cdot h(x_0) + f(x_0) \cdot g'(x_0) \cdot h(x_0) + f(x_0) \cdot g(x_0) \cdot h'(x_0).$$

**Example 2.1.21.** If $u(x) = x^2 \cdot \sin x \cdot 2^x$, then

$$\begin{aligned} u'(x) &= (x^2)' \cdot \sin x \cdot 2^x + x^2 \cdot (\sin x)' \cdot 2^x + x^2 \cdot \sin x \cdot (2^x)' = \\ &= 2x \cdot \sin x \cdot 2^x + x^2 \cdot \cos x \cdot 2^x + x^2 \cdot \sin x \cdot 2^x \cdot \ln 2. \end{aligned}$$

**Corollary 2.1.22.** If $n \in \mathbb{N}$ and the functions $f_1, f_2, \ldots, f_n\colon I \to \mathbb{R}$ are differentiable at the point $x_0 \in I$ then $f_1 \cdot f_2 \cdot \ldots \cdot f_n$ is also differentiable at $x_0$ and

$$\begin{aligned} &(f \cdot f_2 \cdot \ldots \cdot f_n)'(x_0) = f_1'(x_0) \cdot f_2(x_0) \cdot \ldots \cdot f_n(x_0) + \\ &+ f_1(x_0) \cdot f_2'(x_0) \cdot \ldots \cdot f_n(x_0) + \ldots + f_1(x_0) \cdot f_2(x_0) \cdot \ldots \cdot f_n'(x_0). \end{aligned}$$

**Theorem 2.1.23.** If the functions $f, g\colon I \to \mathbb{R}$ are differentiable at the point $x_0 \in I$ and $g(x) \neq 0$ $(x \in I)$, then $\frac{f}{g}$ is also differentiable at point $x_0$ and

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0) \cdot g(x_0) - f(x_0) \cdot g'(x_0)}{\big(g(x_0)\big)^2}.$$

**Example 2.1.24.** If $u(x) = \frac{x^2}{\sin x}$, then

$$u'(x) = \frac{(x^2)' \cdot \sin x - x^2 \cdot (\sin x)'}{\sin^2 x} = \frac{2x \sin x - x^2 \cos x}{\sin^2 x}.$$

**Theorem 2.1.25.** If $I$ and $J$ are open intervals and the function $g\colon I \to J$ is differentiable at the point $x_0$ and the function $f\colon J \to \mathbb{R}$ is differentiable at $g(x_0)$ then $f \circ g\colon I \to \mathbb{R}$ is also differentiable at $x_0$ and

$$(f \circ g)'(x_0) = f'\big(g(x_0)\big) \cdot g'(x_0).$$

**Example 2.1.26.** If $u(x) = \sin(x^2+1)$, then $u = f \circ g$ where:

$$\begin{aligned} f(x) &= \sin x & f'(x) &= \cos x \\ g(x) &= x^2+1 & g'(x) &= 2x. \end{aligned}$$

If we apply the above differentiation rule, we get that

$$u'(x) = \cos(x^2+1) \cdot 2x.$$

**Definition 2.1.27.** If $f\colon I \to \mathbb{R}$ is a differentiable function whose derivative is differentiable at the point $x_0 \in I$, then we say that $f$ is *twice (or 2 times) differentiable* at $x_0$. In this case, the *second derivative* of $f$ at the point $x_0$ is defined by

$$f''(x_0) = (f')'(x_0).$$

If $f$ is twice differentiable at each point of $I$, then it is called *twice (or 2 times) differentiable.*
If for a number $k = 3, 4, \ldots$, the function $f$ is $k-1$ times differentiable, moreover its $(k-1)$-th derivative is differentiable at $x_0 \in I$, then we say that $f$ is *k times differentiable* at $x_0$. In this case, the $k$-th derivative of $f$ at the point $x_0$ is defined by

$$f^{(k)}(x_0) = \left(f^{(k-1)}\right)'(x_0).$$

If $f$ is $k$ times differentiable at each point of $I$, then it is called *k times differentiable*.

**Definition 2.1.28.** A function $f\colon I \to \mathbb{R}$ is called convex if the region in the plane bounded from below by the graph of $f$ is convex. We say that $f$ is concave if $-f$ is convex.

**Theorem 2.1.29.** If $f\colon I \to \mathbb{R}$ is a differentiable function and $f'(x) \geq 0$ $(x \in I)$, then $f$ is monotonically increasing.

If $f\colon I \to \mathbb{R}$ is a differentiable function and $f'(x) \leq 0$ $(x \in I)$, then $f$ is monotonically decreasing.

If $f\colon I \to \mathbb{R}$ is a twice differentiable function and $f''(x) \leq 0$ $(x \in I)$, then $f$ is convex.

If $f\colon I \to \mathbb{R}$ is a twice differentiable function and $f''(x) \leq 0$ $(x \in I)$, then $f$ is concave.

**Definition 2.1.30.** Let $x_0 \in I$ be a point and $f\colon I \to \mathbb{R}$ be a function. We say that $f$ has a local maximum (minimum) at $x_0$ if there is a number $\delta > 0$ such that $f(x) \leq f(x_0)$ $(f(x) \geq f(x_0))$ for each number $x \in I \cap\, ]x_0 - \delta; x_0 + \delta[$.

We say that $f$ has a local extremum at $x_0$ if it has a local maximum or minimum there.

**Theorem 2.1.31.** Let $x_0 \in I$ be a point and $f\colon I \to \mathbb{R}$ be a function which is differentiable at $x_0$. If $f$ has a local extremum at $x_0$, then $f'(x_0) = 0$. If $f'(x_0) = 0$ and there exists a number $\delta > 0$ such that $f|_{]x_0-\delta;x_0[}$ is monotonically increasing and $f|_{]x_0;x_0+\delta[}$ is monotonically decreasing ($f|_{]x_0-\delta;x_0[}$ is monotonically decreasing and $f|_{]x_0;x_0+\delta[}$ is monotonically increasing), then $f$ has a local maximum (minimum) at $x_0$.

**Theorem 2.1.32.** Let $x_0 \in I,\ n = 2, 3, \ldots$ be numbers and $f$ be an $n$ times differentiable function such that

$$f'(x_0) = \ldots = f^{(n-1)}(x_0) = 0, \quad f^{(n)}(x_0) \neq 0.$$

If $n$ is odd, then $f$ has no local extremum at $x_0$. If $n$ is even and $f^{(n)}(x_0) < 0$ ($f^{(n)}(x_0) > 0$), then $f$ has a local maximum (minimum) at $x_0$.

**Solved exercises**

**Exercise 18.** A particle is moving in a straight line. Its position-time function is

$$s(t) = t^3 - 3t\,[\mathrm{m}],$$

where $t$ is the time in seconds.

a) Find a formula for the particle's velocity-time function.

b) Find a formula for the particle's acceleration-time function.

c) Calculate the position, velocity and acceleration of the particle at $t = 0$.

d) Calculate the position, velocity and acceleration of the particle at $t = 1$.

e) Calculate the position, velocity and acceleration of the particle at $t = 2$.

f) Find the position of the particle when it changes the direction of the motion.

g) Give the time interval(s) in which the speed of the particle is increasing.

h) What is the covered distance in the time interval $[0; 2]$?

i) Find the average velocity in the time interval $[0; 1]$.

j) Find the average acceleration in the time interval $[0; 1]$.

**Solution:**

a) The velocity-time function is the derivative of the position-time function

$$v(t) = s'(t) = 3t^2 - 3.$$

b) The acceleration-time function is the derivative of the velocity-time function

$$a(t) = v'(t) = 6t.$$

c) If $t = 0$ then

$$s(0) = 0^3 - 3 \cdot 0 = 0\,[\mathrm{m}]$$
$$v(0) = 3 \cdot 0^2 - 3 = -3\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$$
$$a(0) = 6 \cdot 0 = 0\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right].$$

d) If $t = 1$ then

$$s(1) = 1^3 - 3 \cdot 1 = -2\,[\mathrm{m}]$$
$$v(1) = 3 \cdot 1^2 - 3 = 0\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$$
$$a(1) = 6 \cdot 1 = 6\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right].$$

e) If $t = 2$ then

$$s(2) = 2^3 - 3 \cdot 2 = 2\,[\mathrm{m}]$$
$$v(2) = 3 \cdot 2^2 - 3 = 12 - 3 = 9\left[\frac{\mathrm{m}}{\mathrm{s}}\right]$$
$$a(2) = 6 \cdot 2 = 12\left[\frac{\mathrm{m}}{\mathrm{s}^2}\right].$$

f) The solution of the equation $v(t) = 0$:

$$3t^2 - 3 = 0 \qquad \Rightarrow \qquad t = \pm 1.$$

Since the sign of the function $v(t)$ changes when $t = 1\,[\mathrm{s}]$, a change in the direction of the motion occurs at that instant. In our case

$$s(1) = 1^3 - 3 \cdot 1 = -2,$$

thus when $s = -2\,[\mathrm{m}]$ the direction of the motion of the particle changes.

g) The speed is increasing when $v(t)$ and $a(t)$ have the same sign. Since $a(t) \geq 0$ when $t \geq 0$ and $v(t) \geq 0$ exactly when $t \geq 1$, we get that the speed is increasing when $t \geq 1$.

h) The total covered distance is

$$\big|s(1) - s(0)\big| + \big|s(2) - s(1)\big| = \big| - 2 - 0\big| + \big|2 - (-2)\big| = 6\,[\mathrm{m}].$$

i) Since $s(1) = -2$ and $s(0) = 0$ the average velocity in time interval $[0; 1]\,[\mathrm{s}]$ is

$$\frac{s(1) - s(0)}{1 - 0} = \frac{-2 - 0}{1 - 0} = -2\left[\frac{\mathrm{m}}{\mathrm{s}}\right].$$

j) Since $v(1) = 0$ and $v(0) = -3$ the average acceleration in the time interval $[0; 1]\,[\mathrm{s}]$ is

$$\frac{v(1) - v(0)}{1 - 0} = \frac{0 - (-3)}{1 - 0} = 3\left[\frac{\mathrm{m}}{\mathrm{s}}\right].$$

**Exercise 19.** A particle moves in a straight line according to the position-time function

$$s(t) = t^3 - 7t^2 + 11t - 3\,[\text{cm}],$$

where $t$ is the time in seconds.

a) Find a formula for the particle's velocity-time function.

b) Find a formula for the particle's acceleration-time function.

c) Find the instant(s) when the velocity is equal to zero.

d) Calculate the position, velocity and acceleration of the particle at $t = 0$.

e) Calculate the position, velocity and acceleration of the particle at $t = 1$.

f) Calculate the position, velocity and acceleration of the particle at $t = 2$.

g) Describe the monotonicity properties of the function $s(t)$.

h) Calculate the covered distance in the time interval $[0; 3]$.

**Solution:**

a) The velocity-time function is the time derivative of the position-time function

$$v(t) = s'(t) = 3t^2 - 14t + 11.$$

b) The acceleration-time function is the time derivative of the velocity-time function

$$a(t) = v'(t) = 6t - 14.$$

c) We have to solve the equation

$$3t^2 - 14t + 11 = 0.$$

Applying the quadratic formula, we get that

$$t_{1,2} = \frac{14 \pm \sqrt{196 - 132}}{6} = \frac{14 \pm 8}{6}.$$

Thus the solutions of the quadratic equation are $t = 1$ [s] and $t = \frac{11}{3}$ [s].

d) If $t = 0$ then

$$s(0) = 0^3 - 7 \cdot 0^2 + 11 \cdot 0 - 3 = -3\,[\text{cm}];$$
$$v(0) = 3 \cdot 0^2 - 14 \cdot 0 + 11 = 11 \left[\frac{\text{cm}}{\text{s}}\right];$$
$$a(0) = 6 \cdot 0 - 14 = -14 \left[\frac{\text{cm}}{\text{s}^2}\right].$$

Thus the particle is at a distance of $3\,[\mathrm{cm}]$ from the point $O$ on its left-hand side and it is moving to the right at a speed of $11\left[\frac{\mathrm{cm}}{\mathrm{s}}\right]$.

e) If $t = 1$, we get that

$$s(1) = 1^3 - 7 \cdot 1^2 + 11 \cdot 1 - 3 = 2\,[\mathrm{cm}];$$
$$v(1) = 3 \cdot 1^2 - 14 \cdot 1 + 11 = 0\left[\frac{\mathrm{cm}}{\mathrm{s}}\right];$$
$$a(1) = 6 \cdot 1 - 14 = -8\left[\frac{\mathrm{cm}}{\mathrm{s}^2}\right].$$

f) If $t = 2$, we get that

$$s(2) = 2^3 - 7 \cdot 2^2 + 11 \cdot 2 - 3 = -1\,[\mathrm{cm}];$$
$$v(2) = 3 \cdot 2^2 - 14 \cdot 2 + 11 = -5\left[\frac{\mathrm{cm}}{\mathrm{s}}\right];$$
$$a(2) = 6 \cdot 2 - 14 = -2\left[\frac{\mathrm{cm}}{\mathrm{s}^2}\right].$$

g) The monotonicity properties of the function $s(t)$ are as follows.

| | $t<1$ | $t=1$ | $1<t<\frac{11}{3}$ | $t=\frac{11}{3}$ | $t>\frac{11}{3}$ |
|---|---|---|---|---|---|
| $s'(t)$ | $+$ | 0 | $-$ | 0 | $+$ |
| $s(t)$ | $\nearrow$ | loc. max. | $\searrow$ | loc. min. | $\nearrow$ |

h) The total distance covered by the particle in the time interval [0;3] is

$$s = [s(1) - s(0)] + [s(3) - s(1)]\,.$$

Since

$$s(1) - s(0) = 2 - (-3) = 5,$$

and

$$s(3) - s(1) = -6 - 2 = -8,$$

consequently the total covered distance is

$$5 + 8 = 13\,[\mathrm{m}].$$

**Exercise 20.** A particle is moving in a straight line, according to the position-time function

$$s(t) = t^2 - 4t + 2\,[\mathrm{m}],$$

where $t$ is the time in seconds.

a) Find the average velocity in the time interval $[2;5]\,[\mathrm{s}]$.

b) Find a formula for the particle's velocity-time function.
c) Find the average acceleration in the time interval $[2; 5]$ [s].
d) Find a formula for the particle's acceleration-time function.
e) Calculate the position, velocity and acceleration of the particle at $t = 0$ and describe its motion at the above instant.
f) Calculate the position, velocity and acceleration of the particle at $t = 1$ and describe its motion at the above instant.
g) Describe the motion of the particle at $t = 2$ [s].
h) Find the position of the particle at that moment when its direction of motion changes.
i) Plot the position-time, velocity-time and acceleration-time functions in the time interval $[0; 6]$ in the same coordinate system.

**Solution:**

a) Since

$$s(5) = 5^2 - 4 \cdot 5 + 2 = 25 - 20 + 2 = 7$$

and

$$s(2) = 2^2 - 4 \cdot 2 + 2 = 4 - 8 + 2 = -2$$

thus the average velocity in the time interval $[2; 5]$ [s] is

$$\frac{s(5) - s(2)}{5 - 2} = \frac{7 - (-2)}{3} = 3 \left[\frac{\text{m}}{\text{s}}\right].$$

b) The velocity-time function is the derivative of the position-time function

$$v(t) = s'(t) = 2t - 4.$$

c) Since

$$v(5) = 2 \cdot 5 - 4 = 6$$

and

$$v(2) = 2 \cdot 2 - 4 = 0$$

thus the average acceleration in the time interval $[2; 5]$ [s]

$$\frac{v(5) - v(2)}{5 - 2} = \frac{6 - 0}{3} = 2 \left[\frac{\text{m}}{\text{s}^2}\right].$$

d) The acceleration-time function is the derivative of the velocity-time function

$$a(t) = v'(t) = 2.$$

e) When $t = 0$

$$s(0) = 0^2 - 4 \cdot 0 + 2 = 2\,[\mathrm{m}];$$
$$v(0) = 2 \cdot 0 - 4 = -4 \left[\frac{\mathrm{m}}{\mathrm{s}}\right];$$
$$a(0) = 2 \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right].$$

We get that the particle is at a distance of $2$ $[\mathrm{m}]$ to the right of the point $O$ and moving to the left at a speed of $4 \left[\frac{\mathrm{m}}{\mathrm{s}}\right]$.

f) When $t = 1$

$$s(1) = 1^2 - 4 \cdot 1 + 2 = -1\,[\mathrm{m}];$$
$$v(1) = 2 \cdot 1 - 4 = -2 \left[\frac{\mathrm{m}}{\mathrm{s}}\right];$$
$$a(1) = 2 \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right].$$

We get that the particle is at a distance of $1$ $[\mathrm{m}]$ to the left of the point $O$ and moving to the left at a speed of $2 \left[\frac{\mathrm{m}}{\mathrm{s}}\right]$.

g) When $t = 2$

$$s(2) = 2^2 - 4 \cdot 2 + 2 = -2\,[\mathrm{m}];$$
$$v(2) = 2 \cdot 2 - 4 = 0 \left[\frac{\mathrm{m}}{\mathrm{s}}\right];$$
$$a(2) = 2 \left[\frac{\mathrm{m}}{\mathrm{s}^2}\right].$$

We get that the particle is at a distance of $2$ $[\mathrm{m}]$ to the left of the point $O$ and it is not moving.

h) The solution of equation $v(t) = 0$:

$$2t - 4 = 0 \qquad \Rightarrow \qquad t = 2.$$

The function $v(t)$ changes sign when $t = 2\,[\mathrm{s}]$, and

$$s(2) = -2,$$

so the particle changes direction at a distance of $2\,[\mathrm{m}]$ to the left of the point $O$.

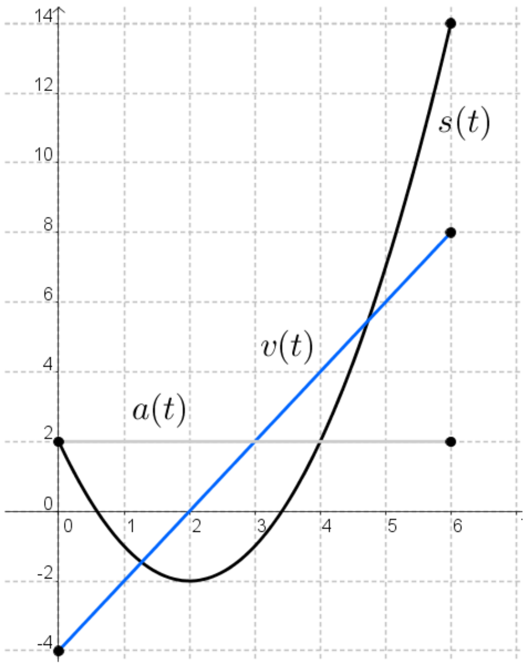
i) The position-time function is

$$s(t) = t^2 - 4t + 2 = (t-2)^2 - 4 + 2 = (t-2)^2 - 2.$$

Thus the graph of the function $s(t)$ is a parabola.
The velocity-time function is

$$v(t) = 2t - 4.$$

Thus the graph of the function $v(t)$ is a line.
The acceleration-time function is a constant function.
The position-time, velocity-time and acceleration-time functions are



# 4. Applications of the least squares method

## Theoretical summary

**Remark 2.2.1.** The least squares method is usually credited to Carl Friedrich Gauss but it was first published by Adrien-Marie Legendre.

The first clear and concise exposition of the method of least squares was published by Legendre in $1\,805$. The technique is described as an algebraic procedure for fitting linear equations to data and Legendre demonstrated the new method by analyzing the same data as Laplace for the shape of the Earth. The value of Legendre's method of least squares was immediately recognized by leading astronomers and geodesists of the time.

The least squares method has a wide range of applications for the regression of data point series. The model function, which is applied for the regression usually has theoretical basis.

