UNIT 2

Introduction to logic

Contents

1. Introduction

1.1. Thought

The term ***thought*** is commonly used to define the ideas or opinions produced by the mind. The process of thinking always meets a specific motivation, which can be caused by the natural, social, or cultural environment, or by the thinking subject himself.

There are many types or classifications of the thought process: analytic reasoning, creativity, systematic thinking… This subject is focused on the study of the scientific thinking; specifically, the logical deductive thinking.

Scientific thinking refers to a systematic process which is based on a system of concepts, judgments, and reasonings about the objects and laws of humans and of the external world. ***Logical thinking*** is characterized by operating through concepts. This type of thinking always follows a specific direction, whose objective is to search a conclusion or solution for a problem. The direction is not always a straight line. In fact, it can be zigzagging with advances, stops, detours, and even reversals.

Deductive thinking moves from the general to the particular. This type of thinking focuses on reaching a conclusion from one or several premises. In contrast, ***inductive thinking*** focuses on the contrary process moving from the particular to the general. Given an assumption, if something is true in some cases, it will be true in other similar cases although we cannot observe it.

The thought and the language are strongly related. The language is necessary to communicate the concepts, the judgements, and the reasonings of the thought. The thought is preserved through the language. The language empowers the thought to become more tangible.

1.2. Mathematical logic

There is not an easy way to provide a precise definition of logic including all its parts and only its parts. We propose to define logic by means of the following keys ideas:

1. The **theory of the definitions** is part of **logic** and consist of precise definitions of specific concepts.

Descriptive definitions *are not very useful for the experts, but they are useful for everyday life.* ***Precise definitions*** *can be useful for experts, but they are unintelligible and useless for laymen in certain subjects. (i.e., mathematical definitions)*

2. **Logic** refers to the **knowledge of the truths** for a specific reality.

Logic *does not deal with the elaboration of thought from all the information that our senses can transmit. All this information cannot be managed in clear and precise ways as the logic demands.*

3. **L**ogic deals with the part of the knowledge which is **expressible in written language.**

Logic does also not deal with knowledge that is expressible in natural language which is usually not very precise and ambiguous. Logic only deals with the part of the knowledge expressible in a language that allows us to communicate scientific knowledge. Therefore, to satisfy the demands of precision, we must use a largely artificial language, with a precise grammar, without ambiguities or exceptions. The type of "sentences" to be used will be able to be evaluated as true or false, without ambiguities.

4. **Logic deals with Truths.**

There are truths of fact, which are clearly verifiable from observation, and truths of reason, which are obtained by reasoning from the truths of fact.

5. **Logic** deals with the study and analysis of **the structure of the precise or valid reasonings** and **the possible laws governing them**: LOGICAL LAWS OR DEDUCTIVE RULES.

The reasoning process can be understood as the process of moving from some truths to others through a chain of statements, being some of them obtained from the others.

Therefore, logic is a mathematical model of the deductive thinking (at least in its principles). It researches about the relation between the premises and the consequences of valid and correct arguments.

Example

All humans are mortal.

All Cantabrian people are humans.

Therefore, all Cantabrian people are mortal.

Comments

✓ Logic is fundamental in the process of modelling different situations for their subsequent analysis. Given an object or a situation, the design of a model consists of selecting some of the main characteristics from the original object or situation and building a similar object or situation that is more manageable. The success of a model depends on the selection of relevant or significant characteristics.

✓ To define an object, we need a precise language (LOGIC LANGUAGE) and some starting truth that is extracted from the main characteristics of the real objects (AXIOMS).

✓ Furthermore, we must use the axiomatic deductive method (DEDUCTIVE RULES) to prove other different truths that provide more knowledge (e.g., mathematical results).

2. b

Propositional Logic

2.1. Propositions

Definition

A ***proposition*** is a sentence that expresses whether an idea is true or false in an exclusive way. Each sentence is associated to a truth value: T if the sentence is true or F if the sentence is false. Propositions, like sets, are denoted by capital letters.

> *Propositions – potential premises and conclusions and the bearers of truth and falsity – are declarative sentences(i.e. sentences like 'Jack kicks the ball' as opposed to interrogatives like 'Does Jack kick the ball?' or imperatives like 'Jack, kick the ball!').*[1]

The truth value which is associated to a proposition is called ***truth assignment***.

Examples

1. A='Two is an even number'. It is a proposition, and its truth assignment is T.
2. B='A car is red'. It is not a proposition because "a car" is unspecific.
3. C='This sentence is false'. It is not a proposition. If it was true the sentence would be false, and if it was false the sentence would be true.

2.2. Logical Connective

Logical connective			
Type	**Name**	**Notation**	**Read**
Unary	Negation	¬	not
Binary	Conjunction	∧	and
	Disjunction	∨	or
	Conditional	→	if…then
	Biconditional	↔	if and only if

Logical connectives are the operators or functions that we can apply on one or several propositions to obtain other propositions. They can be unitary when one proposition is used to obtain another proposition, or binary when two propositions are used to obtain a new proposition.

A ***truth table*** shows all the possible truth assignments for an expression composed of propositions and logical connectives that are collected through a systematic process. To build this table, we have to consider all the truth assignments, and then, we have to apply the own rules of the operators used to build it. If we consider the following set of basic propositions $\{A, B\}$, there will be a total of $2^2 = 4$ possible truth assignments:

A	B
T	T
T	F
F	T
F	F

In general, in a set of n basic propositions, there is a total of 2^n possible truth assignments.

