Collection of exercises to practice notation and the theoretical contents of the unit. You can search for the solution or ask for help through the bibliography and the corresponding forums. Use the forum for each unit to share the solutions to the proposed problems with other students. In these forums, all students can pose or answer questions. Peer collaboration is a very powerful tool for improving abilities associated to problem solving.

Exercise 1. Indicate which of the following are propositions:

a) $x^2 + y > 0$
b) Earth is a star of the Sun.
c) That guy is pretty boring.
d) Seventy is an even number.
e) This sentence is false.

Exercise 2. Calculate the truth value for the following propositions:

a) 9 is odd or 11 is even
b) 9 is odd and 11 is even
c) 9 is even or 11 is even
d) If 9 is even, then 11 is even
e) If 9 is odd, then 11 is even

Exercise 3. In each of the following items, three possible ways of negating the proposition are given. Indicate which is the correct one:

a) The solution is either 2 or 3
- i) Neither 2 nor 3 are the solution
- ii) The solution is not 2 or is not 3
- iii) The solution is not 2 and the solution is not 3

b) Computers have a keyboard and a screen
- i) Computers have a keyboard and do not have a screen
- ii) Computers do not have either keyboard or screen
- iii) Computers do not have a keyboard or do not have a screen

c) $3 < 9$ y 5 is odd
- i) $3 \geq 9$ and 5 is even
- ii) $3 \geq 9$ or 5 is even
- iii) $3 > 9$ or 5 is even

Exercise 4. Rewrite the following propositions symbolically. Indicate the inverse, reciprocal, and contraposed propositions to each of them. Calculate the truth value of each of the given and calculated propositions.

a) $|-1| < 3$ if $-3 < -1 < 3$

b) $|5| < 4$ if $-4 < 5 < 4$

c) If $2 < 7$ then $6 > 8$

d) If $2 > 7$ then $6 < 8$

Notes:

Be it R a ***conditional*** proposition of the form $P \to Q$. The proposition $\neg P \to \neg Q$ is the ***inverse*** proposition of $\boldsymbol{R}$, the proposition $Q \to P$ is the ***reciprocal*** proposition of $\boldsymbol{R}$ and the proposition $\neg Q \to \neg P$ is the ***contraposed*** proposition of R.

Exercise 5. Write the truth table of the following propositions (Well-formed formulas):

a) $P \wedge Q \leftrightarrow \neg P \wedge Q$

b) $(P \to Q) \leftrightarrow (Q \to P)$

c) $\big((P \vee Q) \vee R\big) \to \big(P \vee (Q \vee R)\big)$

Exercise 6. Verify through two different arguments that the following well-formed formulas are tautologies:

a) $\neg(P \wedge Q) \leftrightarrow (\neg P) \vee (\neg Q)$

b) $\big((P \to Q) \wedge (R \to S)\big) \to \big((P \wedge R) \to (Q \wedge S)\big)$

Exercise 7. Verify that the following well-formed formula is a tautology: $\big(P \to (Q \to R)\big) \leftrightarrow \big((P \to Q) \to R\big)$. Do not use a truth table to that extent.

Exercise 8. Prove that the operator $\to$ can be expressed in terms of the operators $\vee$ and $\neg$. You can try to achieve it by proving that $P \to Q$ is logically equivalent to $(\neg P) \vee Q$.

Exercise 9. Reason whether the following pairs of well-formed formulas are logically equivalent:

a) $(P \to Q) \to Q$ and $(\neg Q \to \neg P) \to Q$

b) $P \to Q$ and $\neg P \vee (P \wedge Q)$

c) $(P \to Q) \wedge (Q \to P)$ and $(P \wedge Q) \to (P \wedge Q)$

Exercise 10. Check the validity of the following reasonings using deductive chains. The usage of other methods beyond deductive chains will be positively valued.

a) All humans are mortal. All Spanish people are human. Hence, the Spanish are mortal.

b) If a program does not fail, then it starts and ends. The program starts and fails. Hence, the program does not end.

c) If José won the race, then either Pedro was second or Ramón was second. If Pedro was second, then José did not win the race. If Carlos was second, then Ramón was not second. José won the race. Hence, Carlos was not second.

d) If a program is efficient, then it runs quickly. The program is either efficient or has a virus. The program is not running quickly. Therefore, the program has a virus.

e) If I study, I will not fail this subject. If I don't play video games with my computer, I will study. I failed the subject. Therefore, I played video games with my computer.

f) The hacker that obtained the data from the Bill Gates' account was intelligent. Either Windows was not safe, or Bill Gates committed a mistake with his account. Bill did not commit a mistake, but if Windows' software engineers did not fail, then Windows was safe. Therefore, Windows' software engineers failed, and the hacker was intelligent.