In the following we deal with the simplest case, where there is only one unknown parameter in the model function, so it can be written in the following form:

$$f(x) = f(x, a).$$

When applying the procedure our aim is to determine that value of the parameter $a$ at which the model function gives the best fit to the given series of data points.

In the method the aim is to find the minimum of the following function:

$$H(a) = \sum_{k=1}^{n} \bigl(y_k - f(x_k, a)\bigr)^2$$

for given data points

$$(x_1; y_1), (x_1; y_1), \ldots, (x_n; y_n).$$

This method is called *the least squares method*.

The minimum value can be found by applying the methods that were presented in the section about the calculation of extreme values of real functions.

**Solved exercises**

**Exercise 21.** The propagation of bacteria can be modelled by the function

$$N(t) = N(t, a) = a \cdot 2^t.$$

To determine the number of bacteria, we perform measurements on consecutive days. The results are summarized in the table:

| t[day] | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| N[thousand pieces] | 2.1 | 4 | 7.9 | 16.1 |

Applying the least squares method determine the unknown parameter $a$ and plot the function.

**Solution:**

The function $H$ is

$$H(a) = (2.1 - 2a)^2 + (4 - 4a)^2 + (7.9 - 8a)^2 + (16.1 - 16a)^2.$$

The derivative of the function $H$ is

$$H'(a) = 2 \cdot (2.1 - 2a) \cdot (-2) + 2 \cdot (4 - 4a) \cdot (-4) + \\ + 2 \cdot (7.9 - 8a) \cdot (-8) + 2 \cdot (16.1 - 16a) \cdot (-16).$$

Simplifying the formula above, we get that

$$H'(a) = 680a - 682.$$

We can determine the value of the parameter $a$ from the equation:

$$680a - 682 = 0 \qquad \Rightarrow \qquad a = \frac{682}{680} = 1.0029.$$

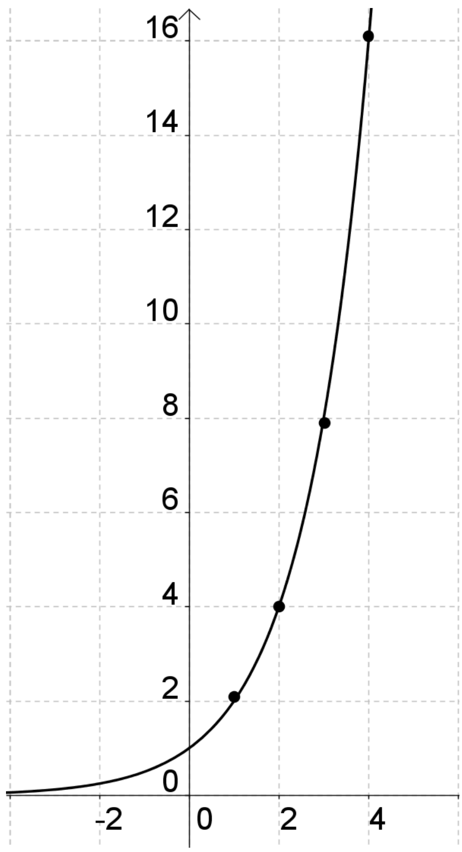
Since

$$H''(a) = 680 > 0,$$

therefore the function $H$ has a minimum at the above value, thus the model function $N$ is

$$N(t) = 1.0029 \cdot 2^t.$$

The figure below shows the function and the points:




**Exercise 22.** The annual profit of a dynamically developing company has been registered in the first four years after its establishment. The obtained data are summarized in the table:

| t[year] | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| P[m $] | 2 | 3 | 6 | 9 |

The experts are assuming a linear relationship between the profit and time. On the basis of the above assumption, applying the least squares method, determine the unknown parameter $m$ in the function below, and then estimate the profit of the company five years after its establishment:

$$P(t) = P(t, m) = m \cdot t.$$

Plot the model function.

**Solution:**

By substituting the obtained data, we get that the function $H(m)$ is

$$H(m) = (2 - m)^2 + (3 - 2m)^2 + (6 - 3m)^2 + (9 - 4m)^2.$$

The derivative of the function above is

$$\begin{aligned} H'(m) = 2 \cdot (2 - m) \cdot (-1) + 2 \cdot (3 - 2m) \cdot (-2) + \\ + 2 \cdot (6 - 3m) \cdot (-3) + 2 \cdot (9 - 4m) \cdot (-4). \end{aligned}$$

Simplifying the above function, we get that

$$H'(m) = 60m - 124.$$

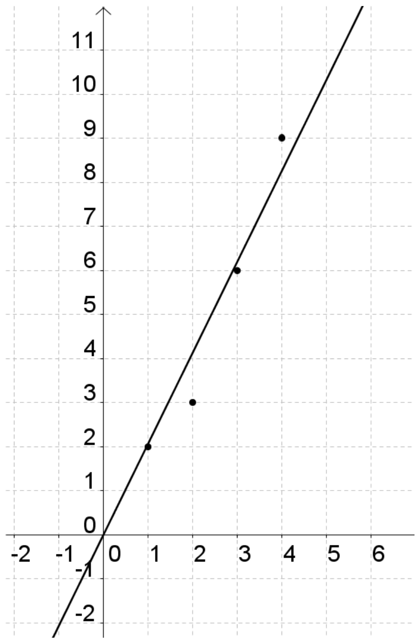
The zero of the function above:

$$60m - 124 = 0 \qquad \Rightarrow \qquad 60m = 124 \qquad \Rightarrow \qquad m = \frac{124}{60} = \frac{31}{15}.$$

Consequently the value of the unknown parameter $m$ is $m = \frac{31}{15}$, thus function the $P$ is

$$P(t) = \frac{31}{15} \cdot t.$$

The graph of the function $P$ is




If $t = 5$ years, we get that

$$P(5) = \frac{31}{15} \cdot 5 = \frac{31}{3} \approx 10.33.$$

Thus the estimation of the profit after five years is 10.33 million dollars.

**Exercise 23.** We intend to measure the electric resistance of the stator coil of an electric motor. In the course of the measurements electric current is flowing through the coil with different intensities and the voltage drop across the coil is measured. The measurement results are summarized in the table:

| I[A] | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| U[V] | 0.1 | 0.22 | 0.3 | 0.41 | 0.49 |

Determine the electric resistance of the stator coil applying the least squares method and Ohm's law.

**Solution:**

Ohm's law can be written in the following form:

$$U = R \cdot I,$$

where $I$ is the intensity of the current flowing through the resistor (coil), $U$ is the voltage through the resistor (across the coil) and $R$ is the electric resistance of the resistor (coil). In the given situation the model function can be written as:

$$U(I) = U(I, R) = R \cdot I.$$

Substituting the data, for the function $H(R)$ (see in the theoretical summary), we get that:

$$H(R) = (0.1 - R)^2 + (0.22 - 2R)^2 + (0.3 - 3R)^2 + \\ + (0.41 - 4R)^2 + (0.49 - 5R)^2.$$

The derivative of the function above is:

$$H'(R) = -2 \cdot (0.1 - R) - 4 \cdot (0.22 - 2R) - 6 \cdot (0.3 - 3R) - \\ - 8 \cdot (0.41 - 4R) - 10 \cdot (0.49 - 5R).$$

The minimum of the function $H(R)$ can be found where its derivative function is zero, thus we have to solve the equation

$$-11.06 + 110R = 0 \qquad \Rightarrow \qquad R \approx 0.1005\,[\Omega].$$

Since

$$H''(R) = 110 > 0,$$

thus the function $H$ has a minimum value at $R \approx 0.1005$. Thus the function $U(I)$ is

$$U(I) = 0.1005 \cdot I.$$

## 5. Differential calculus in economics, marginal cost, marginal revenue

### Theoretical summary

**Remark 2.3.1.** Calculus is applied in basic economic theory in marginal analysis. Economists analyze how small changes, for example increasing the production of a product by a single unit, affect profits, and costs. Marginal analysis quantifies the benefits of performing such an action against the costs.

When the benefits or profits exceed the cost of the action, you can proceed on this course until this situation changes.

**Remark 2.3.2.** The break-even point occurs when the production costs and the total revenue, i.e., the amount of income generated before any deductions are made, are the same.

**Remark 2.3.3.** Notation in this section:

- $x$: number of units produced or sold;
- $P(x)$: *demand* as a function of amount $x$ in units;
- $S(x)$: *supply* as a function of amount $x$ in units;
- $R(x)$: *total revenue* from selling the amount $x$ units;
- $FC$: *fix cost*;
- $VC(x)$: *variable cost*;
- $AFC(x) = \frac{FC}{x}$: *average fix cost*;
- $AVC(x) = \frac{VC(x)}{x}$: *average variable cost*;
- $AC(x) = \frac{C(x)}{x}$: *average total cost*;
- $C(x)$: *total cost* of producing the amount $x$ units;
- $\Pi(x)$: *profit* from selling the amount $x$ units;
- $R'(x)$: *marginal revenue*, which is the extra revenue for selling one extra unit;
- $MC(x) = C'(x)$: *marginal cost* or *marginal cost function*, which is the extra cost for selling one extra unit;
- $M\Pi(x) = \Pi'(x)$: *marginal profit* or *marginal profit function*, which is the extra profit from selling one additional unit;
- $MR(x) = R'(x)$: *marginal revenue* or *marginal revenue function.*

It is clear that

- $C(x) = FC + VC(x)$;
- $AC(x) = AFC + AVC(x)$.

**Definition 2.3.4.** In economics, *demand* is the quantity of a commodity or a service that people are willing or able to buy at a certain price, per unit of time.

In economics, *supply* is the amount of something that firms, consumers, labourers, providers of financial assets, or other economic agents are willing to provide to the marketplace.

The market will reach *equilibrium*, when the quantity demanded and the quantity supplied are the same.

**Definition 2.3.5.** In economics, *elasticity* is the measurement of how an economic variable responds to a change in another. In economics, elasticity is used to determine how changes in product demand and supply related to changes in consumer income or the producer's price.

**Theorem 2.3.6.** The formula for elasticity is

$$E(x) = \frac{x}{f(x)} \cdot f'(x).$$

**Solved exercises**

**Exercise 24.** The cost of manufacturing fishing poles, measured in thousand units, is modeled by

$$C(x) = 3x^3 - 30x^2 + 60x \qquad (0 \leq x \leq 10).$$

The revenue function is modeled by

$$R(x) = 21x + 9.$$

a) Find the fix cost function.
b) Find the variable cost function.
c) Find the profit function.
d) Find the production level that maximize profits.
e) Calculate the marginal cost function.
f) Find the production level, if it exists, that minimizes cost.
g) Find the average cost function.
h) Find the production level, if it exists, that minimizes average cost.

<u>**Solution:**</u>

a) The fix cost function is $0$.

b) The variable cost function is

$$VC(x) = 3x^3 - 30x^2 + 60x.$$

c) The profit function is

$$\Pi(x) = R(x) - C(x) = 21x + 9 - (3x^3 - 30x^2 + 60x) =$$
$$= -3x^3 + 30x^2 - 39x + 9.$$

d) We have to solve the equation $\Pi'(x) = 0$, that is

$$-9x^2 + 60x - 39 = 0 \qquad \Rightarrow \qquad -3x^2 + 20x - 13 = 0.$$

Applying the quadratic formmula, we get that

$$x_{1,2} = \frac{-20 \pm \sqrt{400 - 12 \cdot 13}}{-6}.$$

The solutions of the equation $\Pi'(x) = 0$ are $x_1 \approx 0.73$ and $x_2 \approx 5.94$. The second derivative of the function $\Pi(x)$ is $\Pi''(x) = -18x + 60$.

Since $\Pi''(5.94) < 0$, the maximum profit occurs at the production level of 5 490 fishing poles.

e) The marginal cost function is

$$MC(x) = C'(x) = 9x^2 - 60x + 60.$$

f) We have to solve the equation $C'(x) = 0$, that is

$$9x^2 - 60x + 60 = 0 \qquad \Rightarrow \qquad 3x^2 - 20x + 20 = 0.$$

Applying the quadratic formula, we get that

$$x_{1,2} = \frac{20 \pm \sqrt{400 - 240}}{6} \approx \frac{20 \pm 12.65}{6}.$$

The solutions of the quadratic equation are $x_1 \approx 5.44$ and $x_2 \approx 1.225$. Since $C''(x) = 18x - 60$ and $C''(5.44) > 0$, thus we get that the minimum cost occurs at a production level of approximately 5 440 fishing poles.

g) The average cost function is

$$AC(x) = \frac{C(x)}{x} = \frac{3x^3 - 30x^2 + 60x}{x} = 3x^2 - 30x + 60.$$

h) The derivative function of the average cost is $AC'(x) = 6x - 30$. The solution of the equation $6x - 30 = 0$ is $x = 5$, that is the production level at which the average cost is minimal is $x = 5$.

**Exercise 25.** Suppose that the demand function is

$$f(x) = \frac{200}{x+5}.$$

Find its elasticity function. The price of the product is 5 dollars. By how many percents does the demand change when we increase the price by 1 percent or decrease the price by 5 percents.

**Solution:**

The derivative of the function $f$ is

$$f'(x) = \frac{-200}{(x+5)^2}.$$

The elasticity function is

$$E(x) = \frac{x}{f(x)} f'(x) = \frac{x}{\frac{200}{x+5}} \cdot \frac{-200}{(x+5)^2} = x \cdot \frac{x+5}{200} \cdot \frac{-200}{(x+5)^2} = \frac{-x}{x+5}.$$

The price of the product is $5$ dollars, thus we calculate $E(5)$:

$$E(5) = \frac{-5}{5+5} = -\frac{1}{2}.$$

If the price increases by $1$ percent then the demand decreases by $0.5$ percents. If the price decreases by $5$ percent, then the demand inreases by $5 \cdot 0.5 = 2.5$ percents.

**Exercise 26.** The revenue function is $R(x) = 8\sqrt{x}$. The total cost function is

$$C(x) = x^2.$$

The units are measured in thousand pieces. The revenue and cost are measured in thousand dollars.

Calculate the number of units, when the profit is maximal. Find the maximum value of the profit function.

**Solution:**

The profit function is

$$\Pi(x) = R(x) - C(x) = 8\sqrt{x} - x^2.$$

The derivative of the function $\Pi(x)$ is

$$\Pi'(x) = 8 \cdot \frac{1}{2} \cdot x^{-\frac{1}{2}} - 2x = \frac{4}{\sqrt{x}} - 2x.$$

The solution of the equation $\Pi'(x) = 0$ is

$$\frac{4}{\sqrt{x}} - 2x = 0 \qquad \Rightarrow \qquad 4 = 2x \cdot \sqrt{x}.$$

Applying an algebraic identity, we get that

$$16 = 4x^3 \qquad \Rightarrow \qquad x \approx 1.59.$$

The second derivative of the function $\Pi(x)$ is

$$\Pi''(x) = -2x^{-\frac{3}{2}} - 2$$

which is negative at the latter value, thus the company has to produce approximately $1\,590$ units. Since

$$\Pi(1.59) \approx 7.5595,$$

the maximal profit is $7\,560$ dollars.

**Exercise 27.** Consider the revenue function

$$R(x) = x \cdot \sqrt{13\,800 - 0.2x},$$

where $x$ is the number of units.

Calculate the number of units to be sold to maximize the revenue. Find the maximum value of the revenue function.

**Solution**:

The derivative of the function $R(x)$ is

$$R'(x) = \sqrt{13\,800 - 0.2x} + x \cdot \frac{1}{2} \cdot (13\,800 - 0.2x)^{-\frac{1}{2}} \cdot (-0.2) =$$
$$= \sqrt{13\,800 - 0.2x} - \frac{0.1x}{\sqrt{13\,800 - 0.2x}}.$$

The equation $R'(x) = 0$ is

$$\sqrt{13\,800 - 0.2x} - \frac{0.1x}{\sqrt{13\,800 - 0.2x}} = 0.$$

By writing the equation in an equvivalent form, we get $13\,800 - 0.3x = 0$, that is $x = 46\,000$. The table of the signs of the function $R'$ is as follows:

| | $0 < x < 46\,000$ | $x = 46\,000$ | $x > 46\,000$ |
|---|---|---|---|
| $R'(x)$ | $+$ | $0$ | $-$ |
| $R(x)$ | $\nearrow$ | loc. max. | $\searrow$ |

We get that the company has to produce $46\,000$ units. Then the maximal revenue is

$$R(46\,000) = 46\,000 \cdot \sqrt{13\,800 - 0.2 \cdot 46\,000} \approx 3\,119\,872.$$

**Exercise 28.** The demand function of a product is $D(x) = 126 - 0.003x$ and the cost function is $C(x) = 90x + 3\,600$, where $x$ is the number of units demanded.

a) Give the marginal cost function.

b) Determine the revenue function.

c) Find the marginal revenue function.

d) Give the profit function.

e) Determine the marginal profit function.

f) Calculate the profit for $1\,000$ units.

g) Calculate the number of units that maximizes the profit.

**Solution:**

a) The marginal cost function is
$$MC(x) = C'(x) = 90.$$

b) The revenue function is
$$R(x) = x \cdot D(x) = x \cdot (126 - 0.003x) = 126x - 0.003x^2.$$

c) The marginal revenue function is $MR(x) = R'(x) = 126 - 0.006x$.

d) The profit function is
$$\Pi(x) = R(x) - C(x) = 126x - 0.003x^2 - (90x + 3\,600) =$$
$$= -0.003x^2 + 36x - 3\,600.$$

e) The marginal profit function is $\Pi'(x) = -0.006x + 36$.

f) The profit for 1 000 units is
$$\Pi(1\,000) = -0.003 \cdot 1\,000^2 + 36 \cdot 1\,000 - 3\,600 = 29\,400.$$

g) The solution of the equation $\Pi'(x) = 0$:
$$-0.006x + 36 = 0 \qquad \Rightarrow \qquad x = 6\,000.$$

Since $\Pi''(x) = -0.006 < 0$, the profit function attains its maximum when the company produces 6 000 units. The maximum profit is
$$\Pi(6\,000) = -0.003 \cdot 6\,000^2 + 36 \cdot 6\,000 - 3\,600 = 104\,400.$$

**Exercise 29.** Suppose that the total cost, in dollars, of producing $x$ cell phones is
$$C(x) = 12\,000 - 60x + 2x^3.$$

Find the minimum average cost.

**Solution:**

The average cost function is
$$AC(x) = \frac{12\,000}{x} - 60 + 2x^2.$$

The derivative of function $AC(x)$ is
$$AC'(x) = -\frac{12\,000}{x^2} + 4x.$$

The solution of the equation $AC'(x) = 0$:

$$-\frac{12\,000}{x^2} + 4x = 0 \qquad \Rightarrow \qquad -12\,000 + 4x^3 = 0 \qquad \Rightarrow x \approx 14.42,$$

furthermore

$$AC''(x) = \frac{24\,000}{x^3} + 4,$$

thus

$$AC''(14.42) = \frac{24\,000}{3\,000} + 4 = 12 > 0,$$

that is the function $AC$ has a minimum at $x \approx 14.42$. Thus the company has to produce approximately 14 cell phones.

**Exercise 30.** An apartment complex has 250 apartments to rent. If they rent $x$ apartments then their monthly profit, in dollars, is given by

$$P(x) = -16x^2 + 6\,400x - 160\,000.$$

How many apartments should they rent in order to maximize their profit?

