Block 2. Probability

SOLVED EXAMPLES

Example 2.1. An insurance company randomly chooses an insurance adjuster from the 100 available. 40 of these are experts on car accidents, 25 are experts on home accidents, and 15 are experts in both fields. Find the probability that the insurance adjuster chosen by the insurance company is an expert on car or home accidents.

Solution.

The insurance adjuster is selected at random. Moreover, based on what we know about the experts, we can establish events $C = \{c|\ c\ are\ experts\ on\ car\ accidents\}$ and $H = \{h|\ c\ are\ experts\ on\ home\ accidents\}$.

The probability of C or H is interpreted as $P(C\ \cup\ H)$. With the addition rule: $P(C\ \cup\ H) = P(C) + P(H) - P(C\ \cap\ H)$.

The sample space for the experiment is finite and equiprobable: $P(C) = \frac{40}{100} = \frac{2}{5}$, $P(H) = \frac{25}{100} = \frac{1}{4}$, and $P(C\ \cap\ H) = \frac{15}{100} = \frac{3}{20}$. Then,

$$P(C\ \cup\ H) = \frac{2}{5} + \frac{1}{4} - \frac{3}{20} = \frac{1}{2}$$

The required probability has an occurrence rate of 50%.

Example 2.2. You have a deck of Spanish playing cards with 40 cards. What is the possibility of drawing a knight followed by a three, if you put the first card back into the deck? And if you do not put it back in?

The random experiment consists of randomly drawing two cards—first one and then the other. The sample space comprises all the possible results of the experiment. However, the experiment is conducted in different situations: both **returning** the first card to the deck and **without returning** it. In the two situations, the favourable cases are the same. Therefore, you can designate: $A =$ "$drawing\ a\ knight$" and $B =$ "$drawing\ a\ three$".

If you return the first card to the deck, the events A and B are independent. According to the multiplication rule, $P(A \cap B) = P(A) \cdot P(B)$. Because the sample space is finite and equiprobable, you can use Laplace's law to calculate the probabilities. So,

$$P(A \cap B) = P(A) \cdot P(B) = \frac{4}{40} \cdot \frac{4}{40} = \frac{1}{100}$$

If you do not return the first card to the deck, the events A and B are dependant and, according to the multiplication rule, you get: $P(A \cap B) = P(A) \cdot P(B|A)$. Although the sample space is finite and equiprobable, notice that taking the first card conditions the second. So,

$$P(A \cap B) = P(A) \cdot P(B|A) = \frac{4}{40} \cdot \frac{4}{39} = \frac{2}{195}$$

Notice that, in the experiment, a favourable result has an occurrence possibility of approximately 0.1% and 1.03% with and without returning de first card to the deck, respectively.

Example 2.3. A couple wants to have several children. What is the probability that the couple will have at least 1 girl out of 3 children?

Solution.

For births, we can assume that boys and girls are equally probable and that a child's gender is independent of the gender of any other child.

You must apply the previous procedure:

1. $A = \textit{at least } 1 \textit{ of the } 3 \textit{ children is a girl}.$
2. $\bar{A} = \text{"}\textit{not having at least } 1 \textit{ girl out of } 3 \textit{ children}\text{"}$
3. $P(\bar{A}) = P(\textit{girl and girl and girl}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}.$
4. $P(A) = 1 - P(\bar{A}) = \frac{7}{8}.$

In other words, there is a 87.5% probability that the couple will have at least 1 girl out of 3 children.

Example 2.4. You roll two dice. Find:

a. The probability that one die is a two, if the sum of both is six.
b. The probability $P(A)$ of getting a two.

Solution

a. If the sum is six, you need an event to define this, $E = \{the\ sum\ is\ 6\}$. On the other hand, you need to get a two on at least one of the dice, so $A = \{you\ get\ 2\ on\ least\ one\ of\ the\ dice\}$. According to definition 4.6 or theorem 4.9, $P(A|E) = \frac{n(A \cap E)}{n(E)}$. In E, two pairs belong to A: $A \cap E = \{(2,4),(4,2)\}$. Therefore, $P\ (A|E) = 2/5$.
b. A has 11 points; and S has 36 points. As the space is finite and equiprobable, then, $P\ (A) = 11/36$.