Exercise 11. Prove whether the following conditional well-formed formulas are theorems:

a) $\big((Q \wedge R) \rightarrow P\big) \wedge (Q \rightarrow \neg R) \rightarrow P$

b) $(Q \vee \neg R) \wedge \Big(\big(\neg(R \rightarrow Q)\big) \rightarrow (\neg P)\Big) \rightarrow P$

c) $\big(P \rightarrow (Q \vee R)\big) \wedge (Q \rightarrow S) \wedge (R \rightarrow \neg P) \rightarrow (P \rightarrow S)$

Exercise 12. A new logical operator $\oplus$, called exclusive disjunction, is defined. Through the operator, $P \oplus Q$ is true only when one of the two propositions, that is, either P or Q, is true. Find a way to express this new operator using only conjunction, disjunction, and negation.

Exercise 13. Via the following truth table, find the disjunctive and conjunctive normal forms of G and H.

P	Q	R	G	H
T	T	T	T	F
T	T	F	F	T
T	F	T	F	T
T	F	F	T	F
F	T	T	F	T
F	T	F	F	T
F	F	T	T	F
F	F	F	T	F

Exercise 14. Given WFF $P \wedge \big(Q \vee (\neg R \wedge S)\big)$, build its truth table. From the truth table, build an equivalent WFF.

Exercise 15. Study the validity of the following reasoning using three different methods: *If the band could not play rock or the drinks did not arrive on time, then the New Year's party would have to be cancelled and Alice would get mad. If the party was cancelled, the money would have to be returned. The money was not returned and hence, the band could play rock.*

Exercise 16. Write the following argument in its symbolic form and reason whether it is valid. In case it is valid, go through a deductive argumentation that proves it: *If we go to Asia, we will reach India. If we reach India, we will visit Varanasi. If we visit Varanasi, we will be able to see the Ganges river. Hence, if we go to Asia, we will be able to see the Ganges river.*

Exercise 17. Write the following argument in its symbolic form and reason whether it is valid. In case it is valid, go through a deductive argumentation that proves it: *If x is 1 and y is 2, then z will be 3. In addition, if y is different from 2 or z is equal to 3, then w is 0. Since x is 1, then w will be 0.*

Exercise 18. Given the following proposition (well-formed formula): $\big(P \wedge (Q \rightarrow R)\big) \wedge Q \rightarrow P \wedge R$

a) Prove whether it is a tautology through two different methods.

b) Build its negation. What kind of proposition did you obtain?

Exercise 19. Given well-formed formula: $\big((\neg P \vee Q) \rightarrow R\big) \wedge \big(R \rightarrow (S \vee T)\big) \wedge (\neg S) \wedge (\neg U \rightarrow \neg T) \wedge (\neg U) \rightarrow P$, prove whether it a tautology through three different methods.

Exercise 20. Reason in three different manners whether the following logical argument is a theorem: $(\neg \boldsymbol{P} \vee \boldsymbol{Q}) \wedge (\neg \boldsymbol{Q} \vee \neg \boldsymbol{R}) \wedge (\boldsymbol{P} \vee \neg \boldsymbol{R}) \rightarrow \neg \boldsymbol{R}$

Exercise 21. Reason in three different manners whether the following logical argument is a theorem:

$$\boldsymbol{P} \vee \big((\neg \boldsymbol{P}) \wedge \neg(\boldsymbol{Q} \vee \boldsymbol{R})\big) \rightarrow (\boldsymbol{P} \vee \neg \boldsymbol{R})$$

If P ='Today is Monday', Q ='It is raining', and R ='It is hot in here', translate into natural language the prior symbolic expression and its negation.