**Solution:**

All that we are really being asked to do here is to maximize the profit subject to the constraint that $x$ must be in the range $0 \leq x \leq 250$. The first derivative of $P$ is

$$P'(x) = -32x + 6\,400.$$

The zero $x$ of the function $P'$ satisfies

$$-32x + 6\,400 = 0 \qquad \Rightarrow \qquad x = 200.$$

Since the profit function is continuous and we have an interval with finite bounds we can find the maximum value by simply plugging in the only critical point that we have and the end points of the range:

$$P(0) = -16 \cdot 0^2 + 6\,400 \cdot 0 - 160\,000 = -160\,000$$

$$P(200) = -16 \cdot 200^2 + 6\,400 \cdot 200 - 160\,000 = 480\,000$$

$$P(250) = -16 \cdot 250^2 + 6\,400 \cdot 250 - 160\,000 = 440\,000.$$

The profit function attains its maximum, if they only rent out 200 of the apartments instead of all 250 of them.

**Exercise 31.** The demand function is

$$f(x) = x^2 \cdot \mathrm{e}^{-x^2}.$$

Calculate its elasticity function at $x = 2$.

**Solution:**

The derivative function of $f$ is

$$f'(x) = 2x\mathrm{e}^{-x^2} + x^2 \cdot \mathrm{e}^{-x^2} \cdot (-2x) = \mathrm{e}^{-x^2} \cdot (2x - 2x^3).$$

The elasticity function is

$$E(x) = \frac{x}{f(x)} \cdot f'(x) = \frac{x}{x^2 \cdot \mathrm{e}^{-x^2}} \cdot \mathrm{e}^{-x^2} \cdot (2x - 2x^3) =$$

$$= \frac{2x - 2x^3}{x} = 2 - 2x^2.$$

The elasticity function at $x = 2$ is

$$E(2) = 2 - 2 \cdot 2^2 = -6.$$

If we increase the price by one percent, the demand decreases by 6 percents.

## 6. Extremum problems in geometry, engineering methods and economics

### Theoretical summary

**How can we solve optimization problems?**

1. Read the problem. Collect the given data. Identify the quantity to be optimized.
2. In case of a geometric problem, you can make it much simpler by drawing a figure.
3. Introduce appropriate variables. Formulate all the relevant relations given in the problem and, in case of a geometric task, the ones that can be deduced from the mentioned figure as equations for the variables. Using the obtained equations, express all variables in terms of one of them.
4. Determine a formula for the quantity to be optimized in terms of the latter variable. In this way, that quantity becomes a single variable function $f$ defined on an interval $[a; b]$ $(a \leq b)$.
5. Compute the derivative function of $f$.
6. Find the critical points of $f$, i.e., the the solutions of the equation $f'(x) = 0$ $(a < x < b)$.
7. Make a table of the value of $f$ at the endpoints of its domain and at the critical points.
8. Select the largest or the smallest value in the table.

We remark that the last two steps of this procedure form the Closed Interval Method.

In some optimization problems, the domain of the function $f$ above is not a bounded and closed interval, so the previous method can not be used in such cases. However, there is another way to give the extremum point which works also for that kind of problems.

1. Do the first six steps of the procedure above.
2. Find the second derivative of the function $f$.
3. Let $x_0$ be a critical point of $f$. If $f''(x_0) > 0$, then $x_0$ is a local minimum. If $f''(x_0) < 0$, then $x_0$ is a local maximum.

**Solved exercises**

**Exercise 32.** A farmer has $1\,600\,\mathrm{m}$ of fencing and wants to fence off a rectangular field bounded by a straight river. It needs no fence along the river. Among such fields, what are the dimensions of the one that has the largest area?

**Solution:**

Let's denote the sides of the rectangle by $x$ and $y$.

The total length of the fence is $P = 2y + x$.

Using the equation $P = 1\,600$, we get

$$2y + x = 1\,600 \qquad \Rightarrow \qquad x = 1\,600 - 2y.$$

The area of the rectangle is

$$A = x \cdot y = (1\,600 - 2y) \cdot y = 1\,600y - 2y^2.$$

Thus the function we intend to maximize is

$$A(y) = 1\,600y - 2y^2.$$

Note that $y \geq 0$ and $y \leq 800$, thus the domain of function $A$ is the interval $[0; 800]$.
The derivative of the function $A$ is

$$A'(y) = 1\,600 - 4y,$$

thus to find the extreme values we have to solve the equation $A'(y) = 0$, that is $1\,600 - 4y = 0$.

The solution of the equation is $y = 400$.

The maximum value of $A$ is attained at $400$ or at the endpoints of the domain of the function $A$. Since

$$A(0) = 0 \quad \text{and} \quad A(400) = 320\,000 \quad \text{and} \quad A(800) = 0,$$

the Closed Interval Method gives that the maximum value is attained at 400 and it is

$$A(400) = 320\,000.$$

Hence the dimensions of the field with the largest area are 800 and 400 meters.

**Exercise 33.** A farmer has $800\,[\mathrm{m}]$ of fencing and wants to fence off a rectangular field bounded by a straight river. It needs no fence along the river. Among such fields, what are the dimensions of the one that has the largest area?

**Solution:**

Let's denote the sides of the rectangle by $x$ and $y$.

The total length of the fence is

$$P = 2y + x.$$

From the previous equation, we get that

$$2y + x = 800 \qquad \Rightarrow \qquad x = 800 - 2y.$$

The area of the rectangle is

$$A = x \cdot y = (800 - 2y) \cdot y = 800y - 2y^2.$$

Thus the function we intend to maximize is

$$A(y) = 800y - 2y^2.$$

Note that $y \geq 0$ and $y \leq 400$, thus the domain of function $A$ is the interval $[0; 400]$.

The derivative of the function $A$ is $A'(y) = 800 - 4y$, so to find the extrema, we have to solve the equation $A'(y) = 0$, that is, $800 - 4y = 0$.

The solution of the equation above is $y = 200$.

We compute $A''(y) = -4 < 0$ for all $y$, so $A$ is concave and the local maximum at $y = 200$ is an absolute maximum (here we have used the fact that local maxima of concave functions are global).

Thus the dimensions of the rectangular field with the maximum area are 200 and 400 meters. The maximum area is

$$A = x \cdot y = 400 \cdot 200 = 80\,000\,[\mathrm{m}^2].$$

**Exercise 34.** The editor of a publishing house is designing the layout of a book. The planned margins are $2\,\mathrm{cm}$ wide on the top, on the bottom and on the outer edge of the pages, but the inner margins have to be 4 cm wide because of the binding. The total area of a page is $600\,[\mathrm{cm}^2]$. How should the editor set the dimensions of the pages in order to have the maximal printing area?

**Solution:**

Let's denote the sides of the page by $x$ and $y$. The area of the paper is $x \cdot y =$

600.

Expressing the variable $y$, we have

$$y = \frac{600}{x}.$$

The printing area is

$$\begin{aligned} f(x) &= (x-6)\cdot(y-4) = xy - 4x - 6y + 24 = \\ &= 600 - 4x - \frac{3600}{x} + 24 = 624 - 4x - \frac{3600}{x}. \end{aligned}$$

The domain of the function $f(x)$ is $[6; 150]$.

The first derivative of the function $f(x)$ is

$$f'(x) = -4 + \frac{3600}{x^2}.$$

Solving the equation $f'(x) = 0$, we get that

$$-4 + \frac{3600}{x^2} = 0 \qquad \Rightarrow \qquad \frac{3600}{x^2} = 4 \qquad \Rightarrow \qquad x^2 = 900,$$

The zeros of the function $f'(x)$ are $x = \pm 30$. Since $x \in [6; 150]$, the only solution is $x = 30$.

The second derivative of the function $f(x)$ is

$$f''(x) = -\frac{7200}{x^3} \qquad \Rightarrow \qquad f''(30) = -\frac{7200}{30^3} < 0,$$

thus at $x = 30$, there is a maximum. Consequently the side $y$ is

$$y = \frac{600}{30} = 20.$$

Thus the dimensions of the optimal page are $30\,\mathrm{cm}$ and $20\,\mathrm{cm}$, respectively.

**Exercise 35.** The population of bacteria ($P$) in thousands at a time $t$ in hours can be modelled by

$$P(t) = 100 + \mathrm{e}^t - 3t \qquad (t \geq 0).$$

a) Find the initial population of bacteria.
b) Find the function $P'(t)$.
c) Find the time at which the bacteria are growing at a rate of 6 million per hour.
d) Find the function $P''(t)$ and explain the physical significance of this quantity.

e) Find the minimum number of bacteria, justifying that it is a minimum.

**Solution:**

a) Since
$$P(0) = 100 + \mathrm{e}^0 - 3 \cdot 0 = 101,$$
the initial population of bacteria is 101 000.

b) The derivative of the function $P(t)$ is
$$P'(t) = \mathrm{e}^t - 3.$$

c) We have to solve the equation $P'(t) = 6\,000$, that is,
$$6\,000 = \mathrm{e}^t - 3.$$
The solution of the equation is $t = \ln 6\,003$, that is, $t = 8.7$.

d) The second derivative of the function $P(t)$ is
$$P''(t) = \mathrm{e}^t,$$
which is the rate of change of the growth rate of the bacteria.

e) We have to solve the equation $P'(t) = 0$, that is,
$$\mathrm{e}^t - 3 = 0 \qquad \Rightarrow \qquad \mathrm{e}^t = 3,$$
thus $t = \ln 3 = 1.099$.

The second derivative of the function $P(t)$ is positive for all $t$, thus $P$ has a minimum at $t = 1.099$.

Since
$$P(\ln 3) = 100 + \mathrm{e}^{\ln 3} - 3 \cdot \ln 3 = 99.7,$$
the minimum number of bacteria is 99 700.

**Exercise 36.** A paper aeroplane of weight $w > 1$ will travel at a constant speed of $1 - \frac{1}{\sqrt{w}} \left[\frac{\mathrm{m}}{\mathrm{s}}\right]$ for a time of $\frac{6}{w}$ [s]. What weight will achieve the maximum distance travelled?

**Solution:**

The covered distance is
$$s(w) = \left(1 - \frac{1}{\sqrt{w}}\right) \cdot \frac{6}{w} = \frac{6w^{\frac{1}{2}} - 6}{w^{\frac{3}{2}}}.$$

The derivative function of $s$ is

$$s'(w) = \frac{3 \cdot w^{-\frac{1}{2}} \cdot w^{\frac{3}{2}} - \left(6w^{\frac{1}{2}} - 6\right) \cdot \dfrac{3}{2} \cdot w^{\frac{1}{2}}}{w^3}.$$

Applying algebraic identities, we get that

$$s'(w) = \frac{3 \cdot w - 9w + 9 \cdot w^{\frac{1}{2}}}{w^3} = \frac{-6w + 9 \cdot \sqrt{w}}{w^3}.$$

We have to solve the equation $s'(w) = 0$, that is,

$$\frac{-6w + 9 \cdot \sqrt{w}}{w^3} = 0 \qquad \Rightarrow \qquad -6w + 9 \cdot \sqrt{w} = 0.$$

Factorizing the equation, we have

$$\sqrt{w} \cdot \left(-6\sqrt{w} + 9\right) = 0.$$

Since $w \neq 0$, thus

$$-6\sqrt{w} + 9 = 0 \qquad \Rightarrow \qquad \sqrt{w} = 1.5,$$

thus $w = 2.25$.

The second derivative of the function $s$ is

$$s''(w) = \frac{\left(-6 + \frac{9}{2}w^{-1/2}\right) \cdot w^3 - (-6w + 9\sqrt{w}) \cdot 3w^2}{w^6} =$$

$$= \frac{-6w + \frac{9}{2}\sqrt{w} + 18w - 27\sqrt{w}}{w^4} = \frac{12w - \frac{45}{2}\sqrt{w}}{w^4}.$$

Since $s''(2.25) < 0$, the function $s$ has a maximum at $w = 2.25$.

## 7. Partial derivatives, extrema of functions of several variables and their applications

In the theoretical part of this section, $n$ denotes a natural number, $D \subset \mathbb{R}^n$ stands for an open set, $P \in D$ denotes a point and $f\colon D \to \mathbb{R}$ stands for a function.

**Definition 2.5.1.** We say that the function $f$ has a *local maximum* (*relative maximum*) at $P$ if there is a ball in $\mathbb{R}^n$ centered at $P$ such that

$$f(x_1; \ldots; x_n) \leq f(P)$$

for all points $(x_1; \ldots; x_n) \in D$ that lie inside the ball.

We say that the function $f$ has a *local minimum* (*relative minimum*) at $P$ if there is a ball in $\mathbb{R}^n$ centered at $P$ such that

$$f(x_1; \ldots; x_n) \geq f(P)$$

for all points $(x_1; \ldots; x_n) \in D$ that lie inside the ball. If $f$ has a local minimum or maximum at $P$, then we say that it has a local extremum there.

**Definition 2.5.2.** Let $i = 1, \ldots, n$ be a number. We say that the $i$-th partial derivative of $f$ exists at the point $P = \left(x_1^{(0)}; \ldots; x_n^{(0)}\right)$ if there is an open ball in $\mathbb{R}^n$ centered at $P$ with radius $r > 0$ such that the function

$$x_i \mapsto f\left(x_1^{(0)}; \ldots; x_{i-1}^{(0)}; x_i; x_{i+1}^{(0)}; \ldots; x_n^{(0)}\right) \quad \left(x_i \in \left]x_i^{(0)} - r; x_i^{(0)} + r\right[\right)$$

is differentiable at $x_i^{(0)}$. In this case the derivative of this function at $x_i^{(0)}$ is called the $i$-th partial derivative of $f$ at $P$ and is denoted by $f'_{x_i}(P)$. If any of the functions $f'_{x_1}, \ldots, f'_{x_n}\colon D \to \mathbb{R}$ exists, then it is called a (first order) partial derivative of $f$.

**Definition 2.5.3.** Let $i, j = 1, \ldots, n$ be numbers. If the $j$-th partial derivative of $f$ exists at each point in $D$ and the $i$-th partial derivative of the function $f'_{x_j}$ exists at $P$, then it is denoted by $f''_{x_i x_j}(P)$. If any of the functions

$$f''_{x_1 x_1}, \ldots, f''_{x_1 x_n}, \ldots, f''_{x_n x_1}, \ldots, f''_{x_n x_n}\colon D \to \mathbb{R}$$

exists, then it is called a second order partial derivative of $f$.

**Definition 2.5.4.** We say that the function $f$ is continuously differentiable if all of its first order partial derivatives exist and they are continuous. The function $f$ is termed twice continuously differentiable if each of its second order partial derivatives exists and they are continuous.

**Theorem 2.5.5.** If $f$ is continuously differentiable and it has a local extremum at $P$, then the partial derivatives of $f$ at $P$ are zero.

**Definition 2.5.6.** If the second order derivatives of $f$ exist, then its *Hessian matrix* at $P$ is

$$M(P) = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \dots & x_n \end{array} \\ \begin{pmatrix} f''_{x_1x_1}(P) & f''_{x_1x_2}(P) & \dots & f''_{x_1x_n}(P) \\ f''_{x_2x_1}(P) & f''_{x_2x_2}(P) & \dots & f''_{x_2x_n}(P) \\ \vdots & \vdots & \ddots & \vdots \\ f''_{x_nx_1}(P) & f''_{x_nx_2}(P) & \dots & f''_{x_nx_n}(P) \end{pmatrix} \end{array}.$$

**Theorem 2.5.7.** Suppose that $f$ is twice continuously differentiable and that the partial derivatives of $f$ at $P$ are zero. Let

$$D_i = \det \begin{pmatrix} f''_{x_1x_1}(P) & f''_{x_1x_2}(P) & \dots & f''_{x_1x_i}(P) \\ f''_{x_2x_1}(P) & f''_{x_2x_2}(P) & \dots & f''_{x_2x_i}(P) \\ \vdots & \vdots & \ddots & \vdots \\ f''_{x_ix_1}(P) & f''_{x_ix_2}(P) & \dots & f''_{x_ix_i}(P) \end{pmatrix} \qquad (i = 1, \dots n).$$

If $D_i > 0$ for all $i = 1, 2, \dots, n$, then the $f$ has a local minimum at $P$.

If $(-1)^i \cdot D_i > 0$ for all $i = 1, 2, \dots, n$, then $f$ has a local maximum at $P$. If $D_i < 0$ for an even number $i = 2, 3, ..., n$, then $f$ has no local extremum at $P$.

**Remark 2.5.8.** How can we find a local extremum?

1. We have to calculate the partial derivatives of the function $f$.
2. We have to solve the system of equations

$$\left.\begin{array}{r} f'_{x_1}(x_1; x_2; \dots; x_n) = 0 \\ f'_{x_2}(x_1; x_2; \dots; x_n) = 0 \\ \dots \\ f'_{x_n}(x_1; x_2; \dots; x_n) = 0 \end{array}\right\}.$$

The solutions of the system are called the critical points of $f$.

3. We have to calculate the second order partial derivatives and we have to construct the Hessian matrix:

$$M(x_1;x_2;\ldots;x_n) = \begin{array}{c} \\ x_1 \\ x_2 \\ \vdots \\ x_n \end{array} \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \end{array} \\ \begin{pmatrix} f''_{x_1x_1} & f''_{x_1x_2} & \cdots & f''_{x_1x_n} \\ f''_{x_2x_1} & f''_{x_2x_2} & \cdots & f''_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f''_{x_nx_1} & f''_{x_nx_2} & \cdots & f''_{x_nx_n} \end{pmatrix} \end{array}.$$

4. We have to substitute the critical points in the Hessian matrix and apply Theorem 2.5.7.

**Solved exercises**

**Exercise 37.** Find all critical points of the function

$$f(x;y) = 10x^2 - 20x + 2y^2 - 4y.$$

**Solution:**

The first order partial derivatives of the function $f$ are

$$\begin{aligned} f'_x(x;y) &= 20x - 20 \\ f'_y(x;y) &= 4y - 4. \end{aligned}$$

We have to solve the system of equations

$$\left.\begin{aligned} 20x - 20 = 0 \\ 4y - 4 = 0 \end{aligned}\right\}.$$

The solution of the system is $(1;1)$. The critical point of the function $f$ is $P = (1;1)$.

**Exercise 38.** The critical point for $f(x;y) = x^2 - 8x + y^2 - 10y + 2$ is $P = (4;5)$. Determine if the critical point is a local (relative) maximum or minimum.

**Solution:**

The first order partial derivatives of the function $f$ are

$$\begin{aligned} f'_x(x;y) &= 2x - 8 \\ f'_y(x;y) &= 2y - 10. \end{aligned}$$

The second order partial derivatives are

$$\begin{aligned} f''_{xx}(x;y) &= 2 & \qquad f''_{xy}(x;y) &= 0 \\ f''_{yx}(x;y) &= 0 & \qquad f''_{yy}(x;y) &= 2, \end{aligned}$$

thus the Hessian matrix is as follows:

$$M(x;y) = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \end{array}.$$

If we substitute the coordinates of the point $P$ in the Hessian matrix, we get the same matrix:

$$M(P) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $D_1 = 2$ and

$$D_2 = \det \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4,$$

therefore $D_1$ and $D_2$ are positive, thus $f$ has a local minimum at $P = (4; 5)$.

**Exercise 39.** Let $f(x; y) = 2x^3 + 2y^3 - 6xy + 3$ be a two variable function. Calculate the local extremum of the function.

**Solution:**

The partial derivative functions of $f$ are:

$$f'_x(x; y) = 6x^2 - 6y$$
$$f'_y(x; y) = 6y^2 - 6x.$$

We have to solve the system of equations

$$\left.\begin{aligned} 6x^2 - 6y = 0 \\ 6y^2 - 6x = 0 \end{aligned}\right\}.$$

If we simplify the equations, we get that

$$\left.\begin{aligned} x^2 - y = 0 \\ y^2 - x = 0 \end{aligned}\right\}.$$

From the first equation, we get that $y = x^2$. If we substitute this to the second equation, we get that

$$(x^2)^2 - x = 0 \qquad \Rightarrow \qquad x^4 - x = 0.$$

It follows that

$$x \cdot (x^3 - 1) = 0,$$

thus $x_1 = 0$ and $x_2 = 1$. The values of $y$ are $y_1 = 0$ and $y_2 = 1$. We have two critical points: $P_1 = (0; 0)$ and $P_2 = (1; 1)$.