Definitions

Given two propositions A and B:

The ***negation*** of the proposition A is $\neg A$, which takes truth values opposite to those taken by proposition A. If A is true, $\neg A$ will be false, and if A is false, $\neg A$ will be true. The truth table of this operator is described as follows:

A	$\neg A$
T	F
F	T

The conjunction of A and B, $A \wedge B$, is a proposition whose truth value is only true when A and B are true. In other cases, its truth value is false. The truth table of this operator is described as follows:

A	B	$A \wedge B$
T	T	T
T	F	F
F	T	F
F	F	F

The ***disjunction*** of A and B,$A \vee B$, is a proposition whose truth value is true when some of the propositions A or B are true. In another case, its truth value is false. The truth table of this operator is described as follows:

A	B	$A \vee B$
T	T	T
T	F	T
F	T	T
F	F	F

The ***conditional*** of A and B, $A \rightarrow B$, is a proposition whose truth value is true provided that A is false or when A and B are true. In another case, its truth value is false. The truth table of this operator is described as follows:

A	B	$A \rightarrow B$
T	T	T
T	F	F
F	T	T
F	F	T

The ***biconditional*** of A y B,$A \leftrightarrow B$, is a proposition whose truth value is true provided that A and B have the same truth value; both propositions are true or both propositions are false. In another case, its truth value is false. The truth table of this operator is described as follows:

A	B	$A \leftrightarrow B$
T	T	T
T	F	F
F	T	F
F	F	T

The following table shows the truth tables for all these operators:

A	B	$A \wedge B$	$A \vee B$	$A \rightarrow B$	$A \leftrightarrow B$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

2.3. Propositional Logic Language: Well-formed formulas

Elements of the propositional language: Well-formed formulas.

Definitions

A ***well-formed formula (WFF)*** is a proposition that is built from simple propositions $A, B, \ldots$ and logical connectives used a finite number of times $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ *o* $(A \leftrightarrow B)$.

The elements of the ***formal language*** for the propositional logic are:

1. A numerable set of ***statement*** symbols that represent the propositions. In our case, they are denoted by capital letters.
2. A finite set of ***logical symbols*** that are the logical connectives (¬,∧,∨,→y ↔) and the parentheses to indicate the priority of the logical connectives.
3. Rules to build new expressions, that are well-formed formulas (WFFs), using the previous symbols:
   a. Every statement symbol is a WFF.
   b. If A and B are WFF, then $(\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$, and $(A \leftrightarrow B)$ are also WFFs.
   c. A sequence of symbols is not a formula unless it verifies a. or b.

Comments

✓ For example, considering the previous rules, we can deduce that the expressions with more left parentheses than right parentheses are not formulas.

✓ The parentheses, as they do in the mathematical expressions, indicate the priority of the operators. In other words, the operator within parentheses must be applied before the rest of the operators. The following rules specify the use of the parentheses in WFF.

1. The expression $\neg A$ represents $(\neg A)$.
2. The expression $A \circ B$ represents $(A \circ B)$ where ○ represents ∧, ∨, $\rightarrow$, or $\leftrightarrow$.
3. The expression $A \circ B \circ C$ represents $A \circ (B \circ C)$ where ○ represnts ∧, ∨, $\rightarrow$, or $\leftrightarrow$.
4. The expression $A \circ B \rightarrow C$ represents $(A \circ B) \rightarrow C$ where ○ represents $\wedge$ or $\vee$.
5. The expression $A \rightarrow B \circ C$ represents $A \rightarrow (B \circ C)$ where ○ represents $\wedge$ or $\vee$.

We can also replace → for ↔ in 4. and 5., to obtain similar rules for the biconditional operator.

Example

$\left((A \rightarrow B) \vee ((A \wedge B) \rightarrow C)\right)$ is a WFF.

Exercise

Build a truth table for the previous WFF.

2.4. Tautologies and Contradictions

Definitions

A ***tautology*** is a WFF that always is true. It is denoted by T, and in its truth table, it always has a value of T for any value of the simple propositions that compose it.

A ***contradiction*** is a WFF that always is false. It is denoted by C, and in its truth table, it always has a value of F for any value of the simple propositions that compose it.

Examples

The WFF $A \lor \neg A$ is a tautology. Its truth table is the following:

A	$\neg A$	$A \lor \neg A$
T	F	**T**
T	F	**T**
F	T	**T**
F	T	**T**

The WFF $A \land \neg A$ is a contradiction. Its truth table is the following:

A	$\neg A$	$A \land \neg A$
T	F	**F**
T	F	**F**
F	T	**F**
F	T	**F**

2.4.1. Methods to verify tautologies

To determine whether a WFF A is a tautology, we can use different methods.

The first method is the simplest. It consists of building the truth table of the WFF. If true is the only possible value no matter the values of the propositions that compose the WFF, then A is a tautology.

The second method consists in supposing that the value of A is false for a certain truth value of the propositions that compose the WFF. If a contradiction is deducted from this hypothesis, the tautology will be true. If there is no contradiction, we will have found a truth value where A is false so the WFF A is not a tautology.

Comments

The second method is called reductio ad absurdum. The initial hypothesis considers the opposite case of what we want to prove. Reaching a contradiction indicates that our hypothesis is false, and hence what we want to prove is true.

If there is no contradiction, we will have reached a counterexample so we can conclude that what we want to prove is false.

Examples

1. The WFF $(A \vee \neg B) \to (B \to A)$ is a tautology:

The truth value of this WFF is always true in its truth table:

A	B	$\neg B$	$A \vee \neg B$	$B \to A$	$(A \vee \neg B) \to (B \to A)$
T	T	F	T	T	**T**
T	F	T	T	T	**T**
F	T	F	F	F	**T**
F	F	T	T	T	**T**

Supposing that the WFF has a value of false (F) for some of the truth values, we reach a contradiction:

$$(A \vee \neg B) \to (B \to A)$$

$$F_1$$

$$\boxed{T_2} \qquad F_3 \qquad \left(\xrightarrow{F} \Leftrightarrow T \to F\right)$$

$$T_4 \quad F_5$$

$$F_6 \quad F_7$$

$$\boxed{F_8}$$

The WFF $A \vee \neg B$ in the previous expression is simultaneously true (V_2) and false (F_8). It is a contradiction, so we can state that $(A \vee \neg B) \to (B \to A)$ is a tautology.

2. The WFF $(A \to B) \to (A \vee \neg B)$ is not a tautology:

The truth value of this WFF is not true in its truth table.