Example 2.5. You have 15 balls in a bag; 6 are white and the others are black. You randomly draw 3 balls one after the other. Find the probability that none of the three balls are white.

Solution

• Let $A = \{black\ balls\}$. The probability that the first ball is not white is $p = 9/15$.

• If the first ball is not white, $B = \{black\ balls - 1\}$, the probability that the second one is not white either is $p = 8/14$.

• If the first two balls are not white, $C = \{black\ balls - 2\}$, the probability that the third one is not white either is $p = 7/13$.

Therefore, $P(A \cap B \cap C) = P(A)\ P(B|A)\ P(C|A \cap B) = \frac{9}{15} \cdot \frac{8}{14} \cdot \frac{7}{13} = \frac{12}{65} \approx 0.1846$.

In other words, there is an 18.46% possibility that the three balls are not white.

Example 2.6. You toss a coin until you get heads. What is the probability that you only have to toss the coin 3 times to get this result? What is the probability that you get the desired outcome in an even number of tosses?

Solution

Tossing the coin until you get heads is a random experiment. Its sample space can be represented by $S = \{a_1, a_2, \cdots\}$.

To obtain the probability space, you must assign probabilities to each $a_i \in S \ (i = 1,2,\cdots)$: $P(1) = \frac{1}{2}$, $P(2) = \frac{1}{2^2}$, $P(3) = \frac{1}{2^3}$, … , $P(n) = \frac{1}{2^n}$, … , $P(\infty) = 0$, where n is the number of times you toss the coin.

Let $A = \{n \, is \, 3 \, at \, the \, most\} = \{1, 2, 3\}$ and $B = \{n \, is \, even\}$. So, the probability that the number of times you toss the coin is 3 times at the most is:

$$P(A) = P(1,2,3) = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8}$$

The probability that you get the desired outcome in an even number of tosses is:

$$P(B) = P(2,4,6,\cdots) = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots$$

Notice that $P(B)$ is a geometric series where $a = 1/4$ and $r = 1/4$. Therefore, write:

$$P(B) = \frac{a}{1-r} = \frac{\frac{1}{4}}{1-\frac{1}{4}} = \frac{1}{3}$$

There is 87.5% and 33.33% possibility that you get heads on the third toss and that you get the desired outcome in an even number of tosses, respectively.

Example 2.7. Three boxes, labelled as: X, Y, and Z, contain bulbs, where:

- X has 10, 4 of which are defective.
- Y has 6, 1 of which is defective.
- Z has 8, 3 of which are defective.

You choose a box at random. Then, you take a random bulb out of the selected box. Find the probability that the bulb you took is not defective.

Solution.

There is a **succession** of two random **experiments**, the selection of:

1. one of the boxes.
2. a random bulb from the selected box.

In the first experiment, selecting one box meets the requirements to apply Laplace's law. In this case, the probability assigned to each of the results is $\frac{1}{3}$.

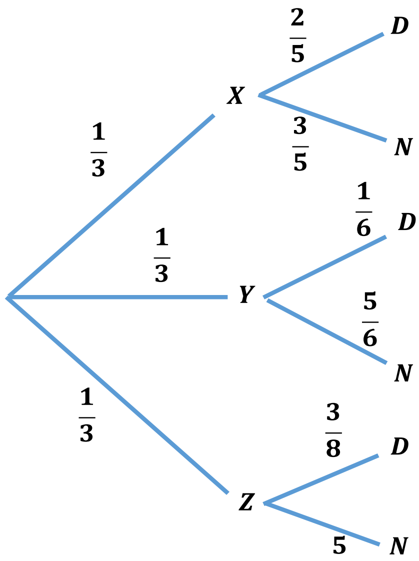

Figure 2.1. Tree diagram.

In the second experiment, the bulb chosen at random may be defective (D) or not (N). In this experiment, Laplace's law is fulfilled and you can assign probabilities to the selection of the bulb. Figure 2.1 shows the assignments of probabilities for taking a defective bulb or a non-defective bulb.