In the general case, the Hessian matrix is as follows:

$$M(x; y) = \begin{matrix} & x & y \\ x & f''_{xx}(x; y) & f''_{xy}(x; y) \\ y & f''_{yx}(x; y) & f''_{yy}(x; y) \end{matrix}.$$

Since

$$f''_{xx}(x;y) = 12x \qquad f''_{xy}(x;y) = -6$$
$$f''_{yx}(x;y) = -6 \qquad f''_{yy}(x;y) = 12y,$$

therefore in this exercise the Hessian matrix is

$$M(x;y) = \begin{array}{c} \\ x \\ y \end{array}\begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} 12x & -6 \\ -6 & 12y \end{pmatrix} \end{array}.$$

If we substitute the point $P_1$ in the Hessian matrix, we get that

$$M(P_1) = \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix}.$$

Since

$$D_2 = \det \begin{pmatrix} 0 & -6 \\ -6 & 0 \end{pmatrix} = 0 - 36 = -36,$$

therefore $f$ has no local extremum at the point $P_1$.
If we substitute the point $P_2$ in the matrix $M(.)$, we get that

$$M(P_2) = \begin{pmatrix} 12 & -6 \\ -6 & 12 \end{pmatrix}.$$

Since $D_1 = 12$ and

$$D_2 = \det \begin{pmatrix} 12 & -6 \\ -6 & 12 \end{pmatrix} = 144 - 36 = 108,$$

we get that both values $D_1$ and $D_2$ are positive, thus $f$ has a local minimum at the point $P_2$. The value of $f$ at the point $P_2$ is

$$f(1;1) = 2 \cdot 1^3 + 2 \cdot 1^3 - 6 \cdot 1 \cdot 1 + 3 = -2 + 3 = 1.$$

**Exercise 40.** Find the local extremum of the function

$$f(x;y) = x^3 - 12x + y^2 - 4y.$$

**Solution:**

The partial derivatives of $f$ are

$$f'_x(x;y) = 3x^2 - 12$$
$$f'_y(x;y) = 2y - 4.$$

The solutions of the system of equations

$$\left.\begin{aligned}3x^2 - 12 = 0\\ 2y - 4 = 0\end{aligned}\right\}$$

are $P_1 = (2; 2)$ and $P_2 = (-2; 2)$.
The second order partial derivatives of the function $f$ are

$$f''_{xx}(x;y) = 6x \qquad f''_{xy}(x;y) = 0$$
$$f''_{yx}(x;y) = 0 \qquad f''_{yy}(x;y) = 2.$$

The Hessian matrix is

$$M(x;y) = \begin{array}{c} \\ x \\ y \end{array} \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{pmatrix} 6x & 0 \\ 0 & 2 \end{pmatrix} \end{array}.$$

The Hessian matrix at the point $P_1$ is

$$M(P_1) = \begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since $D_1 = 12$ and

$$D_2 = \det \begin{pmatrix} 12 & 0 \\ 0 & 2 \end{pmatrix} = 24,$$

therefore $f$ has a local minimum at $P_1$. The value of the function $f$ at $P_1$ is

$$f(2;2) = 2^3 - 12 \cdot 2 + 2^2 - 4 \cdot 2 = -20.$$

The value of the Hessian matrix at the point $P_2$ is

$$M(P_2) = \begin{pmatrix} -12 & 0 \\ 0 & 2 \end{pmatrix}.$$

Since

$$D_2 = \det \begin{pmatrix} -12 & 0 \\ 0 & 2 \end{pmatrix} = -24 < 0,$$

thus $f$ has no local extremum at $P_2$.

**Exercise 41.** The cost function is $f(x;y) = \ln(x^2 + y^2 + 1) + 5$. Find the minimum value of the cost.

**Solution:**

The partial derivatives of $f$ are

$$f'_x(x;y) = \frac{2x}{x^2+y^2+1}$$

$$f'_y(x;y) = \frac{2y}{x^2+y^2+1}.$$

The solution of the system of equations

$$\left.\begin{array}{r} \dfrac{2x}{x^2+y^2+1} = 0 \\[2ex] \dfrac{2y}{x^2+y^2+1} = 0 \end{array}\right\}$$

is $P = (0;0)$.

The second order partial derivatives are

$$f''_{xx}(x;y) = \frac{2\cdot(x^2+y^2+1)-4x^2}{(x^2+y^2+1)^2} = \frac{-2x^2+2y^2+2}{(x^2+y^2+1)^2}$$

and

$$f''_{yy}(x;y) = \frac{2\cdot(x^2+y^2+1)-4y^2}{(x^2+y^2+1)^2} = \frac{2x^2-2y^2+2}{(x^2+y^2+1)^2}$$

and

$$f''_{yx}(x;y) = f''_{xy}(x;y) = \frac{-4xy}{(x^2+y^2+1)^2}.$$

The Hessian matrix is

$$M(x;y) = \begin{array}{c} \\ x \\ \\ y \end{array} \begin{array}{c} \begin{array}{cc} \quad x \quad & \quad y \quad \end{array} \\ \left( \begin{array}{cc} \dfrac{-2x^2+2y^2+2}{(x^2+y^2+1)^2} & \dfrac{-4xy}{(x^2+y^2+1)^2} \\[2ex] \dfrac{-4xy}{(x^2+y^2+1)^2} & \dfrac{2x^2-2y^2+2}{(x^2+y^2+1)^2} \end{array} \right) \end{array}.$$

The Hessian matrix at the point $P$ is

$$M(P) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}.$$

The value of $D_1$ is 2, and

$$D_2 = \det\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 4 - 0 = 4.$$

Since $D_1$ and $D_2$ are positive real numbers, we get that $f$ has a local minimum at the point $P$. The value of the function $f$ at the point $P$ is:

$$f(0;0) = \ln(0^2 + 0^2 + 1) + 5 = 5.$$

**Exercise 42.** In a certain office, the computers $A$, $B$ and $C$ are used for $a$, $b$ and $c$ hours, respectively. If the daily output $f$ is a function of $a$, $b$ and $c$, namely

$$f(a;b;c) = 23a + 29b - 2a^2 - 4b^2 - ab - c^2 + 2c + 100,$$

find the values of $a$, $b$ and $c$ that maximize $f$.

**Solution:**

The partial derivatives of the function $f$ are

$$\begin{aligned} f'_a(a;b;c) &= 23 - 4a - b \\ f'_b(a;b;c) &= 29 - 8b - a \\ f'_c(a;b;c) &= -2c + 2. \end{aligned}$$

We have to solve the system

$$\left.\begin{aligned} 23 - 4a - b = 0 \\ 29 - 8b - a = 0 \\ -2c + 2 = 0 \end{aligned}\right\}.$$

The solution of the system is $P = (5;3;1)$.

The second order partial derivatives are

$$\begin{aligned} &f''_{aa}(a;b;c) = -4 && f''_{ab}(a;b;c) = -1 && f''_{ac}(a;b;c) = 0 \\ &f''_{ba}(a;b;c) = -1 && f''_{bb}(a;b;c) = -8 && f''_{bc}(a;b;c) = 0 \\ &f''_{ca}(a;b;c) = 0 && f''_{cb}(a;b;c) = 0 && f''_{cc}(a;b;c) = -2, \end{aligned}$$

hence the Hessian-matrix is

$$M(a;b;c) = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} -4 & -1 & 0 \\ -1 & -8 & 0 \\ 0 & 0 & -2 \end{pmatrix} \end{array}.$$

The Hessian-matrix at the point $P$ is

$$M(P) = \begin{pmatrix} -4 & -1 & 0 \\ -1 & -8 & 0 \\ 0 & 0 & -2 \end{pmatrix}.$$

Since $D_1 = -4$ and

$$D_2 = \det \begin{pmatrix} -4 & -1 \\ -1 & -8 \end{pmatrix} = 32 - 1 = 31,$$

and

$$D_2 = \det \begin{pmatrix} -4 & -1 & 0 \\ -1 & -8 & 0 \\ 0 & 0 & -2 \end{pmatrix} = -2 \cdot 31 = -62,$$

thus the function $f$ has a maximum at $P$.

The value of the function at the point $P$ is

$$f(5;3;1) = 23 \cdot 5 + 29 \cdot 3 - 2 \cdot 5^2 - 4 \cdot 3^2 - 15 - 1^2 + 2 \cdot 1 + 100 = 202.$$

# Chapter 3.

# Integral calculus and its applications

# 8. Techniques for the calculation of primitive functions, applications in economical problems

## Theoretical summary

**Remark 3.1.1.** In this section, $I$ is an open interval.

**Definition 3.1.2.** *A primitive function* of a function $f\colon I \to \mathbb{R}$ is a differentiable function $F\colon I \to \mathbb{R}$ whose derivative is equal to the original function $f$. Notation: $\int f = F$, $\int f(x)\,\mathrm{d}x = F(x)$.

**Theorem 3.1.3.** If $f, F$ are as above and $F$ is a primitive function of $f$, then $F + c$ is also a primitve function of $F$ for all $c \in \mathbb{R}$.

**Example 3.1.4.** If $f(x) = 5x^4$ then

$$\int 5x^4\,\mathrm{d}x = x^5 + c$$

for all $c \in \mathbb{R}$.

**Theorem 3.1.5.** If $f, g\colon I \to \mathbb{R}$ are functions having primitive functions, then $f + g$ has primitive functions and

$$\int \big(f(x) + g(x)\big)\,\mathrm{d}x = \int f(x)\,\mathrm{d}x + \int g(x)\,\mathrm{d}x.$$

**Example 3.1.6.**

$$\int 5x^4 + 4x^3\,\mathrm{d}x = \int 5x^4\,\mathrm{d}x + \int 4x^3\,\mathrm{d}x = x^5 + x^4 + c.$$

**Theorem 3.1.7.** If $f\colon I \to \mathbb{R}$ is a function having primitive functions, then $\lambda \cdot f$ has primitive functions and

$$\int \lambda \cdot f(x)\,\mathrm{d}x = \lambda \cdot \int f(x)\,\mathrm{d}x,$$

for all $\lambda \in \mathbb{R}$.

**Example 3.1.8.**

$$\int 3x\,\mathrm{d}x = 3\int x\,\mathrm{d}x = 3 \cdot \frac{x^2}{2} + c$$

The primitive functions of some basic elementary functions are given in the table below.

| $f(x)$ | $D_f$ | $\int f(x)\,\mathrm{d}x$ |
|---|---|---|
| $\alpha$ | $\mathbb{R}$ | $\alpha \cdot x + c$ |
| $1$ | $\mathbb{R}$ | $x + c$ |
| $x^r \ (r \in \mathbb{R} \setminus \{-1\})$ | $]0; \infty[$ | $\frac{x^{r+1}}{r+1} + c$ |
| $\sin x$ | $\mathbb{R}$ | $-\cos x + c$ |
| $\cos x$ | $\mathbb{R}$ | $\sin x + c$ |
| $\frac{1}{\cos^2 x}$ | $\mathbb{R} \setminus \{\frac{\pi}{2} + k \cdot \pi \mid k \in \mathbb{Z}\}$ | $\tan x + c$ |
| $-\frac{1}{\sin^2 x}$ | $\mathbb{R} \setminus \{\pi + k \cdot \pi \mid k \in \mathbb{Z}\}$ | $\cot x + c$ |
| $\mathrm{e}^x$ | $\mathbb{R}$ | $\mathrm{e}^x + c$ |
| $a^x \ (a > 0)$ | $\mathbb{R}$ | $\frac{a^x}{\ln a} + c$ |
| $\frac{1}{x}$ | $]0; \infty[$ | $\ln \lvert x \rvert + c$ |
| $\frac{1}{x \cdot \ln a}$ | $]0; \infty[$ | $\log_a x + c$ |
| $\frac{1}{\sqrt{1-x^2}}$ | $]-1; 1[$ | $\arcsin x + c$ |
| $-\frac{1}{\sqrt{1-x^2}}$ | $]-1; 1[$ | $\arccos x + c$ |
| $\frac{1}{1+x^2}$ | $\mathbb{R}$ | $\arctan x + c$ |
| $\sinh x$ | $\mathbb{R}$ | $\cosh x + c$ |

| | | |
|---|---|---|
| $\cosh x$ | $\mathbb{R}$ | $\sinh x + c$ |
| $\frac{1}{\cosh^2 x}$ | $\mathbb{R}$ | $\tanh x + c$ |
| $-\frac{1}{\sinh^2 x}$ | $\mathbb{R} \setminus \{0\}$ | $\coth x + c$ |
| $\frac{1}{\sqrt{1+x^2}}$ | $\mathbb{R}$ | $\operatorname{arsinh} x + c$ |
| $\frac{1}{\sqrt{x^2-1}}$ | $[1; \infty[$ | $\operatorname{arcosh} x + c$ |
| $\frac{1}{1-x^2}$ | $]-1; 1[$ | $\operatorname{artanh} x + c$ |
| $-\frac{1}{1-x^2}$ | $]-\infty; -1[\cup]1; \infty[$ | $\operatorname{arcoth} x + c$ |

**Theorem 3.1.9.** If $F\colon I \to \mathbb{R}$ is a primitive function of the function $f\colon I \to \mathbb{R}$, then

$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a} \cdot F(ax+b) + c$$

for all $a, b, c \in \mathbb{R},\ a \neq 0$.

**Example 3.1.10.**

$$\int \cos(2x+4)\,\mathrm{d}x = \frac{\sin(2x+4)}{2} + c.$$

**Theorem 3.1.11.** If $f\colon I \to \mathbb{R}$ is a nowhere 0 differentiable function, then the primitive function of the function $\frac{f'(x)}{f(x)}$ exists and

$$\int \frac{f'(x)}{f(x)}\,\mathrm{d}x = \ln\left|f(x)\right| + c.$$

**Example 3.1.12.**

$$\int \frac{3x^2}{x^3+6}\,\mathrm{d}x = \ln|x^3+6| + c$$

**Theorem 3.1.13.** Let $n \in \mathbb{R} \setminus \{-1\}$ and $f\colon I \to ]0, \infty[$ be a differentiable function. Then the primitve function of the function $f^n(x) \cdot f'(x)$ exists and

$$\int f^n(x) \cdot f'(x)\,\mathrm{d}x = \frac{f^{n+1}(x)}{n+1} + c.$$

**Example 3.1.14.**

$$\int \sin^5 x \cdot \cos x\,\mathrm{d}x = \frac{\sin^6 x}{6} + c$$

**Theorem 3.1.15** (Integration by parts)**.** If $f, g\colon I \to \mathbb{R}$ are continuously differentiable functions, then $f'g, fg'$ have primitive functions and

$$\int f'g = fg - \int fg'.$$

**Example 3.1.16.** We compute

$$\int x \cos x\,\mathrm{d}x.$$

Using the notation of the previous theorem, let $f, g\colon \mathbb{R} \to \mathbb{R}$ be such that $f'(x) = \cos x,\ g(x) = x\ (x \in \mathbb{R})$. Then $f(x) = \sin x$ and

$$\int x \cos x\,\mathrm{d}x = x \sin x - \int 1 \cdot \sin x\,\mathrm{d}x = x \sin x + \cos x + c.$$

**Remark 3.1.17.** If the marginal revenue function for a manufacturer's product is $MR$, then by integrating this function and using the initial condition $R(0) = 0$, we can find the revenue function.

The revenue function is given by the general relationship

$$R(q) = p \cdot q,$$

where $p$ is the price per unit.

If the marginal cost function for a manufacturer's product is $MC$, then by integrating this function and using the initial condition $C(0) = FC$, we can find the cost function.

**Solved exercieses**

**Exercise 43.** Compute the integrals below:

a) $\int 3x^2 + 6x + 5 \,\mathrm{d}x$

b) $\int \frac{1}{x^2} \,\mathrm{d}x$

c) $\int \sqrt[3]{x} \,\mathrm{d}x$

d) $\int \frac{x^2 + x}{x} \,\mathrm{d}x$

e) $\int 4^x + 5^x \,\mathrm{d}x$

f) $\int \frac{x^2 - 16}{x - 4} \,\mathrm{d}x$

g) $\int \frac{x^2 + 4x + 4}{x + 2} \,\mathrm{d}x$

**Solution:**

a)

$$\int 3x^2 + 6x + 5 \,\mathrm{d}x = 3 \cdot \int x^2 \,\mathrm{d}x + 6 \cdot \int x \,\mathrm{d}x + \int 5 \,\mathrm{d}x =$$
$$= 3 \cdot \frac{x^3}{3} + 6 \cdot \frac{x^2}{2} + 5x + c = x^3 + 3x^2 + 5x + c$$

b) Since

$$\frac{1}{x^2} = x^{-2},$$

we get that

$$\int \frac{1}{x^2} \,\mathrm{d}x = \int x^{-2} \,\mathrm{d}x = \frac{x^{-1}}{-1} + c = -\frac{1}{x} + c.$$

c) Since $\sqrt[3]{x} = x^{\frac{1}{3}}$, we get that

$$\int \sqrt[3]{x} \,\mathrm{d}x = \int x^{\frac{1}{3}} \,\mathrm{d}x = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} = \frac{3}{4} \cdot \sqrt[3]{x^4} + c.$$

d)

$$\int \frac{x^2 + x}{x} \,\mathrm{d}x = \int \frac{x^2}{x} + \frac{x}{x} \,\mathrm{d}x = \int x + 1 \,\mathrm{d}x = \frac{x^2}{2} + x + c.$$

e)
$$\int 4^x + 5^x \,\mathrm{d}x = \frac{4^x}{\ln 4} + \frac{5^x}{\ln 5} + c.$$

f) Since $x^2 - 16 = (x-4)\cdot(x+4)$, we get that
$$\int \frac{x^2-16}{x-4}\,\mathrm{d}x = \int \frac{(x-4)\cdot(x+4)}{x-4}\,\mathrm{d}x = \int x+4\,\mathrm{d}x = \frac{x^2}{2} + 4x + c.$$

g) Since $x^2 + 4x + 4 = (x+2)^2$, we get that
$$\int \frac{x^2+4x+4}{x+2}\,\mathrm{d}x = \int \frac{(x+2)^2}{x+2}\,\mathrm{d}x = \int x+2\,\mathrm{d}x = \frac{x^2}{2} + 2x + c.$$

**Exercise 44.** Let
$$f(x) = \frac{x^2-4}{x-2} \qquad (x \in \mathbb{R} \setminus \{2\}).$$

Determine the primitve function $F$ of $f$ satisfying $F(3) = 20$.