A	B	$\neg B$	$A \to B$	$A \vee \neg B$	$(A \to B) \to (A \vee \neg B)$
T	T	F	T	T	**T**
T	F	T	F	T	**T**
F	T	F	T	F	**F**
F	F	T	T	T	**T**

Supposing that the WFF has a value of false (F) for some of the truth values, we do not reach a contradiction. In fact, we reach the truth value of the last row:

$$(A \rightarrow B) \rightarrow (A \vee \neg B)$$

$$F_1$$

$$T_2 \qquad\qquad F_3 \qquad \left(\xrightarrow{F} \Leftrightarrow T \rightarrow F\right)$$

$$F_4 \quad F_5$$

$$F_6 \quad T_7$$

$$T_8$$

We do not reach a contradiction in the previous expression. In fact, the obtained truth values of the propositions (A false and B true) cause that the WFF has a value of F, (F_1).

2.5. Logical equivalences and implications

Definitions

Two WFF A and B are ***logically equivalent*** when $A \leftrightarrow B$ is a tautology. It is denoted by $A \Leftrightarrow B$.

Two WFF A ***logically implies*** B when WFF $A \rightarrow B$ is a tautology. It is denoted by $A \Rightarrow B$.

Comments

✓ If $A \Leftrightarrow B$, the truth tables of A and B are equal, and vice versa.

✓ Therefore, to prove that two WFFs are logically equivalent, we have two options. The first one requires to use the previous methods to prove that the WFF $A \leftrightarrow B$ is a tautology. The second one requires to prove that the truth tables of A and B are equal.

✓ To prove that $A \Rightarrow B$, we have to verify that $A \rightarrow B$ is a tautology.

Exercises

1. Prove that $A \wedge (A \rightarrow B) \Leftrightarrow B$.
2. Prove that the logical equivalences of the following table are tautologies.

Table of some logical equivalences

Given the arbitrary WFFs P, Q and R, where T and C are a tautology and a contradiction respectively:

Name	**Tautology**	
Double negation	$\neg(\neg P) \Leftrightarrow P$	
Commutative Laws	$P \wedge Q \Leftrightarrow Q \wedge P$ $P \vee Q \Leftrightarrow Q \vee P$	
Associative Laws	$P \wedge (Q \wedge R) \Leftrightarrow (P \wedge Q) \wedge R$ $P \vee (Q \vee R) \Leftrightarrow (P \vee Q) \vee R$	
Distributive Laws	$P \wedge (Q \vee R) \Leftrightarrow (P \wedge Q) \vee (P \wedge R)$ $P \vee (Q \wedge R) \Leftrightarrow (P \vee Q) \wedge (P \vee R)$	
De Morgan's Laws	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$	
Contraposition	$P \to Q \Leftrightarrow \neg Q \to \neg P$	
Identity and domination laws	$P \vee C \Leftrightarrow P$ $P \wedge C \Leftrightarrow C$	$P \vee T \Leftrightarrow T$ $P \wedge T \Leftrightarrow P$
Idempotent Laws	$(P \wedge P) \Leftrightarrow P$ $(P \vee P) \Leftrightarrow P$	
Logical Equivalence involving conjunction and implication	$P \wedge Q \Leftrightarrow \neg(P \to \neg Q)$	
Logical Equivalence involving disjunction and implication	$P \vee Q \Leftrightarrow \neg P \to Q$	

2.6. Propositional calculus

2.6.1. Formal logical systems

Definitions

A ***formal logical system*** is composed of specific WFFs that are considered ***axioms*** (they are generally tautologies) and of ***inference rules*** that allow us to deduce new WFFs from the others.

A ***formal proof or derivation*** is composed of a sequence of WFFs where each WFF is an axiom, an assumption, or is the result of applying an inference rule on an WFFs in the sequence. When there is no assumption in a formal proof, the last element of a formal proof is called ***theorem*** and the formal proof is the proof of the theorem.

A ***formal logical system*** is ***sound*** when all theorem in this system is a tautology.

A ***formal logical system*** is ***complete*** when we can prove that all its tautologies are theorems using its axioms and its inference rules.

Examples

1. The following formal logical system was developed by A. N. Whitehead and B. Russel:

   Axioms: $\forall P, Q, R$ WFFs verify:

   1. $(\boldsymbol{P} \vee \boldsymbol{P}) \rightarrow \boldsymbol{P}$

   2. $\boldsymbol{Q} \rightarrow (\boldsymbol{P} \vee \boldsymbol{Q})$

   3. $(\boldsymbol{P} \vee \boldsymbol{Q}) \rightarrow (\boldsymbol{Q} \vee \boldsymbol{P})$

   4. $(\boldsymbol{P} \rightarrow \boldsymbol{Q}) \rightarrow \left((\boldsymbol{R} \vee \boldsymbol{P}) \rightarrow (\boldsymbol{R} \vee \boldsymbol{Q})\right)$

   All of them are also tautologies.

   Inference rules

   1. Rules of replacement: The result of replacing any element of theorem with a WFF is a theorem.

   For example, replacing P *with* $A \wedge B$ *in the axiom 1.* $(\boldsymbol{P} \vee \boldsymbol{P}) \rightarrow \boldsymbol{P}$*, we obtain that* $\left((\boldsymbol{A} \wedge \boldsymbol{B}) \vee (\boldsymbol{A} \wedge \boldsymbol{B})\right) \rightarrow (\boldsymbol{A} \wedge \boldsymbol{B})$ *is another theorem of this formal logical system.*

   2. Modus ponens: $P \wedge (P \rightarrow Q) \rightarrow Q$. If P and $P \rightarrow Q$ are theorems, then Q is a theorem.

