Collection of exercises to practice the notation and the theorical contents of the unit. You can search the solutions in the bibliography of the unit or ask for help through the forum of the unit.

Use the forum for each unit to share the solutions to the proposed problems with other students. In these forums, all students can pose or answer questions. Peer collaboration is a very powerful tool for improving abilities associated to problem solving.

Sets

Exercise 1. Given $I = \{3,4,5,6,7\}$ where for each $i \in I$, $A_i = \{1,2,\ldots,i\} \subseteq U = \mathbb{Z}^+$, calculate:

a) $\bigcup_{i=3}^{7} A_i$

b) $\bigcap_{i=3}^{7} A_i$

Exercise 2. Given $U = \mathbb{R}$ and $I = \mathbb{N}_0 = \mathbb{N}^* = \{0,1,2,3,\ldots\}$ for each $r \in I, A_r = [-r, r]$, calculate:

a) $\bigcup_{r \in I} A_r$

b) $\bigcap_{r \in I} A_r$

Exercise 3. Given $U = \{1,2,3,4,5,6,7,8,9,10\}$, the sets are $A = \{1,2,3,4,5\}$, $B = \{1,2,3,4\}$, $C = \{1,2,3,4,7\}$, and $D = \{2,4,6,8\}$. Calculate the following sets:

a) $(A \cup B) \cap C$

b) $A \cup (B \cap C)$

c) $C^c \cup D^c$

d) $(C \cap D)^c$

e) $(A \cup B) - C$

f) $A \cup (B - C)$

g) $(B - C) - D$

h) $B - (C - D)$

i) $(A \cup B) - (C \cap D)$

Exercise 4. Given $A, B, C, D, E \subseteq \mathbb{Z}$ and considering $A = \{2n, n \in \mathbb{Z}\}, B = \{3n, n \in \mathbb{Z}\}, C = \{6n, n \in \mathbb{Z}\}$, $D = \{4n, n \in \mathbb{Z}\}$, $E = \{8n, n \in \mathbb{Z}\}$:

a) Indicate and justify which of the following statements are true and which ones are false:

i) $E \subseteq C \subseteq A$

ii) $A \subseteq C \subseteq E$

iii) $B \subseteq D$

iv) $D \subseteq B$

v) $D \subseteq A$

vi) $D^c \subseteq A^c$

b) Determine the following sets:

i) $C \cap E$

ii) $B \cup D$

iii) $A \cap B$

iv) $B \cap D$

v) A^c

vi) $A \cap E$

Exercise 5. Given $A = [0,3], B = 2,7)$, and $U = \mathbb{R}$, determine:

a) $A \cap B$

b) $A \cup B$

c) A^c

d) $A \Delta B$

e) $A - B$

f) $B - A$

Exercise 6. Determine sets A and B with $A - B = \{1,3,7,11\}, B - A = \{2,6,8\}$ and $B \cap A = \{4,9\}$.

Exercise 7. Based on the properties of set algebra (i.e., the laws of set algebra), simplify the following expressions:

a) $A \cap (B - A)$

b) $(A - B) \cup (A \cap B)$

c) $(A \cap B) \cup (A \cap B \cap C^c \cap D) \cup (A^c \cap B)$

d) $A^c \cup B^c \cup (A \cap B \cap C^c)$

e) $A^c \cup (A \cap B^c) \cup (A \cap B \cap C^c) \cup (A \cap B \cap C \cap D^c) \cup \ldots$

Exercise 8. Given the universal set U and the sets $A, B \subseteq U$, sort the following lists of sets regarding their cardinalities:

a) $|A \cup B|, |B|, |\emptyset|, |A \cap B|, |U|$

b) $|A - B|, |\emptyset|, |A \triangle B|, |A \cup B|, |U|$

c) $|A - B|, |\emptyset|, |A|, |A| + |B|, |A \cup B|$

Exercise 9. Given the set of all people as universal set, A is the set of all the system analysts, B is the set of all the accountants, C is the set all the women, and D is the set of all the people over 40 years old. Express using the proper notation of the set theory the following sets:

a) The set of all the women who are both system analysts and accountants.

b) The set of all the men over 40 years old who are accountants.

c) The set of all the women under 40 years old who are system analysts and of all the accountants who are under 40 years old.

d) The set of all the men under 40 years old who are accountants and who are not system analysts.

e) The set of all the men under 40 years old who are neither system analysts, nor accountants.