<u>**Solution:**</u>

Since $x^2 - 4 = (x-2)\cdot(x+2)$, we obtain that
$$\frac{x^2-4}{x-2} = \frac{(x-2)\cdot(x+2)}{x-2} = x+2,$$

therefore
$$\int \frac{x^2-4}{x-2}\,\mathrm{d}x = \int x+2\,\mathrm{d}x = \frac{x^2}{2} + 2x + c.$$

We get that
$$F(x) = \frac{x^2}{2} + 2x + c.$$

Since $F(3) = 20$, thus
$$20 = \frac{3^2}{2} + 2\cdot 3 + c \qquad \Rightarrow \qquad c = 9.5.$$

The desired primitve function is
$$F(x) = \frac{x^2}{2} + 2x + 9.5.$$

**Exercise 45.** Let $f(x) = (3x+1)^2$. Find the primitive function $F$ of $f$ satisfying $F(2) = 10$.

**Solution:**

Since

$$(a+b)^2 = a^2 + 2ab + b^2$$

thus

$$(3x+1)^2 = 9x^2 + 6x + 1.$$

We get that

$$F(x) = \int 9x^2 + 6x + 1 \,\mathrm{d}x = \frac{9x^3}{3} + \frac{6x^2}{2} + x + c = 3x^3 + 3x^2 + x + c.$$

Since $F(2) = 10$, therefore

$$24 + 12 + 2 + c = 10 \qquad \Rightarrow \qquad c = -28,$$

thus the primitve function is

$$F(x) = 3x^3 + 3x^2 + x - 28.$$

**Exercise 46.** Calculate the inetgrals below:

a) $\displaystyle\int \frac{x}{x^2+5}\,\mathrm{d}x$

b) $\displaystyle\int \frac{x+2}{x^2+4x+7}\,\mathrm{d}x$

c) $\displaystyle\int \frac{\mathrm{e}^x}{\mathrm{e}^x+2}\,\mathrm{d}x$

d) $\displaystyle\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}\,\mathrm{d}x$

e) $\displaystyle\int 5 \cdot \frac{\cos x}{\sin x}\,\mathrm{d}x$

**Solution:**

a) If $f(x) = x^2 + 5$, then $f'(x) = 2x$, thus

$$\int \frac{x}{x^2+5}\,\mathrm{d}x = \frac{1}{2} \cdot \int \frac{2x}{x^2+5}\,\mathrm{d}x = \frac{1}{2} \cdot \ln(x^2+5) + c.$$

b)

$$\int \frac{x+2}{x^2+4x+7}\,\mathrm{d}x = \frac{1}{2} \cdot \int \frac{2x+4}{x^2+4x+7}\,\mathrm{d}x = \frac{1}{2} \cdot \ln\left|x^2+4x+7\right| + c.$$

c) If $f(x) = \mathrm{e}^x + 2$, then $f'(x) = \mathrm{e}^x$, thus

$$\int \frac{\mathrm{e}^x}{\mathrm{e}^x + 2}\,\mathrm{d}x = \ln(\mathrm{e}^x + 2) + c.$$

d)

$$\int \frac{\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}\,\mathrm{d}x = \int \frac{1}{2} \cdot \frac{2\mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}\,\mathrm{d}x = \frac{1}{2} \cdot \int \frac{2 \cdot \mathrm{e}^{2x}}{\mathrm{e}^{2x}+1}\,\mathrm{d}x =$$
$$= \frac{1}{2} \cdot \ln(e^{2x} + 1) + c.$$

e)

$$\int \frac{5 \cdot \cos x}{\sin x}\,\mathrm{d}x = 5 \cdot \int \frac{\cos x}{\sin x}\,\mathrm{d}x = 5 \cdot \ln|\sin x| + c.$$

**Exercise 47.** Calculate the integrals below:

a) $\int \sin^6 x \cdot \cos x \,\mathrm{d}x$

b) $\int \cos^3 x \cdot \sin x \,\mathrm{d}x$

c) $\int \frac{\sin x}{\cos^5 x}\,\mathrm{d}x$

d) $\int \frac{\cos x}{\sqrt[4]{\sin^3 x}}\,\mathrm{d}x$

**<u>Solution:</u>**

a)

$$\int \sin^6 x \cdot \cos x \,\mathrm{d}x = \frac{\sin^7 x}{7} + c.$$

b)

$$\int \cos^3 x \cdot \sin x \,\mathrm{d}x = -\int \cos^3 x \cdot (-\sin x)\,\mathrm{d}x = -\frac{\cos^4 x}{4} + c.$$

c)

$$\int \frac{\sin x}{\cos^5 x}\,\mathrm{d}x = \int \sin x \cdot \cos^{-5} x \,\mathrm{d}x = -\int -\sin x \cdot \cos^{-5} x \,\mathrm{d}x =$$
$$= -\frac{\cos^{-4} x}{-4} + c = \frac{1}{4 \cdot \cos^4 x} + c.$$

d)

$$\int \frac{\cos x}{\sqrt[4]{\sin^3 x}}\,\mathrm{d}x = \int \cos x \cdot (\sin x)^{-\frac{3}{4}}\,\mathrm{d}x = \frac{\sin^{\frac{1}{4}} x}{\frac{1}{4}} + c = 4 \cdot \sqrt[4]{\sin x} + c.$$

**Exercise 48.** Calculate the integrals below:

a) $\int \cos(6x+3)\,\mathrm{d}x$

b) $\int \sin(6x+3)\,\mathrm{d}x$

c) $\int \mathrm{e}^{4x+5}\,\mathrm{d}x$

d) $\int (2x+1)^{20}\,\mathrm{d}x$

**Solution:**

a) Since $(\sin x)' = \cos x$, we get that

$$\int \cos(6x+3)\,\mathrm{d}x = \frac{\sin(6x+3)}{6} + c.$$

b)

$$\int \sin(6x+3)\,\mathrm{d}x = -\frac{\cos(6x+3)}{6} + c$$

c) Since $(\mathrm{e}^x)' = \mathrm{e}^x$, we get that

$$\int \mathrm{e}^{4x+5}\,\mathrm{d}x = \frac{\mathrm{e}^{4x+5}}{4} + c.$$

d) Since

$$\int x^{20}\,\mathrm{d}x = \frac{x^{21}}{21} + c,$$

therefore

$$\int (2x+1)^{20}\,\mathrm{d}x = \frac{(2x+1)^{21}}{21 \cdot 2} + c = \frac{(2x+1)^{21}}{42} + c.$$

**Exercise 49.** If the marginal revenue function for a manufacturer's product is

$$MR(q) = 3\,000 - 20q - 3q^2,$$

find the revenue function.

**Solution:**

By integrating the marginal revenue function and using the initial condition $R(0) = 0$, we can find the revenue function. Since

$$\begin{aligned} R(q) &= \int 3\,000 - 20q - 3q^2\,\mathrm{d}q = \\ &= \int 3\,000\,\mathrm{d}q - \int 20q\,\mathrm{d}q - \int 3q^2\,\mathrm{d}q = \\ &= 3\,000q - 10q^2 - q^3 + c \end{aligned}$$

and $R(0) = 0$, thus

$$3\,000 \cdot 0 - 10 \cdot 0^2 - 0^3 + c = 0,$$

so we get that $c = 0$. Consequently, the revenue function is

$$R(q) = -q^3 - 10q^2 + 3\,000q.$$

## 9. Riemann integral, its economical applications and the center of mass

### Theoretical summary

**Definition 3.2.1.** Let $a \leq b$ be real numbers and $f\colon [a;b] \to \mathbb{R}$ be a bounded function. We say that $f$ is *Riemann integrable* if there is a unique number $I \in \mathbb{R}$ such that for each sequence $m_n \in \mathbb{N}$ of integers and corresponding sequence of points

$$t_0^n = a \leq \xi_0^n \leq t_1^n \leq \ldots \leq t_{m_n-1}^n \leq \xi_{m_n-1}^n \leq t_{m_n}^n = b \qquad (n \in \mathbb{N})$$

satisfying $\max\limits_{i=0,\ldots,m_n-1}(t_{i+1}^n - t_i^n) \to 0 \ (n \to \infty)$, one has

$$\sum_{i=0}^{m_n-1} (t_{i+1}^n - t_i^n) f(\xi_i^n) \to I \ (n \to \infty).$$

In this case, $I$ is called the *Riemann integral* of $f$ and is denoted by $\int\limits_a^b f$ or $\int\limits_a^b f(x)\,\mathrm{d}x$.

It is worth noting that the terms of the last displayed sum are the areas of the rectangles with vertices

$$(t_i^n;0);\ (t_{i+1}^n,0);\ (t_{i+1}^n;f(\xi_i^n));\ (t_i^n;f(\xi_i^n)) \qquad (i = 0,\ldots,m_n-1).$$

Moreover, $I$ is the signed area of the region bounded by the graph of $f$ and the lines $y=0,\ x=a,\ x=b$.

**Theorem 3.2.2** (Newton-Leibniz)**.** If $a,b,f$ are as above and $f$ is continuous, then it is Riemann integrable. Moreover, if $F$ is a primitive function of $f$, then

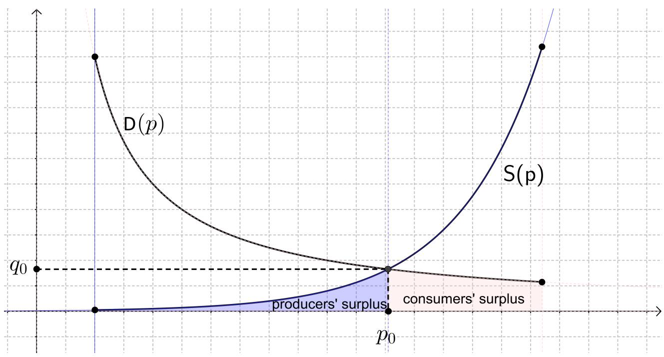
$$\int\limits_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b \doteq F(b-) - F(a+).$$

**Definition 3.2.3.** If $D(p)$ is a demand function, $S(p)$ is a supply function and $\big(p_0; q_0\big)$ is the equlibrium point then the *producers' surplus* is

$$\int\limits_{p_{min}}^{p_0} S(p)\,\mathrm{d}p,$$

and the *consumers' surplus is*

$$\int\limits_{p_0}^{p_{max}} D(p)\,\mathrm{d}p.$$

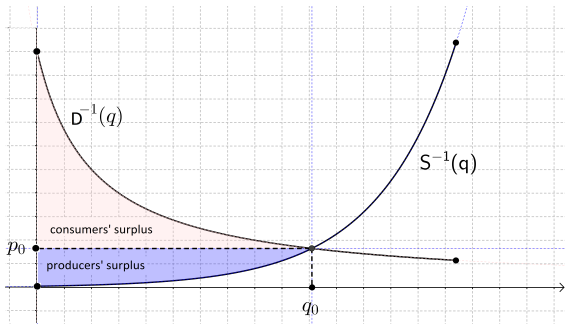



**Definition 3.2.4.** If $D^{-1}(q)$ is an inverse demand function and $S^{-1}(q)$ is an inverse supply function, and $\big(p_0; q_0\big)$ is the equlibrium point, then the producers' surplus is

$$p_0 \cdot q_0 - \int\limits_0^{q_0} S^{-1}(q)\,\mathrm{d}q,$$

and the consumers' surplus is

$$\int\limits_0^{q_0} D^{-1}(q)\,\mathrm{d}q - p_0 \cdot q_0.$$




**Theorem 3.2.5.** If the mass distribution of a plane sheet is homogeneous and it is bounded by the lines $x = a$, $x = b$, $y = 0$ and by the graph of a function $f\colon [a; b] \to \mathbb{R}$ $(a, b \in \mathbb{R},\ a < b)$, then the center of mass of the sheet is $M = (x_s; y_s)$, where

$$x_s = \frac{\int\limits_a^b x \cdot f(x)\,\mathrm{d}x}{\int\limits_a^b f(x)\,\mathrm{d}x}; \qquad y_s = \frac{\frac{1}{2} \cdot \int\limits_a^b \big(f(x)\big)^2\,\mathrm{d}x}{\int\limits_a^b f(x)\,\mathrm{d}x}\ .$$

**Solved exercieses**

**Exercise 50.** The demand function for a product is

$$D(p) = \frac{180}{p} - 4 \qquad (1 \le p \le 45)$$

the supply function is

$$S(p) = 2p - 2 \qquad (1 \le p \le 45).$$

The variable $p$ is given in dollars.

a) Calculate the equilibrium price and quantity.
b) Find the consumers' surplus.
c) Calculate the producers' surplus.

**Solution:**

a) The equilibrium price is the solution of the equation $D(p) = S(p)$, that is,

$$\frac{180}{p} - 4 = 2p - 2.$$

Applying algebraic transformations, we get that

$$2p^2 + 2p - 180 = 0 \qquad \Rightarrow \qquad p^2 + p - 90 = 0.$$

Applying the quadratic formula, we have

$$p_{1,2} = \frac{-1 \pm \sqrt{1 + 360}}{2} = \frac{-1 \pm 19}{2}.$$

Since $p > 0$, we get that $p = 9$.

The equilibrium quantity is

$$D(9) = \frac{180}{9} - 4 = 16\,\text{units}.$$

b) Since

$$\int_9^{45} \frac{180}{p} - 4\,\mathrm{d}p = [180 \ln p - 4p]_9^{45} =$$

$$= 180 \ln 45 - 180 - (180 \ln 9 - 36) \approx 145.7,$$

the consumers' surplus is $\$145.7$.

c) Since

$$\int_1^9 2p - 2\,\mathrm{d}p = \left[\frac{1}{2}\cdot 2p^2 - 2p\right]_1^9 =$$
$$= 81 - 18 - (1-2) = 64,$$

the producers' surplus is $64.

**Exercise 51.** Suppose the demand function for a product is

$$D(p) = 4\,000 - 40p \qquad (10 \le p \le 100),$$

and the supply function is

$$S(p) = 20p - 200 \qquad (10 \le p \le 100).$$

The variable $p$ is given in dollars.

a) Calculate the equilibrium price and quantity.
b) Find the consumers' surplus.
c) Calculate the producers' surplus.

**Solution:**

a) The equilibrium price is the solution of the equation $D(p) = S(p)$.
Since

$$20p - 200 = 4\,000 - 40p \qquad \Rightarrow \qquad p = 70,$$

then the equilibrium price is $70.
The equilibrium quantity is

$$D(70) = 20 \cdot 70 - 200 = 1\,200\,\text{units}.$$

b) Since

$$\int_{70}^{100} 4\,000 - 40p\,\mathrm{d}p = [4\,000p - 20p^2]_{70}^{100} =$$
$$= 4\,000\cdot 100 - 20\cdot 100^2 - (4\,000\cdot 70 - 20\cdot 70^2) = 18\,000,$$

the consumers' surplus is $18 000.

c) Since

$$\int_{10}^{70} 20p - 200\,\mathrm{d}p = [10p^2 - 200p]_{10}^{70} =$$
$$= 10 \cdot 70^2 - 200 \cdot 70 - (10 \cdot 10^2 - 200 \cdot 10) = 36\,000,$$

the producers' surplus is $\$36\,000$.

**Exercise 52.** A manufacturer's marginal cost function is

$$MC(q) = 1.2q + 4.$$

If the production is presently set at $q = 80$ units per week, how much more would it cost to increase the production to $100$ units per week?

**Solution**:

We have to calculate the value of the difference

$$C(100) - C(80).$$

According to Newton-Leibniz theorem, we get that

$$C(100) - C(80) = \int_{80}^{100} C'(q)\,\mathrm{d}q = \int_{80}^{100} MC(q)\,\mathrm{d}q =$$
$$= \int_{80}^{100} 1.2q + 4\,\mathrm{d}q = [0.6q^2 + 4q]_{80}^{100} =$$
$$= (0.6 \cdot 100^2 + 4 \cdot 100) - (0.6 \cdot 80^2 + 4 \cdot 80) =$$
$$= 6\,400 - 4\,160 = 2\,240.$$

The cost of increasing the production from $80$ to $100$ units is $\$2\,240$.

**Exercise 53.** Calculate the center of mass of the plane sheet bounded by the graph of the function $f(x) = \sqrt{x}$ $(x \in [0;4])$ and the lines $y = 0,\ x = 0,\ x = 4$.

**Solution:**

Since

$$\int_0^4 \sqrt{x}\,\mathrm{d}x = \int_0^4 x^{\frac{1}{2}}\,\mathrm{d}x = \left[\frac{2}{3} \cdot x^{\frac{3}{2}}\right]_0^4 = \left[\frac{2}{3} \cdot \sqrt{x^3}\right]_0^4 = \left(\frac{2}{3} \cdot 8\right) = \frac{16}{3}$$

and

$$\int_0^4 x \cdot \sqrt{x}\,\mathrm{d}x = \int_0^4 x \cdot x^{\frac{1}{2}}\,\mathrm{d}x = \int_0^4 x^{\frac{3}{2}}\,\mathrm{d}x = \left[\frac{2}{5} \cdot x^{\frac{5}{2}}\right]_0^4 =$$

$$= \left[\frac{2}{5} \cdot \sqrt{x^5}\right]_0^4 = \frac{2}{5} \cdot 32 = \frac{64}{5},$$

therefore

$$x_s = \frac{\frac{64}{5}}{\frac{16}{3}} = \frac{64}{5} \cdot \frac{3}{16} = \frac{12}{5}.$$

On the other hand

$$\int_0^4 x\,\mathrm{d}x = \left[\frac{x^2}{2}\right]_0^4 = 8,$$

thus

$$y_s = \frac{4}{\frac{16}{3}} = \frac{3}{4}.$$

The center of mass is

$$M = \left(\frac{12}{5}; \frac{3}{4}\right).$$

# 10. Geometric applications of the Riemann integral

## Theoretical summary

**Theorem 3.3.1.** If $f(x)$ and $g(x)$ are continuous real functions on an interval $[a;b]$ $(a,b \in \mathbb{R},\ a \le b)$ such that $f(x) \le g(x)$ $(x \in [a;b])$, then the area of the region bounded by their graphs and by the lines $x=a,\ x=b$ can be calculated as follows:

$$A = \int_a^b \big(g(x) - f(x)\big)\,\mathrm{d}x.$$

Before applying the formula above it is useful to sketch the graphs of the functions. This way we can see which one is the upper $\big(g(x)\big)$ and which is the lower $\big(f(x)\big)$ function. If we have to determine the area of the region bounded by the graphs of two real functions $f$ and $g$, sketching them also helps to find the limits of integration. To be able to do that, it is necessary to find the intersection points of the graphs of the functions $f$ and $g$, thus we have to solve the equation $f(x) = g(x)$. Then we have to integrate the function $g - f$ between the minimum and the maximum of the $x$ coordinates of the intersections, and finally we get the area in question by taking the absolute value of the obtained integral.

**Definition 3.3.2.** The solid generated by rotating a region on a plane about an axis in that plane is called a *solid of revolution*.