2. This is another formal logical system.

   Axioms: $\forall P, Q, R$ WFFs verify:

   1. $\boldsymbol{P} \rightarrow (\boldsymbol{Q} \rightarrow \boldsymbol{P})$

   2. $\left(\boldsymbol{P} \rightarrow (\boldsymbol{Q} \rightarrow \boldsymbol{R})\right) \rightarrow \left((\boldsymbol{P} \rightarrow \boldsymbol{Q}) \rightarrow (\boldsymbol{P} \rightarrow \boldsymbol{R})\right)$

   3. $(\neg \boldsymbol{Q} \rightarrow \neg \boldsymbol{P}) \rightarrow (\boldsymbol{P} \rightarrow \boldsymbol{Q})$

   All of them are also tautologies.

<u>**Inference rules**</u>**:** the same as in the previous example.

1. **Rules of replacement:** The result of replacing any element of theorem with a WFF is a theorem.

For example, replacing P *with* $A \wedge B$ *in the axiom 1.* $(\boldsymbol{P} \vee \boldsymbol{P}) \rightarrow \boldsymbol{P}$*, we obtain that* $((\boldsymbol{A} \wedge \boldsymbol{B}) \vee (\boldsymbol{A} \wedge \boldsymbol{B})) \rightarrow (\boldsymbol{A} \wedge \boldsymbol{B})$ *is another theorem of this formal logical system.*

2. **Modus ponens:** $P \wedge (P \rightarrow Q) \rightarrow Q$. If P and $P \rightarrow Q$ are theorems, then Q is a theorem.

Comments

✓ The two logical systems of the previous examples are correct and complete. That means that all their tautologies are theorems, and all their theorems are tautologies. Therefore, we can use the tautologies that are described by the table of logical equivalences in any formal proof.

✓ The set of theorems that can be proved with a logical system will depend on the number of axioms and the inference rules that we consider. If we select few axioms and inference rules, there will be few candidates for theorems. In contrast, if we select many axioms and inference rules, any WFFs will be probably a theorem.

2.6.2. Deductive arguments

In an argumentation where we use a logical deduction, we start from a set of ***hypotheses,*** $\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots, \boldsymbol{A_n}$, that are propositions whose truth value is considered T, to reach a conclusion or ***thesis,*** B, that is another proposition.

Using natural language, we describe an argumentation as follows:

'If A_1 *happens, and* A_2 *happens, … and* A_n *happens, then* B *will happen'*

Example

'If it rains (A_1*) and I do not have an umbrella (*A_2*), then I will get wet (*B*)'*

To turn this into the logic language, we will write: $A_1 \wedge A_2 \rightarrow B$.

In general, a deductive argumentation is turned into the logic language using the form:

$$A_1 \wedge A_2 \wedge \ldots . . \wedge A_n \rightarrow B$$

Definition

The ***argumentation*** $A_1 \wedge A_2 \wedge \ldots . . \wedge A_n \rightarrow B$ is ***valid*** if it is a tautology.

Comments

✓ To prove that an argumentation $A_1 \wedge A_2 \wedge \ldots . . \wedge A_n \rightarrow B$ is valid, we will consider that the hypotheses $\boldsymbol{A_1}, \boldsymbol{A_2}, \ldots, \boldsymbol{A_n}$ have a truth assignment of T. We will use any of the formal logical systems in the example to obtain, trough the formal proof, the thesis B as a theorem.

✓ If instead of the form $A_1 \wedge A_2 \wedge \ldots . \wedge A_n \rightarrow B$ we have an argumentation with the form $A_1 \wedge A_2 \wedge \ldots . \wedge A_n \rightarrow (A_{n+1} \rightarrow B)$ (in natural language, it will be *'If A_1 happens, and A_2 happens, …, and A_n happens, then if A_{n+1} happens, B will happen'*), we would use the formula $A_1 \wedge A_2 \wedge \ldots . \wedge A_n \wedge A_{n+1} \rightarrow B$, which is logically equivalent to $A_1 \wedge A_2 \wedge \ldots . \wedge A_n \rightarrow (A_{n+1} \rightarrow B)$, according to the tautology $P \rightarrow (Q \rightarrow R) \leftrightarrow P \wedge Q \rightarrow R$ and the rule of replacement.

✓ Since the formal logical system that we have seen are complete and correct, we can summarize the tautologies in the formal proofs.

Example

1. To prove the validity of the argument:

'If I study, I will pass the subjects; and if I pass the subjects, I will be a computer engineer. Since I study, I will be a computer engineer´

We follow these steps:

Step 1. Assign the symbols to the propositions that compose the argument:

$$A = \text{'I study'}$$
$$B = \text{'I pass'}$$
$$C = \text{'I will be a computer engineer'}$$

Step 2. Determine the WFF associated to the argument:

$$(A \rightarrow B) \wedge (B \rightarrow C) \rightarrow (A \rightarrow C)$$
$$\Updownarrow_{(P \rightarrow (Q \rightarrow R)) \leftrightarrow P \wedge Q \rightarrow R + Sustitution}$$
$$\underbrace{(A \rightarrow B) \wedge (B \rightarrow C) \wedge A}_{\text{Hypothesis}} \rightarrow \underbrace{C}_{\text{Thesis}}$$

Step 3. Build a formal proof that ends up in the thesis of our argument:

1. $A \rightarrow B$ (hypothesis)
2. $B \rightarrow C$ (hypothesis)
3. A (hypothesis)
4. B (1. 3. and modus ponens $(P \wedge (P \rightarrow Q) \rightarrow Q)$)
5. C (2., 4. and modus ponens)

2. To prove the validity of the argument:

'If I study or I am a genius, I will pass Discrete Mathematics. If I pass Discrete Mathematics, I will be able to pass the course. Therefore, if I cannot pass the year, then I am not a genius.