Exercise 10. Given the universal set $U = \{1,2,3,4,5,6\}$ and subsets $A = \{1,3,4\}$, $B = \{1,4,6\}$, $C = \{1,3,5\}$, and $D = \{2,4,6\}$:

a) Calculate $A \cup B, A \cap B, C \cup D$ y $C \cap D$. How are the sets C and D with respect to U?

b) Calculate $A^c, B^c, (A \cap B)^c, (A \cup B)^c$

c) Check if the following expressions fulfill the DeMorgan's laws:

i) $(A \cap B)^c = A^c \cup B^c$

ii) $(A \cup B)^c = A^c \cap B^c$

d) Check that the following expressions is verified: $|A \cup B| + |A \cap B| = |A| + |B|$

e) Check that the sets $R = A \cap B^c, S = A^c \cap B, V = A \cap B$ y $W = (A \cup B)^c$ form a partition of the U, which contains all the previous sets.

Comment: Remember that a partition of a set X is a collection of sets $\{X_i\}_{i \in I}$, where I is a set of indexes that verifies:

i) $\bigcup_{i \in I} X_i = X$

ii) $X_i \cap X_j = \emptyset$ si $i \neq j \forall i, j \in I$

Exercise 11. In each section, indicate and justify which expression is true and which expression is false. There may be sections where both expressions are true or both expressions are false.

a) $\{x\} \subseteq \{x\}$ or $\{x\} \in \{x\}$

b) $\{x\} \subset \{x, \{x\}\}$ or $\{x\} \in \{x, \{x\}\}$

c) $P(\{x\}) = \{\emptyset, \{x\}\}$ or $P(\{x\}) = \{\emptyset, x\}$

d) $P(\{x, \{x\}\}) = \{\emptyset, \{x\}, \{x, \{x\}\}\}$ or $P(\{x, \{x\}\}) = \{\emptyset, \{x\}, \{\{x\}\}, \{x, \{x\}\}\}$

Exercise 12. If X is a set of 5 elements:

a) How many members does $P(X)$ have?

b) How many proper subsets does$X$ have?

c) Answer the previous questions if X is a set of n elements.

Exercise 13. Prove the following properties:

a) $A \subseteq A \cup B$

b) $A \cap B \subseteq A$

c) $B \subseteq A \Leftrightarrow A \cup B = A$

d) $B \subseteq A \Leftrightarrow A \cap B = B$

e) $A - B = A \cap B'$

f) $A \subseteq B \Leftrightarrow B' \subseteq A'$

Relations

Exercise 14. Given the binary relations over $A = \{1,2,3,6\}$, $(x, y) \in L \leftrightarrow x \leq y$ and $xDy \leftrightarrow x \text{ divides } y (\exists n \in \mathbb{Z}, y = n \cdot x)$.

a) Write $L \text{ and } D$ as sets

b) Calculate $L \cap D$

c) Both are partial orders. Check it and justifiably indicate which of them is a total order.

Exercise 15. Given the relation $A = \{(a, b), (b, a), (b, c), (c, b), (c, d)\}$ and $B = \{(a, 1), (a, 3), (b, 2), (c, 2)\}$, calculate:

a) $A \circ A^{-1}$

b) $(A \circ A) - (A \circ A^{-1})$

c) $(A \circ A) \cup A$

d) $A \circ B$

Exercise 16. H is a set of all people and $P \subseteq H \times H$ represents the relation "father", which is expressed as $(x, y) \in P (xPy) \leftrightarrow x\ is\ the\ father\ of\ y$. Define the relation "cousin" $x\, Pr\, y \leftrightarrow x \text{ is the cousin of } y$, through the relation "father", its inverse relation, and the identity relation.

Exercise 17. For all the following relations, indicate if they are or not reflexive, symmetric, transitive, or antisymmetric. Indicate if some of them is an equivalence relation, a partial order, or a total order.

a) $R_1 \subseteq \mathbb{Z}^2 | xR_1y \leftrightarrow x \text{ divides } y$

b) $R_3 \subseteq H^2 | (x, y) \in R_3 \leftrightarrow x \text{ and } y \text{ are brothers}$, where H is the set of all people.

c) $R_4 \subseteq P(U)^2 | (A, B) \in R_4 \leftrightarrow A \subseteq B$, where U is the universal set.

d) $R_5 \subseteq H^2 | xR_5y \leftrightarrow x \text{ es hijo de } y$, where H is the set of all people.

e) $R_6 \subseteq H^2 | xR_6y \leftrightarrow x$ has the same or less cardinality than y, where H is the set of all people.

f) $R_7 \subseteq H^2 | xR_7y \leftrightarrow x$ is the wife of y, where H is the set of people.

Exercise 18.

a) From the equivalence relation of the previous exercise, indicate which are their equivalence classes and their quotient set.

b) From the orders of the previous part of the exercise, indicate and justify if the orders are partial or total.