**Theorem 3.3.3.** If we rotate the graph of a continuous function $f(x)$ $(x \in [a;b];\ a,b \in \mathbb{R},\ a \le b)$, then the volume of the obtained solid of revolution is:

$$V = \pi \cdot \int_a^b \big(f(x)\big)^2\,\mathrm{d}x.$$

**Definition 3.3.4.** A *surface of revolution* is formed when a curve is rotated about a line.

**Theorem 3.3.5.** If a function $f(x)$ $(x \in [a;b];\ a,b \in \mathbb{R},\ a < b)$ is continuously differentiable, then the surface area of the surface obtained by rotating the graph of the function $f(x)$ about the $x$-axis equals

$$S = 2\pi \cdot \int_a^b f(x) \cdot \sqrt{1 + \big(f'(x)\big)^2}\,\mathrm{d}x.$$

**Solved exercieses**

**Exercise 54.** Find the area of the region enclosed by the functions $f(x) = 2 - x^2$ and $g(x) = -x$.

**Solution:**

The limits of integration are found by solving the equation

$$2 - x^2 = -x.$$

Applying an algebraic transformation, we get that:

$$x^2 - x - 2 = 0.$$

By the quadratic formula, we get that:

$$x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2}.$$

Thus the solutions of the equation above: $x_1 = -1$ or $x_2 = 2$.
Consequently, the $x$ coordinates of the leftmost and rightmost points of the region are $x = -1$ and $x = 2$, thus the limits of integration are $-1$ and 2.
The area between the curves is:

$$\begin{aligned} A = \int_{-1}^{2} 2 - x^2 - (-x)\,\mathrm{d}x &= \left[2x - \frac{x^3}{3} + \frac{x^2}{2}\right]_{-1}^{2} = \\ &= \left(4 - \frac{8}{3} + 2\right) - \left(-2 + \frac{1}{3} + \frac{1}{2}\right) = \\ &= 6 - \frac{8}{3} + 2 - \frac{5}{6} = \frac{9}{2}. \end{aligned}$$

**Exercise 55.** Find the area of the region enclosed by the functions $f(x) = x^2$ and $g(x) = 4x$.

**Solution:**

The limits of integration are found by solving the equation $f(x) = g(x)$:

$$\begin{aligned} x^2 &= 4x \\ x^2 - 4x &= 0 \\ x \cdot (x - 4) &= 0. \end{aligned}$$

Thus the solutions of the equation above are $x = 0$ and $x = 4$. The area between the curves is:

$$T = \int_0^4 4x - x^2 \, \mathrm{d}x = \left[2x^2 - \frac{x^3}{3}\right]_0^4 =$$
$$= 32 - \frac{64}{3} = \frac{32}{3}.$$

**Exercise 56.** Find the area of the region enclosed by the functions $f(x) = 2x^2$ and $g(x) = 2x + 4$.

**Solution:**

The limits of integration are found by solving the equation $f(x) = g(x)$:

$$2x^2 = 2x + 4.$$

If we write the equation above in another form, we get that:

$$2x^2 - 2x - 4 = 0 \qquad \Rightarrow \qquad x^2 - x - 2 = 0.$$

Applying the quadratic formula, we get that:

$$x_{1,2} = \frac{1 \pm \sqrt{1+8}}{2} = \frac{1 \pm 3}{2},$$

thus $x_1 = -1$ and $x_2 = 2$.
The area is:

$$T = \int_{-1}^2 2x + 4 - 2x^2 \, \mathrm{d}x = \left[x^2 + 4x - \frac{2x^3}{3}\right]_{-1}^2 =$$
$$= \left(2^2 + 4 \cdot 2 - \frac{2 \cdot 2^3}{3}\right) - \left((-1)^2 + 4 \cdot (-1) - \frac{2 \cdot (-1)^3}{3}\right) = 9.$$

**Exercise 57.** Find the area of the region enclosed by the functions $f(x) = x^2 - 4$ and $g(x) = 4 - x^2$.

**Solution:**

The limits of integration are found by solving the equation $f(x) = g(x)$:

$$x^2 - 4 = 4 - x^2$$
$$x^2 = 4,$$

thus $x = -2$ or $x = 2$. The area is:

$$T = \int_{-2}^{2} (4 - x^2) - (x^2 - 4)\,\mathrm{d}x = \int_{-2}^{2} (8 - 2x^2)\,\mathrm{d}x =$$
$$= \left[8x - \frac{2x^3}{3}\right]_{-2}^{2} =$$
$$= \left(8 \cdot 2 - 2 \cdot \frac{2^3}{3}\right) - \left(8 \cdot (-2) - \frac{(-2)^3}{3}\right) =$$
$$= \frac{32}{3} - \left(-\frac{32}{3}\right) = \frac{64}{3}.$$

**Exercise 58.** The region bounded by the graph of the function $f(x) = 2x + 1$ ($x \in [0; 2]$) and the lines $x = 0,\ x = 2$ is revolved about the $x$-axis to generate a solid. Find its volume.

**Solution:**

The volume is:

$$V = \pi \cdot \int_{0}^{2} (2x + 1)^2\,\mathrm{d}x = \pi \cdot \left[\frac{(2x + 1)^3}{6}\right]_{0}^{2} =$$
$$= \pi \cdot \left(\frac{(2 \cdot 2 + 1)^3}{6}\right) - \pi \cdot \left(\frac{(2 \cdot 0 + 1)^3}{6}\right) =$$
$$= \pi \cdot \frac{124}{6} = \frac{62}{3} \cdot \pi.$$

**Exercise 59.** The region bounded by the graph of the function $f(x) = 2x + 6$ ($x \in [0; 2]$) and the lines $x = 0,\ x = 2$ is rotated about the $x$-axis to generate a geometric body. Find its surface area.

**Solution:**

The surface area of the surface of revolution is

$$S = 2\pi \cdot \int_{0}^{2} (2x + 6) \cdot \sqrt{5}\,\mathrm{d}x = 2\pi \cdot \sqrt{5}\left[\frac{(2x + 6)^2}{2 \cdot 2}\right]_{0}^{2} =$$
$$= 2\pi \cdot \sqrt{5} \cdot \left(\frac{(2 \cdot 2 + 6)^2}{4} - \frac{(2 \cdot 0 + 6)^2}{4}\right) =$$
$$= 2\pi \cdot \sqrt{5} \cdot (25 - 18) = 14\pi \cdot \sqrt{5}.$$

# Chapter 4.

# Complex numbers, Fourier transformation and Laplace transformation

# 11. Complex numbers and their applications

The reason for the introduction of complex numbers is that on the set of real numbers, one cannot take $n$th roots of negative numbers provided that $n \in \mathbb{N}$ is an even number, thus the question arises: can we construct a set of numbers for which this is possible?

In certain engineering computations, one has to take such roots of negative numbers.

For example, notice that an engineering problem can occur in which one should solve the equation $x^2 + 1 = 0$. During the solution, the square root of the number -1 should be taken, which though is not possible on the set of real numbers, since there are no real numbers whose square is negative.

One can define such a complex number whose square is negative.

The elements of the set of real numbers correspond to the points of the number line. The elements of the set of complex numbers correspond to the points of the Cartesian plane ($\mathbb{R} \times \mathbb{R}$). In this case, the plane is called *complex (or Gauss) plane*, the horizontal axis is termed *real axis*, the vertical one is referred to as *imaginary axis*. Let us denote the point $(0; 1)$ on the complex plane by $i$. This number is called the *imaginary unit*. The numbers of the form $(a; 0)$ $(a \in \mathbb{R})$ are lying on the real number line, thus these numbers correspond to real numbers. Therefore we denote the complex number $(a; 0)$ simply by $a$.

**Definition 4.1.1.** Let us introduce two operations, an addition and a multiplication on the set $\mathbb{C} = \mathbb{R} \times \mathbb{R}$ in the following way:

$$(a; b) + (c; d) = (a + c; b + d)$$
$$(a; b) \cdot (c; d) = (ac - bd; bc + ad).$$

We call the set $\mathbb{C}$ equipped with the two operations defined so the *set of complex numbers*.

**Theorem 4.1.2.** The structure $(\mathbb{C}, +, \cdot)$ is a field.

**Definition 4.1.3.** The quotient $\frac{z_1}{z_2}$ $(z_1, z_2 \in \mathbb{C};\ z_2 \neq 0)$ is defined by

$$\frac{z_1}{z_2} = z_1 \cdot z_2^{-1}.$$

**Definition 4.1.4.** A *complex number* is a number of the form $z = a + b \cdot i$, where $a$ and $b$ are real numbers, and $i$ satisfies $i^2 = -1$. The real number $a$ is called the *real part* Re $z$ of the complex number $z$ and the real number $b$ is called its *imaginary part* Im $z$.

**Definition 4.1.5.** The *conjugate* of the complex number $z = a+b\cdot i$ $(a, b \in \mathbb{R})$ is $\overline{z} = a - b \cdot i$.

**Definition 4.1.6.** The *absolute value* or *modulus* of the complex number $z = a + b \cdot i$ $(a, b \in \mathbb{R})$ is $|z| = \sqrt{a^2 + b^2}$.

**Example 4.1.7.** The modulus of the complex number $z = 1 + i$ is

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}.$$

**Definition 4.1.8.** The *trigonometric form* of the nonzero complex number $z = a + b \cdot i$ $(a, b \in \mathbb{R})$ is

$$z = |z| \cdot (\cos\varphi + i \cdot \sin\varphi),$$

where $\varphi \in [0, 2\pi[$ is the unique number satisfying this equality. This number is called the *argument* of $z$.

The number $\varphi$ is 0 if $z \in \mathbb{R}, z > 0$, it is $\pi$ if $z \in \mathbb{R}, z < 0$ and it satisfies $\tan\varphi = \frac{b}{a}$ otherwise.

**Theorem 4.1.9.** If $z_1, z_2 \in \mathbb{C};\ \varphi_1, \varphi_2 \in \mathbb{R}$ are such that

$$z_1 = |z_1| \cdot (\cos\varphi_1 + i \cdot \sin\varphi_1)$$

and

$$z_2 = |z_2| \cdot (\cos\varphi_2 + i \cdot \sin\varphi_2)$$

then

$$z_1 \cdot z_2 = |z_1| \cdot |z_2| \cdot \big(\cos(\varphi_1 + \varphi_2) + i \cdot \sin(\varphi_1 + \varphi_2)\big).$$

**Theorem 4.1.10.** If $z_1, z_2 \in \mathbb{C};\ z_2 \neq 0;\ \varphi_1, \varphi_2 \in \mathbb{R}$ are such that

$$z_1 = |z_1| \cdot (\cos\varphi_1 + i \cdot \sin\varphi_1)$$

and

$$z_2 = |z_2| \cdot (\cos\varphi_2 + i \cdot \sin\varphi_2)$$

then

$$\frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \cdot \big(\cos(\varphi_1 - \varphi_2) + i \cdot \sin(\varphi_1 - \varphi_2)\big).$$

**Theorem 4.1.11.** If $z \in \mathbb{C},\ \varphi \in \mathbb{R},\ n \in \mathbb{N}$ are such that $z = |z| \cdot (\cos\varphi + i \cdot \sin\varphi)$ then

$$z^n = |z|^n \cdot \big(\cos(n\varphi) + i \cdot \sin(n\varphi)\big).$$

**Definition 4.1.12.** An *nth* $(n \in \mathbb{N})$ *root* of the complex number $z$ is a number $w \in \mathbb{C}$ satisfying $w^n = z$.

**Theorem 4.1.13.** The $n$th ($n \in \mathbb{N}$) roots of the complex number $z = |z| \cdot (\cos\varphi + i \cdot \sin\varphi)$ ($\varphi \in \mathbb{R}$):

$$w_k = \sqrt[n]{|z|} \cdot \left(\cos\frac{\varphi + k \cdot 2\pi}{n} + i \cdot \sin\frac{\varphi + k \cdot 2\pi}{n}\right), \qquad (k = 0, \dots, n-1).$$

Let $z = a + i \cdot b$ ($a, b \in \mathbb{R}$) be a number. We define $e^z = e^a(\cos b + i \cdot \sin b)$.

**Definition 4.1.14.** The *exponential form* of the nonzero complex number $z = |z| \cdot (\cos\varphi + i \cdot \sin\varphi)$ ($\varphi \in [0, 2\pi[$) is

$$z = |z| \cdot \mathrm{e}^{i\varphi}.$$

**Solved exercises**

**Exercise 60.** Let $z_1 = 2 + 3i$ and $z_2 = 3 - 4i$ be complex numbers! Calculate the complex numbers below:

a) $z_1 + z_2$

b) $z_1 \cdot z_2$

c) $\overline{z}_1$

d) $\overline{z}_2$

e) $|z_1|$

f) $\frac{z_1}{z_2}$

g) $\frac{z_2}{z_1}$

h) $\mathrm{Re}\left(\frac{z_1}{z_2}\right)$

i) $\mathrm{Im}\left(\frac{z_1}{z_2}\right)$

j) $z_1 + 5\overline{z}_2 + 6$

k) $z_1^2$

l) $z_2^2$

**Solution:**

a)

$$z_1 + z_2 = (2 + 3i) + (3 - 4i) = 2 + 3i + 3 - 4i = 5 - i.$$

b)

$$z_1 \cdot z_2 = (2 + 3i) \cdot (3 - 4i) = 6 - 8i + 9i - 12i^2 = \\ = 6 + i - 12 \cdot (-1) = 18 + i.$$

c)

$$\overline{z}_1 = 2 - 3i.$$

d)

$$\overline{z}_2 = 3 + 4i.$$

e)

$$|z_1| = \sqrt{2^2 + 3^2} = \sqrt{4 + 9} = \sqrt{13}.$$

f)

$$\frac{z_1}{z_2} = \frac{2 + 3i}{3 - 4i} = \frac{2 + 3i}{3 - 4i} \cdot \frac{3 + 4i}{3 + 4i} = \frac{(2 + 3i) \cdot (3 + 4i)}{(3 - 4i) \cdot (3 + 4i)} = \\ = \frac{6 + 8i + 9i + 12i^2}{9 - 16i^2} = \frac{6 + 17i - 12}{9 + 16} = \frac{-6 + 17i}{25} = -\frac{6}{25} + \frac{17}{25}i.$$

g)

$$\frac{z_2}{z_1} = \frac{3-4i}{2+3i} = \frac{3-4i}{2+3i} \cdot \frac{2-3i}{2-3i} = \frac{(3-4i)\cdot(2-3i)}{(2+3i)\cdot(2-3i)} =$$
$$= \frac{6-8i-9i+12i^2}{4-9i^2} = \frac{6-17i-12}{4+9} = \frac{-6-17i}{13} = -\frac{6}{13} - \frac{17}{13}i.$$

h)

$$\operatorname{Re}\left(\frac{z_1}{z_2}\right) = -\frac{6}{25}.$$

i)

$$\operatorname{Im}\left(\frac{z_1}{z_2}\right) = \frac{17}{25}.$$

j)

$$z_1 + 5\overline{z}_2 + 6 = 2 + 3i + 5\cdot(3+4i) + 6 = 2 + 3i + 15 + 20i + 6 =$$
$$= 23 + 23i$$

k) Since $(a+b)^2 = a^2 + 2ab + b^2$, we get that

$$z_1^2 = (2+3i)^2 = 4 + 12i + 9i^2 = 4 + 12i - 9 = -5 + 12i.$$

l) Since $(a-b)^2 = a^2 - 2ab + b^2$, we get that

$$z_2^2 = (3-4i)^2 = 9 - 24i + 16i^2 = 9 - 24i - 16 = -7 - 24i.$$

**Exercise 61.** Give the trigonometric form of the complex number $z = -3+3i$.

**Solution:**

Since $a = \operatorname{Re}(z) = -3$ and $b = \operatorname{Im}(z) = 3$ then the modulus of the complex number is

$$r = \sqrt{(-3)^2 + 3^2} = \sqrt{18}.$$

Since

$$\operatorname{tg}(\varphi) = \frac{b}{a} = \frac{3}{-3} = -1$$

we get that $\varphi = 135^\circ$, thus the trigonometric form is

$$z = \sqrt{18}\cdot(\cos 135^\circ + i\cdot\sin 135^\circ),$$

that is

$$z = \sqrt{18}\cdot\left(\cos\frac{3\pi}{4} + i\cdot\sin\frac{3\pi}{4}\right).$$

**Exercise 62.** Calculate the 4th roots of 1.

**Solution:**

Since in the general case

$$w_k = \sqrt[n]{r} \cdot \left( \cos \frac{\varphi + 2k\pi}{n} + i \cdot \sin \frac{\varphi + 2k\pi}{n} \right) \qquad (k = 0, \ldots, n-1)$$

are the $n$th ($n \in \mathbb{N}$) roots of $r \cdot (\cos \varphi + i \cdot \sin \varphi)$ $(r \in \mathbb{R}, r \geq 0;\ \varphi \in \mathbb{R})$, thus the 4th roots of 1 are

$$w_0 = \sqrt[4]{1} \cdot \left( \cos \frac{0 + 2 \cdot 0 \cdot \pi}{4} + i \cdot \sin \frac{0 + 2 \cdot 0 \cdot \pi}{4} \right) =$$
$$= \cos 0 + i \cdot \sin 0 = 1;$$

$$w_1 = \sqrt[4]{1} \cdot \left( \cos \frac{0 + 2 \cdot 1 \cdot \pi}{4} + i \cdot \sin \frac{0 + 2 \cdot 1 \cdot \pi}{4} \right) =$$
$$= \cos \frac{\pi}{2} + i \cdot \sin \frac{\pi}{2} = i;$$

$$w_2 = \sqrt[4]{1} \cdot \left( \cos \frac{0 + 2 \cdot 2 \cdot \pi}{4} + i \cdot \sin \frac{0 + 2 \cdot 2 \cdot \pi}{4} \right) =$$
$$= \cos \pi + i \cdot \sin \pi = -1$$

$$w_3 = \sqrt[4]{1} \cdot \left( \cos \frac{0 + 2 \cdot 3 \cdot \pi}{4} + i \cdot \sin \frac{0 + 2 \cdot 3 \cdot \pi}{4} \right) =$$
$$= \cos \frac{3\pi}{2} + i \cdot \sin \frac{3\pi}{2} = -i.$$

**Exercise 63.** Solve the equation $z^2 + 2z + 2 = 0$.

**Solution:**

If we apply the quadratic formula, we get that

$$z_{1,2} = \frac{-2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm \sqrt{-4}}{2} = \frac{-2 \pm 2i}{2} = -1 \pm i.$$

The solutions of the equation are $z_1 = -1 - i$ and $z_2 = -1 + i$.

**Exercise 64.** Find the exponentinal form of the complex number $1 + i$.

**Solution:**

The modulus of the complex number $z = 1 + i$ is

$$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$$

and its argument is
$$\varphi = \frac{\pi}{4}.$$
The exponential form is
$$z = \sqrt{2} \cdot \mathrm{e}^{i \cdot \frac{\pi}{4}}.$$

**Exercise 65.** A 120 $\Omega$ resistor and a capacitor whose capacitance is 10 $\mu$F are connected in parallel to a 100 Hz sinusoidal voltage source with an RMS voltage of 100 V. Compute the total complex impedance of the circuit!

**Solution:**

The angular frequency:
$$\omega = 2\pi \cdot f = 628.32\,\frac{1}{\mathrm{s}}.$$
The effective resistance of the capacitor: $0$. The reactance:
$$X_C = \omega \cdot L = \frac{1}{\omega \cdot C} = \frac{1}{628.32 \cdot 10 \cdot 10^{-6}} = 159.15\,\Omega,$$
hence $Z_C = 159.15i$. The total impedance satisfies
$$\frac{1}{Z} = \frac{1}{Z_C} + \frac{1}{R} = \frac{R \cdot Z_C}{R + Z_C} =$$
$$= \frac{19\,098i}{120 + 159.15i} = \frac{19\,098i}{120 + 159.15i} \cdot \frac{120 - 159.15i}{120 - 159.15i} =$$
$$= \frac{2291760i + 3\,039446.7}{39\,728.72} = 76.51 + 57.69i.$$

## 12. Analytic functions, Cauchy-Riemann equations

From now on, we consider $\mathbb{C}$ as a metric space equipped with the Euclidean distance.

**Definition 4.2.1.** Let $D \subset \mathbb{C}$ be an open set, $z_0 \in D$ be a point and $f\colon D \to \mathbb{C}$ be a function. We say that $f$ is *differentiable* at $z_0$ if the limit

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. In this case, it is called the *derivative* of $f$ at $z_0$ and denoted by $f'(z_0)$. The function $f$ is called *analytic, holomorphic* or *differentiable* if it is differentiable at each point of $D$. In this case, the function $f'\colon D \to \mathbb{C}$ is termed the *complex derivative function* of $f$.