We follow these steps:

Step 1. Assign the symbols to the propositions that compose the argument:

$$A = \text{'I study'}$$
$$B = \text{'I am a genius'}$$
$$C = \text{'I will pass Discrete Mathematics'}$$
$$D = \text{'I will pass the year'}$$

Step 2. Determine the WFF associated to the argument:

$$\big((A \vee B) \to C\big) \wedge (C \to D) \to (\neg D \to \neg B)$$
$$\Updownarrow$$
$$\underbrace{\big((A \vee B) \to C\big) \wedge (C \to D) \wedge \neg D}_{\text{Hypothesis}} \to \underbrace{\neg B}_{\text{Thesis}}$$

Step 3. Build a formal proof that ends up in the thesis of our argument:

1. $(A \vee B) \to C$ (hypothesis)
2. $(C \to D)$ (hypothesis)
3. $\neg D$ (hypothesis)
4. $\neg C$ (2., 3. tautology $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ and rule of replacement)
5. $\neg(A \vee B)$ (1., 4. tautology $(P \to Q) \leftrightarrow (\neg Q \to \neg P)$ and rule of replacement)
6. $\neg A \wedge \neg B$ (5., tautology $\neg(P \vee Q) \leftrightarrow \neg P \wedge \neg Q$ and rule of replacement)
7. $\neg B$ (6., tautology $(P \wedge Q) \to Q$ and rule of replacement)

Comments

✓ To prove that an argument is valid, we can use the different formal proofs. The formal proof which proves the validity of an argument is not unique.

✓ Some arguments are not valid. To prove that an argument expressed by $A_1 \wedge A_2 \wedge \ldots . \wedge A_n \to B$ is not valid, we have several options.

1. The first option is to follow the steps of the previous examples until we build a formal proof which leads to a contradiction. However, this method is not easy because tautologies and inference rules are applied aimlessly, so we might reach neither a contradiction nor a proof.
2. The second option is to follow the previous step until the **step 2**. Then, we must find a truth assignment for the obtained WFF that makes the WFF take a truth value of F. To do this, you must apply the methods to verify tautologies that were before described.

2.7. Disjunctive and Conjunctive normal forms

Given a WFF, we know how to build the truth table. Nevertheless, we can raise the inverse problem: can we find the WFF associated with a truth table?

The answer is affirmative. To achieve it, we can use two equivalent options: (1) to calculate the disjunctive normal form of the table, or (2) to calculate the conjunctive normal form of the table.

Both methods will lead to different but equivalent WFFs (Note that both forms will have the same truth table). These two methods are presented through examples below.

Example

Given the following truth table where P, Q and R are arbitrary propositions:

P	Q	R	¿?
T	T	T	T
T	T	F	F
T	F	T	F
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	F
F	F	F	F

Disjunctive normal form

Step 1. Select the rows of the table where the truth value is T:

P	Q	R	¿?	
T	T	T	T	←
T	T	F	F	
T	F	T	F	
T	F	F	T	←
F	T	T	F	
F	T	F	T	←
F	F	T	F	
F	F	F	F	

Step 2. For each one of these rows, we obtain an element of the form. Specifically, the **conjunction** of the propositions corresponding to a row is an element of the form. If the truth value of a proposition is T, we consider the proposition. If the truth value of a proposition is F, we consider the negation of the proposition.

P	Q	R	¿?	
T	T	T	T	$(P \wedge Q \wedge R)$
T	F	F	T	$(P \wedge \neg Q \wedge \neg R)$
F	T	F	T	$(\neg P \wedge Q \wedge \neg R)$

Step 3. We join all the elements obtained by the previous step using the operator **disjunction**. The obtained WFF is the disjunctive normal form, which is associated with the truth table:

$$(P \wedge Q \wedge R) \vee (P \wedge \neg Q \wedge \neg R) \vee (\neg P \wedge Q \wedge \neg R)$$

Conjunctive normal form

Step 1. Select the rows of the table where the truth value is F:

P	Q	R	¿?	
T	T	T	T	
T	T	F	F	←
T	F	T	F	←
T	F	F	T	
F	T	T	F	←
F	T	F	T	
F	F	T	F	←
F	F	F	F	←

Step 2. For each one of these rows, we obtain an element of the form. Specifically, the **disjunction** of the propositions corresponding to a row is an element of the form. If the truth value of a proposition is T, we consider the negation of proposition. If the truth value of a proposition is F, we consider the proposition.

P	Q	R	¿?	
T	T	F	F	$(\neg P \vee \neg Q \vee R)$
T	F	T	F	$(\neg P \vee Q \vee \neg R)$
F	T	T	F	$(P \vee \neg Q \vee \neg R)$
F	F	T	F	$(P \vee Q \vee \neg R)$
F	F	F	F	$(P \vee Q \vee R)$

Step 3. We join all the elements obtained by the previous step using the operator **conjunction**. The obtained WFF is the disjunctive normal form, which is associated with the truth table:

$$(\neg P \vee \neg Q \vee R) \wedge (\neg P \vee Q \vee \neg R) \wedge (P \vee \neg Q \vee \neg R) \wedge (P \vee Q \vee \neg R) \wedge (P \vee Q \vee R)$$

2.8. Propositional logic and Boolean algebra

Definition

A Boolean algebra is a quadruple of the form $(B,',+,\cdot)$, where:

- ✓ B is a set.
- ✓ $': B \to B | x \to x'$, is a unitary operation.
- ✓ $+: B \times B \to B | (x,y) \to x+y$, is a binary operation.
- ✓ $\cdot: B \times B \to B | (x,y) \to x \cdot y$, is a binary operation.

Verifying that:

1. $x', (x+y), (x \cdot y) \in B \forall x, y \in B$, so they are internal operations.
2. Commutative Laws for binary operations
3. Associative Laws for binary operations
4. Distributive Laws between both binary properties
5. Identity Law for each one of the operations
6. Complementation law

Exercise

Prove that:

1. $(P, \neg, \wedge, \vee)$ is a Boolean algebra where P is a set composed by all WFFs.
2. $(P(U), (\ \ )^c, \cup, \cap)$ is a Boolean algebra where U is an arbitrary set.

Use any of the formal logical systems to prove the properties of the algebra of sets.

3. First-order logic (Predicate logic)

The deductive thinking model of propositional logic is too simple. It is easy to find argumentations that cannot be analyzed with this model: ' $(x < 3) \wedge (y < 0) \rightarrow (x \cdot y < 0)$'. If we consider that x represents a natural number, this statement is true. However, if x represents a real number, this statement is false.