Exercise 19. Each part of this exercises considers a relation over the set $\{1,2,3,4,5\}$. Determine if each relation is an equivalence relation or not. If so, indicate the equivalence classes of the relation.

a) $\{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,1)\}$

b) $\{(1,1), (2,2), (3,3), (4,4), (5,5), (1,3), (3,1), (3,4), (4,3)\}$

c) $\{(1,1), (2,2), (3,3), (4,4)\}$

d) $\{(1,1), (2,2), (3,3), (4,4), (5,5), (1,5), (5,1), (3,5), (5,3), (1,3), (3,1)\}$

e) $\{(x, y) \in A^2, 4 \text{ divides } (x - y)\}$

f) $\{(x, y) \in A^2, 3 \text{ divides } (x + y)\}$

g) $\{(x, y) \in A^2, x \text{ divides } (2 - y)\}$

Exercise 20. On $\mathbb{R}^2$, we consider the relation $(x, y)R(w, z) \leftrightarrow x + y = w + z$.

a) Prove that it is an equivalence relation.

b) Find the expression for the different equivalence classes.

c) How many elements does the quotient set of this equivalence relation have?

Exercise 21. Given the set $A = \{1,2,3,4\}$, determine if the following binary relations on A are equivalence relations or partial or total order:

a) $\{(1,1), (2,2), (3,3), (4,4)\}$

b) $\{(1,1)\}$

c) $A \times A$

d) $\{(1,1), (1,2), (2,1), (2,2), (4,4), (2,2)\}$

e) $\{(1,4), (1,2), (4,3)\}$

Functions

Exercise 22. Given the sets$X = \{1,2,3,4\}$ and $Y = \{a, b, c, d\}$, determine if each of the following relations from X to Y $(R \subseteq X \times Y)$ are functions or not. If they are functions, calculate their images and indicate if they are injective, surjective, or bijective. Obtain the inverse functions, whenever they exist, expressing them as sets of ordered pairs. Indicate the domain and the image of the inverse functions.

a) $\{(1, a), (2, a), (3, c), (4, b)\}$

b) $\{(1, c), (2, a), (3, b), (4, c), (2, d)\}$

c) $\{(1, c), (2, d), (3, a), (4, b)\}$

d) $\{(1, d), (2, d), (4, a)\}$

e) $\{(1, b), (2, b), (3, b), (4, b)\}$

Exercise 23. Given the function $f = \{(a, b), (b, a), (c, b)\}$ from $X = \{a, b, c\}$ to itself:

a) Write the following functions as a set of ordered pairs: $f \circ f$ y $f \circ f \circ f$.

b) Define $f^n = \underbrace{f \circ f \circ \ldots \circ f}_{n}$

Exercise 24. Given three sets $X = \{1,2,3\}$, $Y = \{a, b, c, d\}$, and $Z = \{w, x, y, z\}$, we define $g: X \to Y$ as $g = \{(1, b), (2, c), (3, a)\} \subseteq X \times Y$ and $f: Y \to Z$, such that $f = \{(a, x), (b, x), (c, z), (d, w)\} \subseteq Y \times Z$.

a) Write $f \circ g$ as a set of ordered pairs.

b) Indicate which type of function the following functions are:f, g y $f \circ g$, indicating their domains and their images.

c) Calculate the inverse functions for the previous functions, whenever they exist, and write them as sets of ordered pairs.

Exercise 25. Given the set of natural numbers $\mathbb{N}$ and $A_p = \{1,2, \ldots, p\} \subseteq \mathbb{N}$, determine which of the following functions are injective, which ones are surjective, and which ones are bijective, Justify your answers:

a) $f: \mathbb{Z} \to \mathbb{Z} | f(j) = \begin{cases} \frac{j}{2} & \text{if } j \text{ is even} \\ \frac{j-1}{2} & \text{if } j \text{ is odd} \end{cases}$

b) $f: \mathbb{N} \to \mathbb{N} | f(x) = \text{ greater integer} \leq \sqrt{x}$

c) $f: A_6 \to A_6 | f(x) = \text{remainder of dividing } 3x \text{ by } 7$

d) $f: A_3 \to A_3 | f(x) = \text{remainder of dividing } 3x \text{ by } 4$

e) $f: \mathbb{N} \times \mathbb{N} \to \mathbb{N} | f(x, y) = x + y$

f) $g: \mathbb{N} \times \mathbb{N} \to \mathbb{N} | f(x, y) = x \cdot y$

Exercise 26. Prove that the following property by reductio ad absurdum: $A \times \emptyset = \emptyset$

References

R. Johnsonbaugh, "Discrete mathematics". Prentice Hall, 1997.

W. K. Grassmann and J.P. Tremblay, "Logic and discrete mathematics : a computer science perspective". Prentice Hall, 1996.