Block 3. Probability distribution functions

PROPOSED PROBLEMS

Unknown discrete and continuous distributions

Problem 3.1. A vacuum cleaner manufacturing company measures the total hours that a family uses one of its devices for, in units of 100. An expert establishes that the time of use is random and is described with a probability density function given by:

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 \leq x < 2 \\ 0, & in\ any\ other\ case \end{cases}$$

Find the probability that, over a one-year period, a family uses their vacuum cleaner for

a. less than 120 hours;

between 50 and 120 hours.

A/ a. 0.68 b. 0.375

Problem 3.2. A shipment of seven TVs contains two defective units. A hotel buys three of these TVs at random. If x is the number of defective units the hotel buys, find the probability distribution of X. Express the results using the probability histogram.

A/

x	0	1	2
$f(x)$	2/7	4/7	1/7

Problem 3.3. The following table presents, at random, the number of defects per 10 metres of a synthetic cloth in continuous rolls with a uniform width:

x	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

Construct the cumulative density function of the random variable X.

A/ $$f(x) = \begin{cases} 0, & x < 0 \\ 0.41, & 0 \leq x < 1 \\ 0.78, & 1 \leq x < 2 \\ 0.94, & 2 \leq x < 3 \\ 0.99, & 3 \leq x < 4 \\ 1, & x \geq 4 \end{cases}$$

Problem 3.4. A continuous random variable X that can have values between $x = 1$ and $x = 3$ has a function given by $f(x) = 1/2$. Find:

a. $P(2 < X < 2.5)$

b. $P(X \leq 1.6)$

A/ a. $1/4$ b. 0.3

Problem 3.5. Consider the probability density function

$$f(x) = \begin{cases} k\sqrt{x}, & 0 < x < 1 \\ 0, & in\ any\ other\ case \end{cases}$$

a. Evaluate k.

b. Find $F(X)$ and use this to evaluate $P(0.3 < X < 0.6)$.

A/ a. $3/2$ b. $F(X) = x^{\frac{3}{2}}$; 0.3004.

Joint probability distributions

Problem 3.6. From a bag of fruit that contains three oranges, two apples, and three bananas, a random sample of four fruits is selected. If X is the number of oranges and Y is the number of apples in the sample, find:

a. The joint probability distribution of X and Y.

b. $[P(X,Y) \in A]$; A is the region constituted by $\{(x,y)|x + y \leq 2\}$.

A/

a.

Y \ X	0	1	2	3
0	-	3/70	9/70	3/70
1	2/70	18/70	18/70	2/70
2	3/70	9/70	3/70	-

The joint density function $f(X,Y)$ for the random variables X and Y.

b. 1/2

Problem 3.7. Let X be the reaction time, in seconds, to a given stimulant and Y be the temperature, in °F, at which a given reaction starts. Assume that the random variables X and Y have a joint density function:

$$f(x,y) = \begin{cases} 4xy, & 0 < x < 1; \quad 0 < y < 1 \\ 0, & in\ any\ other\ case \end{cases}$$

Find:

a. $P\left(0 \leq X \leq \frac{1}{2} \text{ y } \frac{1}{4} \leq Y \leq \frac{1}{2}\right)$;

b. $P(X < Y)$.

A/ a. $3/64$ b. $1/2$.

Problem 3.8. In a company, first thing in the morning on any given day, the amount of kerosene (in thousands of litres) in a tank can be considered a random quantity Y. Kerosene is sold during the day, and that quantity can also be considered random$X$. If the tank is not refilled during the day $(x \leq y)$ and you assume that the joint density function of the random variables X and Y is expressed as:

$$f(x,y) = \begin{cases} 2, & 0 < x < 1; \quad 0 < y < 1 \\ 0, & in\ any\ other\ case \end{cases}$$

a. Determine if X and Y are independent.

b. Find $P\left(\frac{1}{4} \leq X \leq \frac{1}{2} \middle| Y = \frac{3}{4}\right)$.

A/ a. Dependant b. $1/3$.

Problem 3.9. On a given day, a company's numerical control machine can fail $1, 2$ or 3 times. The number of times it fails can be considered a random variable X. The number of times they call a technician due to an emergency can be considered a random variable Y. If the joint density function is given by

$f(x,y)$	x	1	2	3
y	1	0.05	0.05	0.1
	2	0.05	0.1	0.35
	3	0	0.2	0.1

a. Evaluate the marginal distribution of X.

b. Evaluate the marginal distribution of Y.

c. Find $P(Y = 3|X = 2)$.

A/

a.

x	1	2	3
$g(x)$	0.10	0.35	0.55

b.

y	1	2	3
$h(y)$	0.20	0.50	0.30

c. 0.2.