In the first experiment, drawing from a box, the events are not simultaneous and are mutually exclusive; that is, you choose only from one box. Therefore, you apply the addition rule. The selection of a defective bulb follows the first event and you apply the product rule. Therefore, the probability that the chosen bulb is defective is $P(N) = P(X)P(D\ in\ X) + P(Y)P(D\ in\ Y) + P(Z)P(D\ in\ Z)$.

Notice that the probabilities are calculated using a tree diagram (Figure 2.1) and multiplication theorem. The final result is the sum of three paths; in other words, $P(N) = \frac{1}{3}\frac{3}{5} + \frac{1}{3}\frac{5}{6} + \frac{1}{3}\frac{5}{8}$. A process described in this way is a **stochastic process**.

Example 2.8. Three companies manufacture electrical devices as follows:
SSSA produces 60% of the devices, of which 4% are defective;
AIS produces 25%, of which 2% are defective;
MAESAZ produces 15%, of which 3% are defective.
Find the probability that a randomly chosen device is defective.

Solution.

To make it easier, change the company names to letters: Change SSSA to X; AIS to Y; and MAESAZ to Z. We make this change because, in the case of probabilities, a name is just a name.

Randomly choosing a defective device is a stochastic process. You should review the solution for Example 4.14. Therefore, you have to assign probabilities. In this case, they are given by percentages.

Let $D = \{defective\ device\}$ and $N = \{non - defective\ device\}$.

The A_i of the problem are: X, Y, and Z. The event E is D. The requirements indicated in the problem situation are met. Therefore, as per theorem 4.11 (law of total probability), randomly choosing a defective electrical device is:

$$P\ (D)\ =\ P(X)\ P\ (D|X)\ +\ P\ (Y)\ P\ (D|Y)\ +\ P\ (Z)\ P\ (D|Z)$$

$$=\ 0.60\ \times\ 0.04\ +\ 0.25\ \times\ 0.02\ \ +\ 0.15\ \times\ 0.03\ =\ 0.0335$$

Taking into account that $P\ (D|j)$ is the probability D if j; $(j = X, Y, Z)$.

There is a 3.35% possibility that the chosen device is defective.

Example 2.9. Consider the statement for Example 2.8. Assume that you find a defective device. Find the probability that the device was manufactured by each one of the companies.

Solution.

We have a two-stage stochastic process. The analysis presented in Example 2.8 is valid. Therefore, you need Bayes' formula, where the total probability is:

$$P\ (D)\ =\ P(X)\ P\ (D|X)\ +\ P\ (Y)\ P\ (D|Y)\ +\ P\ (Z)\ P\ (D|Z)$$

$$= 0.60 \times 0.04 + 0.25 \times 0.02 + 0.15 \times 0.03 = 0.0335$$

The probability that the device was manufactured by each one of the companies is:

$$P\ (X|D) = \frac{P(X)\ P\ (D|X)}{P\ (D)} = \frac{0.60 \times 0.04}{0.0335} = \frac{240}{335} \approx 0.7164$$

$$P\ (Y|D) = \frac{P(Y)\ P\ (D|Y)}{P\ (D)} = \frac{0.25 \times 0.02}{0.0335} = \frac{50}{335} \approx 0.1493$$

$$P\ (Z|D) = \frac{P(Z)\ P\ (D|Z)}{P\ (D)} = \frac{0.15 \times 0.03}{0.0335} = \frac{45}{335} \approx 0.1343$$

There is a 71.64, 14.93, and 13.43% possibility that the companies X, Y, and Z manufactured the defective device, respectively.

Using other counting techniques

Example 2.10. A conference has 80 attendees. 70 of them speak English and 50 speak French. You choose two participants at random and you want to know:

a. What is the probability that they understand each other without an interpreter?
b. What is the probability that they understand each other only in French?
c. What is the probability that they understand each other in a single language?
d. What is the probability that they understand each other in both languages?

Solution.

You choose the two attendees at random. The sample space is made up of 80 attendees. The space is finite and equiprobable, and you can calculate the probabilities using Laplace's law, but the number of favourable cases is not evident for any of the sections.