Exercise 22. A fundamental principle of logical reasoning, the syllogism law, establishes that if *p* implies *q* and *q* implies *r*, then *p* implies *r*.

a) Express the law using logical connectors

b) Prove whether it is a valid reasoning in three different ways, being one of them a deductive chain

Exercise 23. Express the following proposition using logical language, and reason whether it is a tautology in two different ways: *If p and q are verified, then it is not true that p implies that q is not upheld.*

FIRST ORDER LOGIC

Exercise 24. Indicate the truth value of the following first order WFFs, reasoning your response:

- **a)** $(\forall m \in \mathbb{N})(\exists n \in \mathbb{N})(n > 2m)$
- **b)** $(\exists n)(\forall m)(n > 2m)$ in $\mathbb{N}$, that is $(\exists n \in \mathbb{N})(\forall m \in \mathbb{N})(n > 2m)$
- **c)** $(\forall x)(\exists y)(x + y = 0)$ in $\mathbb{Z}$
- **d)** $(\exists y)(\forall x)(x + y = 0)$ in $\mathbb{Z}$
- **e)** $(\exists y)(\forall x)(x \cdot y = 0)$ in $\mathbb{Z}$

Exercise 25. Calculate the truth value for the following WFFs in $\mathbb{Z}$.

- **a)** $(\forall x)(\exists y)(x + y = x)$
- **b)** $(\exists y)(\forall x)(x + y = x)$
- **c)** $(\forall x)(\forall y)(x < y \vee y < x)$
- **d)** $(\forall x)\big(x < 0 \rightarrow (\exists y)(y > 0 \wedge x + y = 0)\big)$
- **e)** $(\exists x)(\exists y)(x^2 = y)$
- **f)** $(\forall x)(x^2 > 0)$

Exercise 26. Given WFF $(\exists x)\Big(A(x) \wedge (\forall y)\big(B(x,y) \rightarrow C(y)\big)\Big)$

- **a)** Calculate its truth value in $\mathbb{Z}$, $A(x) = 'x > 0'$, $B(x,y) = 'x > y'$ and $C(y) = 'y \leq 0'$.
- **b)** Find another interpretation of the WFF for which it takes a truth value equal to false within $\mathbb{Z}$

Exercise 27. For each of the following WFFs, find an interpretation for which they are true and another interpretation for which they are false:

- **a)** $(\forall x)\big[\big(A(x) \vee B(x)\big) \wedge \neg\big(A(x) \wedge B(x)\big)\big]$
- **b)** $(\forall x)(\forall y)\big(P(x,y) \rightarrow P(y,x)\big)$
- **c)** $(\forall x)\Big(P(x) \rightarrow (\exists y)\big(Q(x,y)\big)\Big)$
- **d)** $(\exists x)\big(P(x)\big) \rightarrow (\forall x)\big(P(x)\big)$

Exercise 28. Provide the adequate interpretations to prove that the following WFFs are not valid:

- **a)** $(\exists x)A(x) \wedge (\exists x)B(x) \rightarrow (\exists x)\big(A(x) \wedge B(x)\big)$
- **b)** $(\forall x)(\exists y)P(x,y) \rightarrow (\exists x)(\forall y)P(y,x)$
- **c)** $(\forall x)\big(P(x) \rightarrow Q(x)\big) \rightarrow \big((\exists x)P(x) \rightarrow (\forall x)Q(x)\big)$
- **d)** $(\forall x)\big(\neg A(x)\big) \leftrightarrow \big((\forall x)A(x)\big)$

Exercise 29. Several negation forms are proposed for each of the following predicates. Choose the correct one:

a) Everyone likes chocolate
   i) Nobody likes chocolate
   ii) Some people dislike chocolate
   iii) Everyone hates chocolate

b) Some people like mathematics
   i) Some people do not like mathematics
   ii) Everyone hates mathematics
   iii) Everyone likes mathematics

c) Some computer programs are slow or efficient
   i) All computer programs are slow or efficient
   ii) Some computer programs are quick and efficient
   iii) All computer programs are quick and efficient