Problem 3.10. You roll a balanced die twice. Let X and Y be the number of fours and fives that you get in the two rolls, respectively. Find:

a. The joint density function of X and Y.

b. $[P(X,Y) \in A]$; A is the region constituted by $\{(x,y)|2x + y < 3\}$.

A/ a.

		x		
	$f(x,y)$	0	1	**2**
y	0	$\frac{16}{36}$	$\frac{8}{36}$	$\frac{1}{36}$
	1	$\frac{8}{36}$	$\frac{2}{36}$	
	2	$\frac{1}{36}$		

b. $11/12$

Problem 3.11. Given the joint density function,

$$f(x,y) = \begin{cases} 6 - x - y, & 0 < x < 2; \quad 2 < y < 4 \\ 0, & in\ any\ other\ case \end{cases}$$

Find $P(1 < Y < 3|X = 2)$.

A/ $3/4$.

Problem 3.12. If X, Y and Z have the joint density function

$$f(x,y,z) = \begin{cases} k\,x\,y^2 z, & 0 < x < 1; \quad 0 < y < 1;\ 0 < z < 2 \\ 0, & in\ any\ other\ case \end{cases},$$

a. Find k.

b. Find $P\left(X < \frac{1}{4}, Y > \frac{1}{2}, 1 < Z < 2\right)$.

A/ a. 3 b. $21/512$

Mathematical expectation

Problem 3.13. The density function of the coded measurements of the pitch diameter of the threads in a fitting is

$$f(x)=\begin{cases}\dfrac{4}{\pi(1+x^2)}, & 0<x<1\\ 0, & in\ any\ other\ case\end{cases}$$

Find the expected value of X.

A/ $\ln 4/\pi$.

Problem 3.14. Let X be a random variable with the following probability distribution.

x	-3	6	9
$f(x)$	1/6	1/2	1/3

Find $\mu_{g(X)}$ of the random variable. $g(X)=(2X+1)^2$.

A/ 209

Problem 3.15. The random variables X and Y have the following joint density function:

	$f(x,y)$ x	2	4
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

a. Find the expected value of $g(X,Y)=XY^2$.

b. Find μ_X and μ_Y.

A/ a. 35.2 b. $\mu_X=3.20$; c. $\mu_Y=3.00$

Variance and covariance

Problem 3.16. Find the standard deviation of the random variable $g(X) = (2X + 1)^2$ from problem 3.14.

A/ 118.9

Problem 3.17. Find the covariance of the random variables X and Y from problem 3.9.

A/ $\sigma_{XY} = 0{,}005$

Mean and variances of linear combinations of random variables

Problem 3.18. A random variable X has a probability distribution:

x	0	1	2	3	4	5
$f(x)$	1/15	2/15	2/15	3/15	4/15	3/15

Find $E(X)$ and $E(X^2)$. Then, evaluate $E[(2X+1)^2]$.

A/ 209

Problem 3.19. If a random variable X is defined so that $E[(X-1)^2] = 10$ and $E[(X-2)^2] = 6$. Find μ and σ^2.

A/ $\mu = \frac{7}{2}$ $\sigma^2 = \frac{15}{4}$

Problem 3.20. A random variable X has a mean of $\mu = 12$, a variance of $\sigma^2 = 9$, and an unknown probability distribution. With the help of Chebyshev's theorem, estimate

a. $P(|X-10| \geq 3)$;

b. $P(|X-10| < 3)$;

c. $P(5 < X < 15)$;

d. the value of the constant c, so that $P(|X-10| \geq c) \leq 0.04$.

A/ a. At the most $4/9$ c. At the least $21/25$

b. At the least $5/9$ d. 10

Problem 3.21. Consider that the random variables X and Y represent the numbers that you get when you roll a red die and a green die, respectively. Find:

a. $E(X+Y)$;

b. $E(X-Y)$;

c. $E(X\,Y)$.

A/ a. 7; b. 0; c. 12.25.

Problem 3.22. If the joint density function of X and Y is given by

$$f(x,y)=\begin{cases}\frac{2}{7}(x+2y), & 0<x<1, \quad 1<y<2\\ 0, & in\ any\ other\ case\end{cases}$$

Find the expected value of $g(X,Y)=\frac{X}{Y^3}+X^2Y$.

A/ 1.

Known discrete distributions

Problem 3.23. You roll a six-sided die:

- **a.** Build the distribution model for the random variable X “number obtained”.
- **b.** What is the probability that you obtain a number greater than 4?
- **c.** Calculate the expected value and the variance of the random variable.
- **d.** What is the probability that you get a number between 2 and 4, inclusive?

A/ a. $X \equiv$Uniform distribution of six points (specify the random variable, the probability mass function, and the cumulative distribution function). **b.** ½. **c.** $\frac{21}{6}$ and $\frac{35}{12}$. **d.** ½.