The validity of the previous argumentation depends not only on the relation between the hypothesis and thesis, but also of the objects that appear, of their definition domain, and of the relations that are between them.

The model of first-order logic tackles this kind of statements. This model has a greater capacity that the model of propositional logic and is the one used in mathematics proofs.

3.1. First-order language

To define a first-order language, we need a set of symbols and a set of rules that allow us to build the elements of our language. The rules indicate whether a sequence of symbols (expressions) has a meaning in our language:

1. Set of symbols:
   A. Logical symbols:
      1. Parentheses ( )
      2. Logical connectives: $\neg, \wedge, \vee, \rightarrow$ y $\leftrightarrow$
      3. Variables $x_1, x_2, \ldots$
      4. The equals sign= (Optional)

   B. Parameters
      1. Quantifier ∀ (Universal) ∃ (Existential)
      2. Symbols of constants:$\infty, \emptyset, \ldots, 0, 1, \ldots, \pi,$
      3. Symbol of function: $f, g, \ldots, \times, -, +, /, \ldots$
      4. Symbol of predicate: $P, Q, R, \ldots, \in, <, >, \subset, , \cup, \cap, =, \ldots$

2. Set of **rules** to build a semantic unit:
   A. Terms: Names of objects or their relations.

   A ***term*** is:
      1. A constant or a variable.
      2. If $t_1, t_2, \ldots, t_n$ are terms and f_n is a symbol of a n-ary function (with n arguments), $f(t_1, t_2, \ldots, t_n)$ is a term.
      3. No sequence of symbols (expression) is a term unless bound by 1. or 2.

   B. Formulas or ***well-formed formulas:*** Assertions about objects or their relations.
      1. If $t_1, t_2, \ldots, t_n$ are terms and P_n is a symbol of a n-ary predicate (with n arguments), then $P_n(t_1, t_2, \ldots, t_n)$ is an **well-formed formula.**

2. If A and B are ***well-formed formulas,*** then $(\forall x)(A), (\exists x)(A), (\neg A)$, $(A \wedge B)$, $(A \vee B)$, $(A \rightarrow B)$ and $(A \leftrightarrow B)$ are also ***well-formed formulas***.
3. No sequence of symbols (expression) is a ***well-formed formula*** unless bound by 1. or 2.

Examples

1. The formal language of set theory is composed of the following symbols:

✓ All the logical symbols that appear in the list, even the equals sign.

✓ A symbol of binary predicate: ∈

✓ A symbol of constant: ∅

Some other symbols are usually added to simplify the notation: ⊆, ⊃,∩,∪..

2. 'Exists n that is an even number' would be translated into the first-order language as: $(\exists n)(P(n))$, where n is a variable, ∃ is the quantifier, and $P(n)$ is the predicate 'n *even number*'. The truth value of this WFF depends on the domain of the variable. If the domain is $\mathbb{Z}$, we will say that $(\exists n \in \mathbb{Z})(P(n))$ is a WFF which is true. However, if the domain of the variable is $D = \{2n-1\}_{n\in\mathbb{Z}}$, $(\exists n \in D)(P(n))$ is a false predicate.

3. 'Every real number x verifies that $|x|$ is greater or equal than zero´. It would be expressed as:$(\forall x \in \mathbb{R})(|x| \geq 0)$. If we only write $(\forall x)(P(x))$ where $P(x) = ' \underbrace{\underbrace{\left| \underbrace{x}_{variable} \right|}_{function} \underbrace{\geq}_{symbol\ of\ predicate} \underbrace{0}_{constant}}_{predicate} '$, we are not specifying the domain of the variable.

Comments:

✓ When we are stating general WFFs (to prove their validity, in formal examples…), we will not specify the domain of the variable.

✓ The domain of variable will be specified when we work with particular interpretations of the WFFs.

3.2. Bound variables and free variables

A ***variable*** is ***bound*** when:

1. It is associated with a quantifier e.g., $(\exists x)$ $(\forall x)$

2. The predicate to which the variable belongs is affected by a quantifier $(\forall x)P(x)$

A ***variable*** is ***free*** when it is not bound. It can be understood as a constant or a specific case. In some cases, we will use the first of the alphabet for the free variables, and we will use the last letters of the alphabet for the bound variables.

Examples

1. In the WFF $(\forall x)[(\exists y)P(x,y) \wedge Q(x,y)]$, the variable y is bound for the first two times that it appears (the first time for 1. Is associated with a quantifier, and the second time (in $P(x,y)$) for 2.), because the predicate $P(x,y)$ is affected by the previous quantifier. However, this variable is free in $Q(x,y)$.

2. In the WFF $(\forall x)[P(x,y) \rightarrow (\forall y)Q(y)]$, the variable x is bound the two times that it appears. In contrast, the variable y is free the first time that it appears and is bound the following two times that it appears.

Comments

✓ Remember that the criteria to remove the parentheses can be summarized as follows: The logical operators only affect what immediately follows the operators.

✓ This assertation can be generalized for the quantifiers. A quantifier only affects to the predicate that immediately follows the quantifier.

✓ To modify the range of the quantifiers (i.e., when more than one predicate is affected by the same quantifier), as well as to modify the range of the logical operators, we must use parentheses.

3.3. Validity

Definitions:

A ***WFF*** is ***valid*** when it is true for all their possible interpretations.

A ***WFF*** is ***contradictory*** when it is false for all their possible interpretations.

Comments:

✓ These definitions are similar to the definitions of tautology and contradiction that we have seen in the propositional logic model.

✓ To prove that a WFF is not valid, we must fin an interpretation of the WFF whose value is false.

✓ A WFF interpretation of the first-order logic consists of assigning values to the constants and assertations that appear in it and defining a domain of interpretation for the variables.

Examples

1. The WFF $(\forall x)P(x) \rightarrow (\exists x)P(x)$ is valid.

2. The WFF $(\exists x)P(x) \rightarrow (\forall x)P(x)$ is not valid. For example, given $P(x) = 'x \text{ is even}'$ and its domain of interpretation is $x \in \mathbb{Z}$: although there is an even number, it does not mean that all the integer numbers are even numbers.