Exercise 30. Negate the following WFFs:

a) $(\forall x)(\forall y)(x < z < y)$

b) $(\exists x)(\forall y)(\exists z)(z < y \to z < x^2)$

Exercise 31. Given the following WFF: $(\forall x)(\forall y)\big(x < y \to (\exists z)(x < z < y)\big)$

a) Obtain its negated expression

b) Determine the truth value in:
   i) $\mathbb{N}$
   ii) $\mathbb{Z}$
   iii) $\mathbb{Q}$
   iv) $\mathbb{R}$

Exercise 32. The following WFF is not valid: $(\forall y)(\exists x)P(x,y) \to (\exists x)(\forall y)P(x,y)$

a) Find an interpretation that proves that it is not valid

b) Find the mistake in the proof of the WFF:
   i) $(\forall y)(\exists x)P(x,y)$ (Hypothesis)
   ii) $(\exists x)P(x,y)$ (1. Universal exemplification $(\forall x)P(x) \to P(a)$ and substitution)
   iii) $P(a,b)$ (2. Existential exemplification $(\exists x)P(x) \to P(b)$) and substitution)
   iv) $(\forall y)P(a,y)$ (3. Universal generalization)
   v) $(\exists x)(\forall y)P(x,y)$ (4. Existential generalization $P \to P(x)$)

Exercise 33. Convert the following arguments into symbols and reason whether they are valid. Use a deductive chain to prove the validity of the arguments and an interpretation that makes the WFF that represents them false in order to justify that they are not valid.

a) There is a computer scientist that is not short-sighted. Everyone who wears glasses is short-sighted. In addition, everyone who works with computers either wears glasses or contact lenses. Hence, there is a computer scientist who wears contact lenses.

b) Every member of Walqa's board of directors comes either from a private company or from the DGA's main government body. Everyone who comes from the DGA government and holds a computer science degree is in favor of Walqa. Juan dies not come from a private company and hold a computer science degree. Therefore, if Juan is a member of Walqa's board of directors, he is in favor of Walqa.

Exercise 34. Using predicate calculus, prove the validity of the following WFFs:

a) $(\forall x)P(x) \rightarrow (\exists x)P(x)$

b) $(\forall x)P(x) \rightarrow (\forall x)\big(P(x) \wedge Q(x)\big)$

c) $(\forall x)P(x) \wedge (\exists x)Q(x) \rightarrow (\exists x)\big(P(x) \wedge Q(x)\big)$

d) $(\exists x)(\exists y)P(x,y) \rightarrow (\exists y)(\exists x)P(x,y)$

e) $(\forall x)(\forall y)P(x,y) \rightarrow (\forall y)(\forall x)P(x,y)$

f) $(\exists x)\big(P(x) \wedge Q(x)\big) \rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

g) $(\exists x)\big(P(x) \vee Q(x)\big) \rightarrow (\exists x)P(x) \vee (\exists x)Q(x)$

h) $(\exists x)(\forall y)P(x,y) \rightarrow (\forall y)(\exists x)P(x,y)$

Exercise 35. Write the following statement in a symbolic manner and reason its validity: *Any worker of TID is either a computer scientist or a telecommunications engineer. Everyone who holds the title of telecommunications engineer and knows programming has passed a specific course. Pedro is not a computer scientist but knows programming. If Pedro works in TID, then he has passed a specific course.*

Exercise 36. Prove the validity of the following distributive laws:

a) $(\forall x)\big(P(x) \wedge Q(x)\big) \Leftrightarrow (\forall x)P(x) \wedge (\forall x)Q(x)$

b) $(\exists x)\big(P(x) \vee Q(x)\big) \Leftrightarrow (\exists x)P(x) \vee (\exists x)Q(x)$

c) $(\forall x)P(x) \vee (\forall x)Q(x) \Rightarrow (\forall x)\big(P(x) \vee Q(x)\big)$

d) $(\exists x)\big(P(x) \wedge Q(x)\big) \Rightarrow (\exists x)P(x) \wedge (\exists x)Q(x)$

e) $(\forall x)\big(P(x) \rightarrow Q(x)\big) \Rightarrow (\forall x)P(x) \rightarrow (\forall x)Q(x)$

Exercise 37. Given $P(n) = (\forall n \in \mathbb{N})(n > 3 \rightarrow n^2 \geq 2n + 3)$

a) Build its negation

b) Reason the truth values of $P(n)$ and $\neg P(n)$, providing appropriate proof

References

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