Problem 3.24. A batch of 26 mechanical parts, manufactured by a specialist company, contains 6 defective parts. One part is taken from the batch at random.

- **a.** Construct the distribution model for the random variable X “defective part”.
- **b.** What is the probability that the chosen part is not defective?
- **c.** Calculate the mathematical expectation and the standard deviation of this random variable.

A/ a. Bernoulli distribution (specify the random variable, the probability mass function, and the cumulative distribution function). **b.** $10/13$. **c.** $3/13$ and $\sqrt{30}/13$.

Problem 3.25. A student answers four true/false questions at random.

- **a.** Construct the distribution model for the random variable X “number of correct answers”.
- **b.** What is the probability that they provide 3 correct answers?
- **c.** Calculate the expected value and the variance of this random variable.
- **d.** What is the probability that they provide only two right answers at the most?
- **e.** What is the probability that they provide between 1 and 3 correct answers, inclusive?
- **f.** What is the probability that they provide two or more correct answers?

A/ a. Binomial distribution (specify the random variable, the probability mass function, and the cumulative distribution function). **b.** 0.3125. **c.** 2 and 1. **d.** 0.6875. **e.** 0.875. **f.** 0.6875.

Problem 3.26. A store has an average of 13 customers per hour. The owner leaves the store for 17 minutes, and this action can lead to a loss of customers.

- **a.** Contruct the distribution model for the random variable X "number of customers lost".
- **b.** Calculate the expected number of clients lost and the standard deviation of this random variable.
- **c.** What is the probability that the owner loses no customers?
- **d.** Find the probability that they lose 3 customers.
- **e.** Determine the probability that they lose 3 or more customers.

A/ a. Poisson distribution (specify the random variable, the probability mass function, and the cumulative distribution function). **b.** 3.68 and 1.92. **c.** 0.025. **d.** 0.21. **e.** 0.714.

Problem 3.27. A student answers sixty multiple-choice questions, where each question has eleven possible answers but only one is correct.

- **a.** Construct the distribution model for the random variable X "number of correct answers".
- **b.** Find the probability that that they provide at least 3 correct answers.
- **c.** Find the expected value and the variance of this random variable.

A/ a. Binomial distribution (specify the random variable, the probability mass function, and the cumulative distribution function), but you can use a Poisson distribution. **b.** 0.9. **c.** 5.45 and 2,23 (Bernoulli) or 2.34 (Poisson).

Problem 3.28. A trick coin is tossed 8 times. The probability of getting heads is 2/3, whereas the probability of getting tails is 1/3.

- **a.** Construct the distribution model for the random variable X "number of tosses to get the first tails".
- **b.** Calculate the mean and the variance of this random variable.
- **c.** What is the probability that you get the first tails between the first and the sixth tosses, inclusive?

- **d.** Find the probabilities that: you get a tail on the last toss and you do not even get one in the eight tosses.
- **e.** Determine the probability that you get the first heads on the third toss.

A/ a. Geometric distribution (specify the random variable, the probability mass function, and the cumulative distribution function). **b.** 3 and 6. **c.** 0.36. **d.** 0.02 and 0.04. **e.** 0.07.

Problem 3.29. A trick coin has a $p = 0.25$ probability of landing heads up. You toss the coin until you get 2 heads.

- **a.** Construct the distribution model for the random variable X "number of failed tosses until getting two heads".
- **b.** Calculate the number of expected failures until you get the second success and the standard deviation of the distribution of the random variable.
- **c.** Describe the distribution of the number of tosses needed to get two heads.
- **d.** Find the expected number of tosses until you get the second success and the standard deviation of the distribution of the random variable.
- **e.** Find the probability that you get two heads before the fourth toss.
- **f.** Determine the probability that you get the two heads between the third and sixth tosses.

A/ a. Negative binomial distribution (specify the random variable, the probability mass function, and the cumulative distribution function). **b.** 6 and $2\sqrt{6}$. **c.** A Pascal random variable, which in the same experiment as in the previous random variable measures the number of tosses until the second success. **d.** 8 and $2\sqrt{6}$. **e.** 0.26. **f**. 0.40.

Known continuous distributions

Uniform distribution

Problem 3.30. The daily quantity of coffee, in litres, a machine serves is a random variable X that has a uniform continuous distribution with $A = 7$ and $B = 10$. Find the probability that, on a given day, the quantity of coffee the machine serves is:

a. at the most, 8.8 litres;

b. more than 7.4 litres, but less than 9.5 litres;

a. at least, 8.5 litres;

A/ a. 0.6; b. 0.7; c. 0.5.