3.4. Predicate calculus

The deductive argumentations have the same form as in the formal logical model, We will be able to logically deduct that a WFF is valid through formal proofs.

To argue the first-order formal proofs, we will use a more powerful formal logical system. It includes some axioms and more deductive rules.

Furthermore, to prove that a WFF or an argumentation is not valid, we only need to find an interpretation of the WFF or of the argumentation whose truth value is false.

Formal logical system for the first-order logic

Axioms: $\forall P, Q, R$ WFFs, it is verified that:

1. $P \rightarrow (Q \rightarrow P)$
2. $\big(P \rightarrow (Q \rightarrow R)\big) \rightarrow \big((P \rightarrow Q) \rightarrow (P \rightarrow R)\big)$
3. $(\neg Q \rightarrow \neg P) \rightarrow (P \rightarrow Q)$
4. $(\forall x)\big(P(x) \rightarrow Q(x)\big) \rightarrow \big((\forall x)P(x) \rightarrow (\forall x)Q(x)\big)$ [Distributive]
5. $(\forall x)P(x) \rightarrow P(x)$ or $(\forall x)P(x) \rightarrow P(a)$, where a is any constant of the same definition domain than x. [Universal Exemplification]
6. $(\exists x)P(x) \rightarrow P(t)$, where t is any variable or constant that is not used in the sequence of the proof. [Existential Exemplification]
7. $P(x) \rightarrow (\exists x)P(x)$ or $P(a) \rightarrow (\exists x)P(x)$, where a is any constant and x is a variable that does not appear in $P(a)$. [Existential Generalization]
8. $\neg\big((\exists x)P(x)\big) \leftrightarrow (\forall x)\big(\neg P(x)\big)$ [Negation]

Inference Rules

1. **Rule of replacement:** The result of replacing any element of a theorem by a WFF is a theorem.

   For example, if we replace P by $A \wedge B$ in the axiom 1. $(\boldsymbol{P} \vee \boldsymbol{P}) \rightarrow \boldsymbol{P}$, we will obtain that $\big((\boldsymbol{A} \wedge \boldsymbol{B}) \vee (\boldsymbol{A} \wedge \boldsymbol{B})\big) \rightarrow (\boldsymbol{A} \wedge \boldsymbol{B})$ is another theorem of this formal logical system.
2. **Modus ponens:** $P \wedge (P \rightarrow Q) \rightarrow Q$. If P and $P \rightarrow Q$ are theorems, then Q is a theorem.
3. **Universal Generalization:** $Q(a) \rightarrow \forall x\big(Q(x)\big)$ provided that:
   a. $Q(a)$ has not been deducted from a hypothesis where x is the free variable.
   b. $Q(a)$ has not been deducted through the axiom 6. from a WFF with the form $(\exists y)Q(y)$.

Comments

✓ This formal logical system is complete. In other words, every valid WFF is a theorem in a logical system and is correct. That means that every theorem is a valid formula.

✓ If we had used the axioms of A. N. Whitehead and B. Rusell instead of the first four axioms, we would have obtained a different formal logical system that would be also correct and complete.

✓ Since the formal logical system is correct and complete. In our formal proofs, we will be able to use all the tautologies that we have seen and all the argumentations that are valid according to the proofs previously performed.

Practical summary:

To build formal proofs in first-order logic, we only have to include the following axioms and inference rules to the set of axioms and rules that we used to build formal proofs in propositional logic.
Note that all the WFFs in this section are valid in the first-order logic. This verification is easy from the axioms and inference rules of the previous formal logical order.

- Negation of quantifiers
  1. $\neg\big((\exists \boldsymbol{x})\boldsymbol{P}(\boldsymbol{x})\big) \leftrightarrow (\forall \boldsymbol{x})\big(\neg \boldsymbol{P}(\boldsymbol{x})\big)$
  2. $\neg\big((\forall \boldsymbol{x})\boldsymbol{P}(\boldsymbol{x})\big) \leftrightarrow (\exists \boldsymbol{x})\big(\neg \boldsymbol{P}(\boldsymbol{x})\big)$

- Inference rules of universal and existential generalization and inference rules of universal and existential exemplification. To add and remove quantifiers.

1. Universal Exemplification (UE). $(\forall x)P(x) \rightarrow P(a)$

Given a true statement of the form $(\forall x)P(x)$, provided that the variable x is replaced by constants of its interpretation domain, the statement will be true.

2. Universal Generalization (UG). $P(a) \rightarrow (\forall x)P(x)$

If a predicate $P(a)$ is true for all the constants of its interpretation domain, we can infer the truth of $(\forall x)P(x)$.

3. Existential Exemplification (EE). $(\exists x)P(x) \rightarrow P(a)$

Given a true statement of the form $(\exists x)P(x)$, we can infer out of it a substitution case of the propositional function, with the restriction of using a constant or variable (in this case a) that has not appeared before within the proof. That is, a constant or variable that has not been specified of delimited.

4. Existential Generalization (EG). $P(a) \rightarrow (\exists x)P(x)$

If a predicate $P(a)$ is true for at least one of its substitution cases as constants of its interpretation domain, we can infer the truth of $(\exists x)P(x)$.

Comments:

✓ The restriction in EE has the function of avoiding invalid inferences such as:
"Some men are wise, so Peter is wise."
We want to prove that:

$$(\exists x)\big(M(x) \wedge W(x)\big) \rightarrow W(p)$$

So we construct the next formal proof:

1. $(\exists x)\big(M(x) \wedge W(x)\big)$ (Hypothesis)
2. $M(p) \wedge W(p)$ (1 and EE, if the restriction did not exist).
3. $W(p)$ (2 and the simplification of the conjunction).

✓ Since the constant p has already appeared in the proposition to be proved, the restriction prevents that it to be used in the second step. When EE and UE have to be applied in a proof as in the example, it is necessary to apply EE (it is the rule with restrictions), and then, UE can be applied on the same constant without problems.