With the notation and conditions above, we define the real and imaginary parts $u$ and $v$, resp. of $f$ by

$$u, v\colon D \to \mathbb{R}, \quad u(x;y) = \mathrm{Re} f(x;y),\ v(x;y) = \mathrm{Im} f(x;y) \qquad ((x;y) \in D).$$

**Theorem 4.2.2.** Let $D \subset \mathbb{C}$ be an open set. Then the complex function

$$f(z) = f(x + i \cdot y) = u(x;y) + i \cdot v(x;y) \qquad (z = (x;y) \in D)$$

is differentiable if and only if its real and imaginary parts are continuously differentiable and they satisfy the *Cauchy-Riemann equations*:

$$\frac{\partial u}{\partial x}(x;y) = \frac{\partial v}{\partial y}(x;y) \qquad \text{and} \qquad \frac{\partial v}{\partial x}(x;y) = -\frac{\partial u}{\partial y}(x;y).$$

Moreover, the complex derivative function of $f$ is then given by

$$f'(z) = \frac{\partial u}{\partial x}(x;y) + i \cdot \frac{\partial v}{\partial x}(x;y).$$

**Solved exercises**

**Exercise 66.** Prove that that the function $f(z) = z^2$ $(z \in \mathbb{C})$ is differentiable.

**Solution:**

Let $z = x + i \cdot y$ $(x, y \in \mathbb{R})$. Then

$$f(z) = f(x + i \cdot y) = (x + i \cdot y)^2 = x^2 + 2xy \cdot i - y^2.$$

Let $u(x; y) = x^2 - y^2$ and $v(x; y) = 2xy$. Since

$$\frac{\partial u}{\partial x}(x; y) = 2x \qquad \text{and} \qquad \frac{\partial v}{\partial y}(x; y) = 2x$$

and

$$\frac{\partial v}{\partial x}(x; y) = 2y \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x; y) = -2y$$

thus the Cauchy-Riemann equations hold.

**Exercise 67.** Prove that the function $f(z) = z^3$ $(z \in \mathbb{C})$ is differentiable.

**Solution:**

Let $z = x + i \cdot y$ $(x, y \in \mathbb{R})$. Then

$$\begin{aligned} f(z) = f(x + i \cdot y) &= (x + i \cdot y)^3 = \\ &= x^3 + 3x^2 y \cdot i + 3xy^2 \cdot i^2 + i^3 \cdot y^3 = \\ &= x^3 + 3x^2 y \cdot i - 3xy^2 - y^3 \cdot i. \end{aligned}$$

Let $u(x; y) = x^3 - 3xy^2$ and $v(x; y) = 3x^2 y - y^3$. Since

$$\frac{\partial u}{\partial x}(x; y) = 3x^2 - 3y^2 \qquad \text{and} \qquad \frac{\partial v}{\partial y}(x; y) = 3x^2 - 3y^2$$

and

$$\frac{\partial v}{\partial x}(x; y) = 6xy \qquad \text{and} \qquad \frac{\partial u}{\partial y}(x; y) = -6xy$$

thus the Cauchy-Riemann equations hold.

# 13. Fourier series and transformation and their applications in digital signal processing

In mathematical analysis, the Fourier series and transformation are usually defined for so-called Lebesgue integrable functions. However their theory is not covered by this textbook, therefore we chose to define that series and transformation only for Riemann integrable and GR-integrable functions, respectively.

**Definition 4.3.1.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function with period $2\pi$ such that $f|_{[-\pi,\pi]}$ is Riemann integrable. Then the real numbers

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(mx)\,\mathrm{d}x,$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(mx)\,\mathrm{d}x \qquad (m \in \mathbb{N}\cup\{0\},\ n \in \mathbb{N})$$

are called the *Fourier coefficients* of $f$. Moreover, the *Fourier series* of $f$ is defined by the formal expression

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(nx) + b_n\sin(nx) \qquad (x \in \mathbb{R}).$$

We emphasize that the series in the last displayed sum is not necessarily convergent for each number $x \in \mathbb{R}$. Indeed, one of the main objectives of Fourier theory, an area of mathematical analysis, is to provide conditions for the convergence of Fourier series of functions. The series of the squared Fourier coefficients of a so-called square integrable function with period $2\pi$ is convergent by the next statement.

**Theorem 4.3.2** (Parseval's identity)**.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function with period $2\pi$ such that $f^2|_{[-\pi,\pi]}$ is Riemann integrable. Then

$$\frac{a_0^2}{2} + \sum_{n=1}^{\infty} a_n^2 + b_n^2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x.$$

Fourier series have a crucial role in the investigation of discrete signals. In the case of continuous ones, that role is played by the so-called Fourier transforms. In order to define them, we will need the following notions.

**Definition 4.3.3.** Let $a, b \in \mathbb{R}$ be real numbers and $f\colon [a,\infty[\to \mathbb{R}$ ($f\colon ]-\infty, b] \to \mathbb{R}$) be a function. We say that $f$ is *GR-integrable* if $f|_{[a,x]}$ ($f|_{[x,b]}$) is Riemann integrable for all numbers $x \in [a,\infty[$ ($x \in ]-\infty, b]$) and the limit $\lim\limits_{x\to\infty} \int\limits_a^x f$ $\left(\lim\limits_{x\to-\infty} \int\limits_x^b f\right)$ exists. In this case, the latter limit is denoted by

$$\int\limits_a^\infty f = \int\limits_a^\infty f(t)\,\mathrm{d}t \quad \left(\int\limits_{-\infty}^b f = \int\limits_{-\infty}^b f(t)\,\mathrm{d}t\right)$$

and is called the *improper integral* of $f$.

**Definition 4.3.4.** We say that a function $f\colon \mathbb{R} \to \mathbb{R}$ is *GR-integrable* if $f|_{]-\infty,0]}, f|_{[0,\infty[}$ are GR-integrable. In this case, we define the *improper integral* of $f$ by

$$\int\limits_{-\infty}^\infty f = \int\limits_{-\infty}^\infty f(t)\,\mathrm{d}t = \int\limits_{-\infty}^0 f + \int\limits_0^\infty f.$$

We remark that the latter sum equals $\lim_{r\to\infty} \int\limits_{-r}^r f$.

**Definition 4.3.5.** Let $a, b \in \mathbb{R}$ be numbers, $D$ be one of the sets $[a,\infty[$, $]-\infty, b], \mathbb{R}$ and $f\colon D \to \mathbb{C}$ be a function. Then we say that $f$ is *GR-integrable* if the functions $u, v\colon D \to \mathbb{R}$ defined by

$$u(t) = \operatorname{Re} f(t), \quad v(t) = \operatorname{Im} f(t) \qquad (t \in D)$$

are GR-integrable. In this case, we define the *improper integral* of $f$ by $I(u) + i \cdot I(v)$ where $I(u)$ and $I(v)$ is the improper integral of $u$ and $v$, respectively. Moreover, according to the domain of $f$, its improper integral is denoted by the corresponding symbols appearing in the previous two definitions.

Notice that if $f\colon \mathbb{R} \to \mathbb{R}$ is a GR-integrable function and $\lambda \in \mathbb{R}$ is a number, then $f(t)\mathrm{e}^{-i\lambda t}$ is also GR-integrable.

**Definition 4.3.6.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be a GR-integrable function. Then the *Fourier transform* of $f$ is the function $\mathcal{F}(f)\colon \mathbb{R} \to \mathbb{C}$ defined by

$$\mathcal{F}[f](\lambda) = \int\limits_{-\infty}^\infty f(t)\mathrm{e}^{-i\lambda t}\,\mathrm{d}t \qquad (\lambda \in \mathbb{R}).$$

The map $f \mapsto \mathcal{F}[f]$ ($f\colon \mathbb{R} \to \mathbb{R}$ is GR-integrable) is called the *Fourier transformation*.

**Theorem 4.3.7.** If $f\colon \mathbb{R} \to \mathbb{R}$ is a GR-integrable function, then $\mathcal{F}[f]$ is continuous, bounded and

$$\lim_{\lambda \to \pm\infty} \mathcal{F}[f](\lambda) = 0.$$

**Theorem 4.3.8.** If $f, g\colon \mathbb{R} \to \mathbb{R}$ are GR-integrable functions, then $f + g$ is also GR-integrable and

$$\mathcal{F}[f + g] = \mathcal{F}[f] + \mathcal{F}[g].$$

*Proof:* Since

$$\mathcal{F}[f + g](\lambda) = \int_{-\infty}^{\infty} (f + g)(t) \cdot \mathrm{e}^{-i\lambda t} \, \mathrm{d}t = \int_{-\infty}^{\infty} f(t) \cdot \mathrm{e}^{-i\lambda t} \, \mathrm{d}t +$$

$$+ \int_{-\infty}^{\infty} g(t) \cdot \mathrm{e}^{-i\lambda t} \, \mathrm{d}t = \mathcal{F}[f](\lambda) + \mathcal{F}[g](\lambda) \qquad (\lambda \in \mathbb{R}),$$

therefore the proof is complete. ■

**Theorem 4.3.9.** If $f\colon \mathbb{R} \to \mathbb{R}$ is a GR-integrable function and $\alpha \in \mathbb{R}$ is a number, then $\alpha \cdot f$ is also GR-integrable and

$$\mathcal{F}[\alpha \cdot f] = \alpha \cdot \mathcal{F}[f].$$

*Proof:* Since

$$\mathcal{F}[\alpha \cdot f](\lambda) = \int_{-\infty}^{\infty} (\alpha \cdot f)(t) \cdot \mathrm{e}^{-i\lambda t} \, \mathrm{d}t = \alpha \cdot \int_{-\infty}^{\infty} f(t) \cdot \mathrm{e}^{-i\lambda t} \, \mathrm{d}t = \alpha \cdot \mathcal{F}[f](\lambda)$$

$(\lambda \in \mathbb{R})$, therefore the proof is complete. ■

**Corollary 4.3.10.** If $f, g\colon \mathbb{R} \to \mathbb{R}$ are GR-integrable functions and $\alpha, \beta \in \mathbb{R}$ are numbers, then $\alpha \cdot f + \beta \cdot g$ is also GR-integrable and

$$\mathcal{F}[\alpha \cdot f + \beta \cdot g] = \alpha \cdot \mathcal{F}[f] + \beta \cdot \mathcal{F}[g].$$

*Proof:* Since

$$\mathcal{F}[\alpha \cdot f + \beta \cdot g] = \mathcal{F}[\alpha \cdot f] + \mathcal{F}[\beta \cdot g] = \alpha \cdot \mathcal{F}[f] + \beta \cdot \mathcal{F}[g] \qquad (\lambda \in \mathbb{R}),$$

therefore the proof is complete. ■

**Theorem 4.3.11.** The Fourier transformation is injective.

**Theorem 4.3.12.** If $f$ is a differentiable function such that $f$ and $f'$ are GR-integrable, then

$$\mathcal{F}[f'](\lambda) = i\lambda \cdot \mathcal{F}[f](\lambda) \qquad (\lambda \in \mathbb{R}).$$

**Theorem 4.3.13.** If $f\colon \mathbb{R} \to \mathbb{R}$ is a GR-integrable function and $\alpha \in \mathbb{R}$ is a number, then $\mathrm{e}^{i\alpha \cdot t} \cdot f(t)$ is also GR-integrable and

$$\mathcal{F}\left[\mathrm{e}^{i\alpha \cdot t} \cdot f(t)\right](\lambda) = \mathcal{F}[f](\lambda - \alpha) \qquad (\lambda \in \mathbb{R}).$$

*Proof:* Since

$$\mathcal{F}\left[\mathrm{e}^{i\alpha \cdot t} \cdot f(t)\right](\lambda) = \int\limits_{-\infty}^{\infty} \mathrm{e}^{i\alpha \cdot t} \cdot f(t) \cdot \mathrm{e}^{-i\lambda t}\,\mathrm{d}t = \int\limits_{-\infty}^{\infty} f(t) \cdot \mathrm{e}^{-i(\lambda - \alpha) \cdot t}\,\mathrm{d}t =$$

$$= \mathcal{F}[f](\lambda - \alpha) \qquad (\lambda \in \mathbb{R}),$$

thus the proof is complete. ∎

**Solved exercises**

**Exercise 68.** Let $a > 0$ be a number and $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(t) = 1$ if $t \in \mathbb{R},\ |t| \le a$ and $f(t) = 0$ otherwise. Compute $\mathcal{F}[f]$.

**Solution:**

$$\mathcal{F}[f](\lambda) = \int_{-\infty}^{\infty} f(t) \cdot \mathrm{e}^{-i\lambda t}\,\mathrm{d}t = \int_{-a}^{a} \mathrm{e}^{i(-\lambda)t}\,\mathrm{d}t = \int_{-a}^{a} \cos(-\lambda t)\,\mathrm{d}t + i\int_{-a}^{a} \sin(-\lambda t)\,\mathrm{d}t$$

$$= \left[\frac{\sin(\lambda t)}{\lambda}\right]_{-a}^{a} + i\left[\frac{\cos(\lambda t)}{\lambda}\right]_{-a}^{a} = 2\frac{\sin(a\lambda)}{\lambda} \qquad (\lambda \in \mathbb{R} \setminus \{0\})$$

The first two equalities above are valid also when $\lambda = 0$, hence we get

$$\mathcal{F}[f](0) = \int_{-a}^{a} 1\,\mathrm{d}t = 1 \cdot (a - (-a)) = 2a.$$

**Exercise 69.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(t) = 1 - |t|$ if $t \in [-1, 1]$ and $f(t) = 0$ otherwise. Compute $\mathcal{F}[f]$.

**Solution:**

$$\mathcal{F}[f](\lambda) = \int_{-1}^{1} (1 - |t|)\mathrm{e}^{i(-\lambda)t}\,\mathrm{d}t = \int_{-1}^{1} (1 - |t|)\cos(-\lambda t)\,\mathrm{d}t$$

$$+i\int_{-1}^{1} (1 - |t|)\sin(-\lambda t)\,\mathrm{d}t.$$

The function in the last integral is odd, so that integral is 0. The one preceding it is

$$2\int_{0}^{1} (1 - |t|)\cos(-\lambda t)\,\mathrm{d}t = 2\int_{0}^{1} (1 - t)\cos(\lambda t)\,\mathrm{d}t,$$

since the function in it is even. To determine the latter integral, for an arbitrary number $\lambda \in \mathbb{R} \setminus \{0\}$, we compute

$$\begin{aligned}\int (1-t)\cos(\lambda t)\,\mathrm{d}t &= \int ((1/\lambda)\sin(\lambda t))'(1-t)\,\mathrm{d}t \\ &= (1/\lambda)\sin(\lambda t)\cdot(1-t) - \int (1/\lambda)\sin(\lambda t)\cdot(-1)\,\mathrm{d}t \\ &= (1/\lambda)(\sin(\lambda t)\cdot(1-t) - (1/\lambda)\cos(\lambda t)) + c.\end{aligned}$$

We obtain that

$$\mathcal{F}[f](\lambda) = 2[(1/\lambda)(\sin(\lambda t)\cdot(1-t) - (1/\lambda)\cos(\lambda t))]_0^1 = (2/\lambda^2)(1-\cos\lambda).$$

Finally

$$\mathcal{F}[f](0) = 2\int_0^1 (1-t)\,\mathrm{d}t = 2\cdot[t - t^2/2]_0^1 = 1.$$

## 14. Laplace transformation and its applications in the theory of differential equations

**Definition 4.4.1.** Let $f$ be a function satisfying the following property:

(*) $f\colon [0,\infty[\to \mathbb{R}$ is such that $f(t)\mathrm{e}^{\gamma t}$ ($t \in [0,\infty[$) is GR-integrable for some number $\gamma \in \mathbb{R}$.

The *Laplace-transform* of $f$ is the complex-valued function

$$\mathcal{L}[f](s) = \int_0^\infty f(t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t.$$

The map $f \mapsto \mathcal{L}[f]$ is called the *Laplace transformation*.

**Theorem 4.4.2.** If $f, g$ are functions satisfying (*), then $f+g$ also satisfies it and

$$\mathcal{L}[f+g] = \mathcal{L}[f] + \mathcal{L}[g].$$

*Proof:* Since

$$\mathcal{L}[f+g](s) = \int_0^\infty (f+g)(t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t = \int_0^\infty f(t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t + \\ + \int_0^\infty g(t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t = \mathcal{L}[f](s) + \mathcal{L}[g](s),$$

therefore the proof is complete. ■

**Theorem 4.4.3.** If $f$ is a function satisfying (*) and $\alpha \in \mathbb{R}$ is a number, then $\alpha \cdot f$ also satisfies (*) and

$$\mathcal{L}[\alpha \cdot f] = \alpha \cdot \mathcal{L}[f].$$

*Proof:* Since

$$\mathcal{L}[\alpha \cdot f](s) = \int_0^\infty (\alpha \cdot f)(t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t = \alpha \cdot \int_0^\infty f(t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t = \alpha \cdot \mathcal{L}[f](s),$$

therefore the proof is complete. ■

**Corollary 4.4.4.** If $f, g$ are functions satisfying (*) and $\alpha, \beta \in \mathbb{R}$ are numbers, then $\alpha \cdot f + \beta \cdot g$ also satisfies (*) and

$$\mathcal{L}[\alpha \cdot f + \beta \cdot g] = \alpha \cdot \mathcal{L}[f] + \beta \cdot \mathcal{L}[g]$$

*Proof:* Since

$$\mathcal{L}[\alpha \cdot f + \beta \cdot g] = \mathcal{L}[\alpha \cdot f] + \mathcal{L}[\beta \cdot g] = \alpha \cdot \mathcal{L}[f] + \beta \cdot \mathcal{L}[g],$$

therefore the proof is complete. ■

**Theorem 4.4.5.** The Laplace transformation is injective.

**Theorem 4.4.6.** If $f$ is a continuously differentiable function satisfying (*), then $f'$ also satisfies it and

$$\mathcal{L}[f'](s) = s \cdot \mathcal{L}[f](s) - f(0).$$

**Theorem 4.4.7.** If $f$ is a function satisfying (*) and $\lambda \in \mathbb{R}$ is a number, then $\mathrm{e}^{\lambda \cdot t} \cdot f(t)$ also satisfies it and

$$\mathcal{L}\left[\mathrm{e}^{\lambda \cdot t} \cdot f(t)\right](s) = \mathcal{L}[f](s - \lambda).$$

*Proof:* Since

$$\mathcal{L}\left[\mathrm{e}^{\lambda \cdot t} \cdot f(t)\right](s) = \int\limits_0^\infty \mathrm{e}^{\lambda \cdot t} \cdot f(t) \cdot \mathrm{e}^{-st}\,\mathrm{d}t = \int\limits_0^\infty f(t) \cdot \mathrm{e}^{s-(\lambda)\cdot t}\,\mathrm{d}t =$$
$$= \mathcal{L}[f](s - \lambda),$$

therefore the proof is complete. ■

**Theorem 4.4.8.**

$$\mathcal{L}[1] = \frac{1}{s}.$$

*Proof:* Since

$$\mathcal{L}[1] = \int\limits_0^\infty \mathrm{e}^{-st}\,\mathrm{d}t = \lim_{c\to\infty} \int\limits_0^c \mathrm{e}^{-st}\,\mathrm{d}t =$$
$$= \lim_{c\to\infty} \left[\frac{\mathrm{e}^{-s\cdot t}}{-s}\right]_0^c = \lim_{c\to\infty} \frac{\mathrm{e}^{-s\cdot c}}{-s} + \frac{1}{s} = \frac{1}{s},$$

therefore the proof is complete. ■

**Theorem 4.4.9.**

$$\mathcal{L}[t] = \frac{1}{s^2}.$$

*Proof:* We compute

$$\begin{aligned}\mathcal{L}[t](s) &= \int_0^\infty t\cdot \mathrm{e}^{-st}\,\mathrm{d}t = \lim_{c\to\infty}\int_0^c t\cdot\mathrm{e}^{-st}\,\mathrm{d}t = \\ &= \lim_{c\to\infty}\left[t\cdot\frac{\mathrm{e}^{-s\cdot t}}{-s}\right]_0^c - \lim_{c\to\infty}\int_0^c \frac{\mathrm{e}^{-st}}{-s}\,\mathrm{d}t = \\ &= \lim_{c\to\infty}\left[t\cdot\frac{\mathrm{e}^{-s\cdot t}}{-s} - \frac{\mathrm{e}^{-st}}{s^2}\right]_0^c = \\ &= \lim_{c\to\infty}\left(c\cdot\frac{\mathrm{e}^{-s\cdot c}}{-s} - \frac{\mathrm{e}^{-s\cdot c}}{s^2}\right) + \frac{1}{s^2},\end{aligned}$$

and

$$\lim_{c\to\infty}\frac{\mathrm{e}^{-s\cdot c}}{s^2} = 0.$$

We apply L'Hôpital's rule to get that

$$\begin{aligned}\lim_{c\to\infty} c\cdot\frac{\mathrm{e}^{-s\cdot c}}{-s} &= \lim_{c\to\infty} -\frac{c}{\mathrm{e}^{s\cdot c}\cdot s} = \\ &= \lim_{c\to\infty} -\frac{1}{\mathrm{e}^{s\cdot c}\cdot s^2} = 0,\end{aligned}$$

and hence we conclude that the theorem holds. ■

**Theorem 4.4.10.** Let $n\in\mathbb{N}$. Then

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}.$$

**Example 4.4.11.** If

$$f(t) = t^3 - t^2 + 5t + 2,$$

then

$$\begin{aligned}\mathcal{L}[f](s) &= \mathcal{L}[t^3 - t^2 + 5t + 2](s) = \mathcal{L}[t^3](s) - \mathcal{L}[t^2](s) + 5\mathcal{L}[t](s) + \mathcal{L}[2](s) = \\ &= \frac{3!}{s^4} - \frac{2!}{s^3} + 5\cdot\frac{1}{s^2} + \frac{2}{s} = \\ &= \frac{6 - 2s + 5s^2 + 2s^3}{s^4} = \frac{2s^3 + 5s^2 - 2s + 6}{s^4}.\end{aligned}$$