Normal distribution

Problem 3.31. In a standard normal distribution, determine the area under the curve that is:

a. to the left of $z = 1.43$;

b. to the right of $z = -0.89$;

c. between $z = -2.16$ and $z = -0.65$

d. to the left of $z = -1.39$;

e. to the right of $z = 1.96$;

f. between $z = -0.48$ and $z = 1.74$

A/ a. 0.9236. b. 0.8133. c. 0.2424. d. 0.0823. e. 0.0250. f. 0.6435

Problem 3.32. For a standard normal distribution, determine the value of k, so that

a. $P(Z < k) = 0.0427$,

b. $P(Z > k) = 0.2946$,

c. $P(-0.93 < Z < k) = 0.7235$

A/ a. -1.72; b. 0.54. c. 1.28.

Problem 3.33. If the random variable X has a normal distribution, with a mean of 18 and a standard deviation of 2.5, find:

a. $P(X < 15)$;

d. the value of k, so that $P(X < k) = 0.2236$;

c. the value of k, so that $P(X > k) = 0.1814$;

d. $P(17 < X < 21)$.

A/ a. 0.1151; b. 16.1; c. 20.275; d. 0.5403.

Normal approximation to the binomial

Problem 3.34. Evaluate $P(1 < X < 4)$ for a binomial variable with $n = 15$ and $p = 0.2$ using

a. tables
b. the approximation of the normal curve

A/ a. 0.8006; b. 0.7803.

Problem 3.35. A process for manufacturing an electrical part produces 1% defective parts. A quality control plan selects 1% of the process, and if there is no defective part, the process continues. Use the normal approximation to the binomial to find:

a. the probability that the process continues with the indicated sampling plan;

b. the probability that the process continues even if it is wrong, in other words, if the frequency of defective parts changes to $5.0\ \%$ defective parts.

A/ a. 0.1574; b. 0.0108.

Problem 3.36. If $20\ \%$ of the inhabitants of a city prefer to buy organic products in the market over any other kind of product, what is the probability that among the following 1000 products entering the market,

a. between 170 and 185, inclusive, are organic?

b. at least 210, but no more than 225, are organic?

A/ a. 0.1171; b. 0.2049.

Gamma distribution and others

Problem 3.37. If a random variable X has a gamma distribution with $\alpha = 2$ and $\beta = 1$. Find the probability $P(1.8 < X < 2.4)$.

A/ $2.8e^{-1.8} - 3.4e^{-2.4} = 0.1545.$

Problem 3.38. The water usage in a city approximately follows a gamma distribution with $\alpha = 2$ and $\beta = 3$. The daily capacity of that city is 9 million litres of water per day.

a. Find the mean and the variance of the daily water usage in the city.

b. According to Chebyshev's theorem, there is a $3/4$ probability that the water usage on a given day will fall within what interval?

A/ a. $\mu = 6$; $\sigma^2 = 18$; b. between 0 and 14.485 million litres.

Problem 3.39. The length of time for a person to be served in a coffee shop is a random variable that has an exponential distribution with a mean of four minutes. What is the probability that a person is served in less than three minutes on at least four of the following six days?

A/ $\sum_{x=4}^{6} \binom{6}{x} \left(1 - e^{-\frac{3}{4}}\right)^{x} \left(e^{-\frac{3}{4}}\right)^{6-x} = 0.3968.$

Problem 3.40. It is assumed that the lifetime, in years, of a hearing-aid battery is a random variable that has a Weibull distribution with $\alpha = 1/2$ and $\beta = 2$.

a. How long can this battery last?

b. What is the probability that this battery will still work after two years?

A/ a. $\sqrt{\pi/2} = 1.2533.$ b. e^{-2}.

Problem 3.41. The lifetime of some automotive seals obeys a Weibull distribution with a failure rate $Z(t) = 1/\sqrt{t}$. Find the probability that a given seal is still in service after four years.

A/ e^{-4}.

Problem 3.42. The response time of a given computer, obtained in research, obeys an exponential distribution with a mean of three seconds.

a. What is the probability that the response time is more than five seconds?

b. What is the probability that the response time is more than ten seconds?

A/ a. 0.1889; b. 0.357.

Problem 3.43. Percentages often follow a log-normal distribution. The average power usage (dB per hour) of a company is studied, fand it is found that it follows the indicated distribution with the parameters $\mu = 4$ and $\sigma = 2$.

a. What is the average power usage?

b. What is the variance?

A/ a. e^6; b. $e^{12}(e^4 - 1)$.

Problem 3.44. The number of vehicles arriving at a crossroad per minute has a Poisson distribution with a mean of 10. Focus on the period of time that must pass before 15 vehicles have reached the crossroad.

a. What is the probability that more than one minute elapses between arrivals?

b. What is the average number of minutes that elapses between arrivals?

A/ a. e^{-10}; b. $\beta = 0.10$.