Example

Show the validity of the following reasoning:

'Dogs are vertebrates and mammals.
Some dogs are guardians.
Then some vertebrates are guardians.'

The representation of this reasoning in the language of the first-order logic is:

$$\Big((\forall x)P(x) \rightarrow \big(V(x) \wedge M(x)\big)\Big) \wedge \Big((\exists x)\big(P(x) \wedge G(x)\big)\Big) \rightarrow (\exists x)\big(V(x) \wedge G(x)\big)$$

From the following formal proof, we will be able to prove the validity of this reasoning:

1. $(\forall x)P(x) \rightarrow \big(V(x) \wedge M(x)\big)$
2. $(\exists x)\big(P(x) \wedge G(x)\big)$
3. $P(a) \wedge G(a)$ EE in 2 (a is a constant / free variable which is particular, but it is not specified).
4. $P(a) \rightarrow \big(V(a) \wedge M(a)\big)$ UE in 1.
5. $P(a)$ simplification of 3.
6. $V(a) \wedge M(a)$ 4 and 5 modus ponens
7. $G(a)$ simplification in 3.
8. $G(a) \wedge V(a)$ conjunction 6 y 7.
9. $(\exists x)\big(V(x) \wedge G(x)\big)$ EG in 8.

❖ Distributive laws.

1. $(\forall x)\big(P(x) \wedge Q(x)\big) \Leftrightarrow (\forall x)P(x) \wedge (\forall x)Q(x)$

2. $(\exists x)\big(P(x) \vee Q(x)\big) \Leftrightarrow (\exists x)P(x) \vee (\exists x)Q(x)$

3. $(\forall x)P(x) \vee (\forall x)Q(x) \Rightarrow (\forall x)\big(P(x) \vee Q(x)\big)$

4. $(\exists x)\big(P(x) \wedge Q(x)\big) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

5. $(\forall x)\big(P(x) \to Q(x)\big) \Rightarrow (\forall x)P(x) \to (\forall x)Q(x)$

Comments:

✓ The reciprocal implication of 3 is not always true. For example, if we suppose that all real numbers are rational or irrational, we will not be able to conclude that all real numbers are rational or that all real numbers are irrational.

✓ The reciprocal implication of 4 is not always true. For example, although there are rational numbers and there are irrational numbers, we cannot conclude that some numbers are rational and irrational simultaneously.

Example

We will write the following argument with symbols, and we will prove its validity through a formal proof.

'Any dragon is green or yellow. The father of a yellow dragon is always yellow. The mother of a green dragon is always green. We know that there is a dragon whose mother is not green, so there is a dragon whose father is yellow'

Step 1. **Interpretation domain**

Dragons

Step 2. **Symbols of function:**

$$p(x) = x's\ father$$

$$m(x) = x's\ mother$$

Step 3. **Creating the symbols of the predicate:**

$$A(x) =' x\ is\ yellow'$$

$$B(x) =' x\ is\ green'$$

Step 4. **Building the well-formed formula (WFF)**

PREDICATES

(HYPOTHESIS) Any dragon is green or yellow:

$$(\forall x)\big(A(x) \vee B(x)\big)$$

(HYPOTHESIS) The father of a yellow dragon is always yellow:

(If a dragon is yellow, its father will be yellow)

$(\forall x)\left(A(x) \to A\big(p(x)\big)\right)$ (It can be biconditional)

(HYPOTHESIS) The mother of a green dragon is always green.

(If a dragon is green, its mother will be green)

$(\forall x)\left(B(x) \to B\big(m(x)\big)\right)$ (It can be also biconditional)

(HYPOTHESIS) There is a dragon whose mother is not green.

$$(\exists x)\left(\neg B(m(x))\right)$$

Therefore →

(THESIS) There is a dragon whose father is yellow.

$$(\exists x)\left(A(p(x))\right)$$

WFF

$$\left((\forall x)(A(x) \vee B(x))\right) \wedge \left((\forall x)\left(A(x) \to A(p(x))\right)\right) \wedge \left((\forall x)\left(B(x) \to B(m(x))\right)\right) \wedge \left((\exists x)\left(\neg B(m(x))\right)\right) \to \left((\exists x)A(p(x))\right)$$

Step 5. Formal proof

1. $(\forall x)(A(x) \vee B(x))$ (HYPOTHESIS)
2. $(\forall x)\left(A(x) \to A(p(x))\right)$ (HYPOTHESIS)
3. $(\forall x)\left(B(x) \to B(m(x))\right)$ (HIPOTHESIS)
4. $(\exists x)\neg B(m(x))$ (HYPOTHESIS)
5. $\neg B(m(d))$ (4+Existential Exemplification x=d)
6. $(A(d) \vee B(d))$ (Universal Exemplification in 1 x=d)
7. $\left(A(d) \to A(p(d))\right)$ (Universal Exemplification in 2 x=d)
8. $\left(B(d) \to B(m(d))\right)$ (Universal Exemplification in 3 x=d)
9. $\left(\neg B(m(d)) \to \neg B(d)\right)$ (8 + contraposition $(P \to Q \leftrightarrow \neg Q \to \neg P)$ + replacement)
10. $\neg B(d)$ (5+9+Modus Ponens + replacement)
11. $A(d)$ (1+10+$\left((P \vee Q) \wedge \neg Q \to P\right)$ + replacement)
12. $A(p(d))$ (11+7 + Modus Ponens + replacement)
13. $(\exists x)A(p(x))$ (12+ Universal Generalization)

3.5. Applications

First-order logic is applied for different purposes in computer science. Among them, first-order logic is applied to develop expert systems that are focused on decision-making problems.

An expert system is a software program that encodes a knowledge model into a narrow field in order to simulate decision-making ability of a human expert (medical diagnostics, machine failure detection, trajectory planning, …)

This kind of studies are developed in subjects such as artificial intelligence or theory of knowledge. In these subjects, one of the main problems is to represent the knowledge that a human expert has and translate it into a computer language.

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