**Theorem 4.4.12.** Let $a \in \mathbb{R}$. The Laplace transform of $f(t) = \mathrm{e}^{a \cdot t}$ is

$$\mathcal{L}[\mathrm{e}^{at}](s) = \frac{1}{s-a}.$$

*Proof:* Since

$$\begin{aligned}\mathcal{L}[\mathrm{e}^{at}](s) &= \int_0^\infty \mathrm{e}^{at} \cdot \mathrm{e}^{-st}\,\mathrm{d}t = \int_0^\infty \mathrm{e}^{at} \cdot \mathrm{e}^{-st}\,\mathrm{d}t = \\ &= \int_0^\infty \mathrm{e}^{(a-s)\cdot t}\,\mathrm{d}t = \lim_{c\to\infty} \int_0^c \mathrm{e}^{(a-s)\cdot t}\,\mathrm{d}t = \lim_{c\to\infty} \left[\frac{\mathrm{e}^{(a-s)\cdot t}}{a-s}\right]_0^c = \\ &= \lim_{c\to\infty} \frac{\mathrm{e}^{(a-s)\cdot c}}{a-s} - \frac{1}{a-s} = \frac{1}{s-a},\end{aligned}$$

therefore the proof is complete. ■

**Example 4.4.13.** The Laplace transform of the function

$$f(t) = \mathrm{e}^{2t} + \mathrm{e}^{-3t}$$

is

$$\begin{aligned}\mathcal{L}[f](s) &= \mathcal{L}[\mathrm{e}^{2t} + \mathrm{e}^{-3t}](s) = \mathcal{L}[\mathrm{e}^{2t}](s) + \mathcal{L}[\mathrm{e}^{-3t}](s) = \\ &= \frac{1}{s-2} + \frac{1}{s+3} = \frac{s+3+s-2}{(s-2)\cdot(s+3)} = \frac{2s+1}{s^2+s-6}.\end{aligned}$$

**Theorem 4.4.14.** Let $\omega \in \mathbb{R}$. The Laplace transform of $f(t) = \sin(\omega \cdot t)$ is

$$\mathcal{L}[\sin(\omega \cdot t)](s) = \frac{\omega}{s^2+\omega^2}.$$

*Proof:*

$$\mathcal{L}[\sin(\omega \cdot t)](s) = \int_0^\infty \sin(\omega \cdot t) \cdot \mathrm{e}^{-st}\,\mathrm{d}t$$

Applying integration by parts we get that

$$\begin{aligned}&\int \sin(\omega \cdot t) \cdot \mathrm{e}^{-st}\,\mathrm{d}t = \sin(\omega \cdot t) \cdot \frac{\mathrm{e}^{-st}}{-s} - \int \omega \cdot \cos(\omega \cdot t) \cdot \frac{\mathrm{e}^{-st}}{-s}\,\mathrm{d}t = \\ &= \sin(\omega \cdot t) \cdot \frac{\mathrm{e}^{-st}}{-s} - \left(\omega \cdot \cos(\omega \cdot t) \cdot \frac{\mathrm{e}^{-st}}{s^2} + \int \omega^2 \cdot \sin(\omega \cdot t) \cdot \frac{\mathrm{e}^{-st}}{s^2}\,\mathrm{d}t\right).\end{aligned}$$

Thus

$$\left(\frac{\omega^2}{s^2}+1\right)\cdot\int \sin(\omega\cdot t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t=\left(\frac{\sin(\omega\cdot t)}{-s}-\frac{\omega\cdot\cos(\omega\cdot t)}{s^2}\right)\cdot \mathrm{e}^{-st},$$

therefore

$$\int \sin(\omega\cdot t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t=$$
$$=\frac{s^2}{\omega^2+s^2}\cdot\left(\frac{\sin(\omega\cdot t)}{-s}-\frac{\omega\cdot\cos(\omega\cdot t)}{s^2}\right)\cdot \mathrm{e}^{-st}+c.$$

We get that

$$\int_0^\infty \sin(\omega\cdot t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t=$$
$$=\frac{s^2}{\omega^2+s^2}\cdot\lim_{c\to\infty}\left[\left(\frac{\sin(\omega\cdot t)}{-s}-\frac{\omega\cdot\cos(\omega\cdot t)}{s^2}\right)\cdot \mathrm{e}^{-st}\right]_0^c=$$
$$=\frac{s^2}{\omega^2+s^2}\cdot\left(\lim_{c\to\infty}\left(\frac{\sin(\omega\cdot c)}{-s}-\frac{\omega\cdot\cos(\omega\cdot c)}{s^2}\right)\cdot \mathrm{e}^{-sc}+\frac{\omega}{s^2}\right)=$$
$$=\frac{s^2}{\omega^2+s^2}\cdot\frac{\omega}{s^2}=\frac{\omega}{\omega^2+s^2},$$

therefore the proof is complete. ■

**Theorem 4.4.15.** Let $\omega\in\mathbb{R}$. The Laplace transform of the function $f(t)=\cos(\omega\cdot t)$ is

$$\mathcal{L}[\cos(\omega\cdot t)](s)=\frac{s}{\omega^2+s^2}.$$

*Proof:* By definition

$$\mathcal{L}[\cos(\omega\cdot t)](s)=\int_0^\infty \cos(\omega\cdot t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t.$$

Applying integration by parts, we get that

$$\int \cos(\omega\cdot t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t=\cos(\omega\cdot t)\cdot\frac{\mathrm{e}^{-st}}{-s}+\int \omega\cdot\sin(\omega\cdot t)\cdot\frac{\mathrm{e}^{-st}}{-s}\,\mathrm{d}t=$$
$$=\cos(\omega\cdot t)\cdot\frac{\mathrm{e}^{-st}}{-s}+\left(\omega\cdot\sin(\omega\cdot t)\cdot\frac{\mathrm{e}^{-st}}{s^2}-\int \omega^2\cdot\cos(\omega\cdot t)\cdot\frac{\mathrm{e}^{-st}}{s^2}\,\mathrm{d}t\right).$$

Thus
$$\left(\frac{\omega^2}{s^2}+1\right)\cdot\int\cos(\omega\cdot t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t=\left(\frac{\cos(\omega\cdot t)}{-s}+\frac{\omega\cdot\sin(\omega\cdot t)}{s^2}\right)\cdot \mathrm{e}^{-st},$$
therefore
$$\int\cos(\omega\cdot t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t=$$
$$=\frac{s^2}{\omega^2+s^2}\cdot\left(\frac{\cos(\omega\cdot t)}{-s}+\frac{\omega\cdot\sin(\omega\cdot t)}{s^2}\right)\cdot \mathrm{e}^{-st}+c.$$
We get that
$$\int_0^\infty\cos(\omega\cdot t)\cdot \mathrm{e}^{-st}\,\mathrm{d}t=$$
$$=\frac{s^2}{\omega^2+s^2}\cdot\lim_{c\to\infty}\left[\left(\frac{\cos(\omega\cdot t)}{-s}+\frac{\omega\cdot\sin(\omega\cdot t)}{s^2}\right)\cdot \mathrm{e}^{-st}\right]_0^c=$$
$$=\frac{s^2}{\omega^2+s^2}\cdot\left(\lim_{c\to\infty}\left(\frac{\cos(\omega\cdot c)}{-s}+\frac{\omega\cdot\sin(\omega\cdot c)}{s^2}\right)\cdot \mathrm{e}^{-sc}+\frac{1}{s}\right)=$$
$$=\frac{s^2}{\omega^2+s^2}\cdot\frac{1}{s}=\frac{s}{\omega^2+s^2},$$
therefore the proof is complete. ■

**Theorem 4.4.16.** Let $n\in\mathbb{N}$ and $a\in\mathbb{R}$. The Laplace transform of the function
$$f(t)=t^n\cdot \mathrm{e}^{at}$$
is
$$\mathcal{L}[f](s)=\frac{n!}{(s-a)^{n+1}}.$$

**Solved exercise**

**Exercise 70.** Let $t \geq 0$. Solve the initial value problem

$$y''(t) - 3y'(t) + 2y(t) = \mathrm{e}^{2t}, \quad y(0) = 0,\ y'(0) = 0,$$

i.e., determine all twice differentiable functions $y\colon [0, \infty[ \to \mathbb{R}$ satisfying the equalities above.

**Solution:**

It is known that functions satisfying equations like the first one above have the property (*). Let $y\colon [0, \infty[ \to \mathbb{R}$ be a twice differentiable function fulfilling all of the last displayed equalities. If we apply the Laplace transformation to both sides of the first one, we get that

$$\mathcal{L}[y''(t) - 3y'(t) + 2y(t)] = \mathcal{L}[\mathrm{e}^{2t}].$$

The Laplace transformation is linear, thus

$$\mathcal{L}[y''] - 3\mathcal{L}[y'] + 2\mathcal{L}[y] = \mathcal{L}[\mathrm{e}^{2t}].$$

Since

$$\mathcal{L}[y''](s) = s^2 \cdot \mathcal{L}[y](s) - s \cdot y(0) - y'(0),$$

and

$$\mathcal{L}[y'](s) = s \cdot \mathcal{L}[y] - y(0),$$

moreover

$$\mathcal{L}[\mathrm{e}^{2t}](s) = \frac{1}{s-2},$$

we get that

$$s^2 \cdot \mathcal{L}[y](s) - s \cdot y(0) - y'(0) - 3 \cdot \big(s \cdot \mathcal{L}[y](s) - y(0)\big) + 2\mathcal{L}[y](s) = \frac{1}{s-2}.$$

Since $y(0) = 0$ and $y'(0) = 0$, therefore

$$s^2 \cdot \mathcal{L}[y](s) - 3s \cdot \mathcal{L}[y](s) + 2\mathcal{L}[y](s) = \frac{1}{s-2}.$$

By applying an algebraic identity, we get that

$$\mathcal{L}[y](s) \cdot (s^2 - 3s + 2) = \frac{1}{s-2},$$

thus

$$\mathcal{L}[y](s) = \frac{1}{(s^2 - 3s + 2) \cdot (s-2)}.$$

The root(s) of the polynomial $s^2 - 3s + 2$ are $s_1 = 1$ and $s_2 = 2$, thus

$$s^2 - 3s + 2 = (s-1)\cdot(s-2),$$

therefore

$$\mathcal{L}[y](s) = \frac{1}{(s-1)\cdot(s-2)^2}.$$

If we apply partial fraction decomposition, we get that

$$\frac{1}{(s-1)\cdot(s-2)^2} = \frac{A}{s-1} + \frac{B}{s-2} + \frac{C}{(s-2)^2}.$$

Thus

$$1 = A\cdot(s-2)^2 + (s-2)\cdot(s-1)\cdot B + (s-1)\cdot C,$$

therefore

$$(A+B)\cdot s^2 + (-4A - 3B + C)\cdot s + 4A + 2B - C = 1.$$

From the previous equation, we get the system of equations below

$$\left.\begin{array}{rcrcrcr} A & + & B & & & = & 0 \\ -4A & - & 3B & + & C & = & 0 \\ 4A & + & 2B & - & C & = & 1 \end{array}\right\}.$$

If we solve it by Gaussian elimination, we get that

$$\left(\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ -4 & -3 & 1 & 0 \\ 4 & 2 & -1 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -2 & -1 & 1 \end{array}\right) \to$$

$$\to \left(\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Thus we get the system of equations below

$$\left.\begin{array}{rcrcrcr} A & + & B & & & = & 0 \\ & & B & + & C & = & 0 \\ & & & & C & = & 1. \end{array}\right\}$$

The solution of the system is $(A; B; C) = (1; -1; 1)$, thus

$$\mathcal{L}[y](s) = \frac{1}{s-1} - \frac{1}{s-2} + \frac{1}{(s-2)^2}.$$

The solution of the initial value problem is

$$y(t) = \mathrm{e}^t - \mathrm{e}^{2t} + t\cdot\mathrm{e}^{2t}.$$

# Chapter 5.

# Miscellaneous exercises without solution

## 15. Miscellaneous exercises without solution

**Exercise 71.** The demand function of a product is

$$D(p) = 300 - 6p.$$

The supply function of this product is

$$S(p) = 4p - 40.$$

a) Find the equilibrium point!

b) Sketch the graph of the demand function and the supply function and mark the equilibrium point!

**Exercise 72.** The motion equation of the simple harmonic motion is

$$y(t) = 16 \cdot \sin(4t).$$

Sketch the graph of the function describing the motion on the interval $[0; 2\pi]$.

**Exercise 73.** Represent the first five terms of the sequence

$$a_n = n^2 - 6n + 5$$

in a coordinate system in the plane!

**Exercise 74.** Represent the first four terms of the sequence

$$a_n = (-1)^n \cdot \big(2^n - n + 2\big)$$

in a coordinate system in the plane!

**Exercise 75.** Calculate the limit of the sequence

$$a_n = \frac{2^{n+1} + n}{3^n}.$$

**Exercise 76.** A particle moves in a straight line according to the position-time function

$$s(t) = 16t - t^2 \text{ [cm]},$$

where $t$ is the time in seconds.

a) Find a formula for the particle's velocity-time function.

b) Find a formula for the particle's acceleration-time function.

c) Find the instant(s) when the velocity is equal to zero.

d) Calculate the position, velocity and acceleration of the particle at $t = 0$.

e) Calculate the position, velocity and acceleration of the particle at $t = 1$.

f) Describe the monotonicity properties of the function $s(t)$.

**Exercise 77.** The annual profit of a dynamically developing company has been registered in the first four years after its establishment. The obtained data are summarized in the table:

| t[year] | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| P[m $] | 3 | 5 | 10 | 15 |

The experts are assuming a linear relationship between the profit and time. On the basis of the above assumption, applying the least squares method, determine the unknown parameter $m$ in the function below, and then estimate the profit of the company five years after its establishment.

$$P(t) = P(t, m) = 2^t.$$

Plot the model function.

**Exercise 78.** The cost of manufacturing fishing poles, measured in thousand units, is modeled by

$$C(x) = 9x^3 - 90x^2 + 180x \qquad (0 \leq x \leq 10).$$

The revenue function is modeled by

$$R(x) = 63x + 27.$$

a) Find the fix cost function.
b) Find the variable cost function.
c) Find the profit function.
d) Find the production level that maximize profits.
e) Calculate the marginal cost function.
f) Find the production level, if it exists, that minimizes cost.
g) Find the average cost function.
h) Find the production level, if it exists, that minimizes average cost.

**Exercise 79.** The population of bacteria ($P$) in thousands at a time $t$ in hours can be modelled by

$$P(t) = 2^t + 20t \qquad (t \geq 0).$$

a) Find the initial population of bacteria.
b) Find the time at which the bacteria are growing at a rate of 6 million per hour.

**Exercise 80.** Find the local extremum of the function

$$f(x;y) = x^2 - 2y^2 + 6x + 8y - 4.$$

**Exercise 81.** Find the local extremum of the function

$$f(x;y) = x^3 + y^3 - 9xy.$$

**Exercise 82.** Find the local extremum of the function

$$f(x;y;z) = x^3 + y^3 - 3xy + 10z^2 - 10z + 2.$$

**Exercise 83.** In a certain office, the computers $A$, $B$ and $C$ are used for $a$, $b$ and $c$ hours, respectively. If the daily output $f$ is a function of $a$, $b$ and $c$, namely

$$f(a;b;c) = 46a + 58b - 4a^2 - 8b^2 - 2ab - 2c^2 + 4c + 600,$$

find the values of $a$, $b$ and $c$ that maximize $f$.

**Exercise 84.** Compute the integrals below:

a) $\int x^3 + 2x + 1 \,\mathrm{d}x$

b) $\int \frac{1}{x^2+4} \,\mathrm{d}x$

c) $\int \sqrt[6]{x^2} \,\mathrm{d}x$

d) $\int \frac{x^4 + x + 1}{x} \,\mathrm{d}x$

e) $\int 2^x + 3^x \,\mathrm{d}x$

**Exercise 85.** If the marginal revenue function for a manufacturer's product is

$$MR(q) = 10\,000 - 50q - 6q^2,$$

find the revenue function.

**Exercise 86.** Suppose the demand function for a product is

$$D(p) = 4\,000 - 40p \qquad (10 \le p \le 100),$$

and the supply function is

$$S(p) = 20p - 200 \qquad (10 \le p \le 100).$$

The variable $p$ is given in dollars.

a) Calculate the equilibrium price and quantity.
b) Find the consumers' surplus.
c) Calculate the producers' surplus.

**Exercise 87.** Find the area of the region enclosed by the functions $f(x) = x^2 + 1$ and $g(x) = 2x + 1$.

**Exercise 88.** The region bounded by the graph of the function $f(x) = 4 - 2x$, where $x \in [0, 2]$, the lines $x = 0$, $x = 2$ and the $x$-axis is rotated about the $x$-axis to generate a geometric body. Find its surface area.

**Exercise 89.** The region bounded by the graph of the function $f(x) = x^2 + 1$, where $x \in [0, 3]$, the lines $x = 0$, $x = 3$ and the $x$-axis is rotated about the $x$-axis to generate a geometric body. Find its volume.

**Exercise 90.** Find the complex numbers $z$, which fulfill the equation $z^2 + 2z + 2 = 0$.

**Exercise 91.** Find the complex numbers $z$, which fulfill the equation $z^3 + 8 = 0$.

**Exercise 92.** Prove that the function $f\colon \mathbb{C} \to \mathbb{C}$, $f(z) = z^2 + 2z + 6$ is differentiable.

**Exercise 93.** Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined by $f(t) = 2 - 2|t|$ if $t \in [-1, 1]$ and $f(t) = 0$ otherwise. Compute $\mathcal{F}[f]$.

**Exercise 94.** Let $t \geq 0$. Solve the initial value problem

$$y''(t) - 4y'(t) = 3t + 2, \quad y(0) = 0,\, y'(0) = 0,$$

that is, determine all twice differentiable functions $y\colon [0, \infty[\to \mathbb{R}$ satisfying the equalities above.

**Exercise 95.** Let $t \geq 0$. Solve the initial value problem

$$y''(t) - 9y'(t) + 20y(t) = t^2 + 2t + 1, \quad y(0) = 0,\, y'(0) = 0,$$

that is, determine all twice differentiable functions $y\colon [0, \infty[\to \mathbb{R}$ satisfying the equalities above.

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