The following Venn diagram (Figure 2.2) is useful for finding the number of favourable cases. Here, "$\boldsymbol{x}$" is the number of attendees who can **speak** both **French** and **English**.

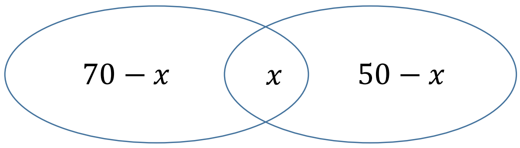

Figure 2.2. Venn diagram

It is true that $(70 - x) + x + (50 - x) = 80$. By solving the system you get $x = 40$. In other words, 40 attendees speak both French and English; 30 speak only English, and 10 speak only French. Therefore:

a. Let$A =$ "*the two attendees understand each other without an interpreter*".
Consider the probability that the two attendees understand each other using interpreters:

$$P(\bar{A}) = \frac{\binom{30}{1} \cdot \binom{10}{1}}{\binom{80}{2}} = \frac{30 \cdot 10}{\frac{80!}{2!\ 78!}} = \frac{15}{158}$$

Then, $P(A) = 1 - P(\bar{A}) = 1 - \frac{15}{158} = \frac{143}{158}$

b. Now let:

$B =$ *"the two attendees understand each other only in French"*

This means that they speak only French **or** that they speak both languages; that is, you apply the addition rule:

$$P(B) = \underbrace{\frac{\binom{10}{2}}{\binom{80}{2}}}_{\substack{only\\ French}} + \underbrace{\overbrace{\frac{\binom{10}{1}\cdot\binom{40}{1}}{\binom{80}{2}}}^{\substack{product\\ rule}}}_{\substack{both\\ languages}}$$

You can apply the product rule to the second addend. Therefore,

$$P(B) = \frac{\frac{10!}{2!\ 8!} + 10 \cdot 40}{\frac{80!}{2!\ 78!}} = \frac{89}{632}$$

c. Let:

$C =$ *"the two attendees understand each other in a single language"*

This means that the two attendees understand each other only in English **or** only in French. You can apply the addition rule. For this, define the events:

$C_1 =$ *"They understand each other only in English"*

$C_2 =$ *"They understand each other only in French"*

Then, write: $P(C) = P(C_1 \cup C_2) = P(C_1) + P(C_2)$, where

$$P(C_1) = \frac{\binom{30}{2}}{\binom{80}{2}} + \frac{\binom{30}{1}\cdot\binom{40}{1}}{\binom{80}{2}} = \frac{\frac{30!}{2!\ 28!} + 30 \cdot 40}{\frac{80!}{2!\ 78!}} = \frac{327}{632}$$

Because the event C_2 coincides with the event in section b, you have already calculated its probability. So,

$$P(C) = P(C_1) + P(C_2) = \frac{327}{632} + \frac{89}{632} = \frac{52}{79}$$

d. Now let:

$D =$ *"the two attendees understand one another in both languages"*

In other words, they understand each other in both English and French. Apply Laplace's law:

$$P(D) = \frac{\binom{40}{2}}{\binom{80}{2}} = \frac{\frac{40!}{2!\ 38!}}{\frac{80!}{2!\ 78!}} = \frac{39}{158}$$

Probability and analogy calculations

Example 2.11. You have three boxes that look the same. In the first, there are 3 white balls and 4 black balls; in the second there are 5 black balls; and in the third there are 2 black balls and 3 white balls. You want to know:

a. If you draw a ball from a box, chosen at random, what is the probability that the ball is black?

b. You drew a black ball from one of the boxes. What is the probability that you drew that ball from the 2nd box?

Solution.

Lets:

$A = \text{“}drawing\ a\ black\ ball\text{”}$

$A_1 = \text{“}drawing\ a\ ball\ from\ the\ first\ box\text{”}$

$A_2 = \text{“}drawing\ a\ ball\ from\ the\ second\ box\text{”}$

$A_3 = \text{“}drawing\ a\ ball\ from\ the\ third\ box\text{”}$

a. Based on the statement, you know that:

$$P(A_1) = P(A_2) = P(A_3) = \frac{1}{3}$$

$$P(A|A_1) = \frac{4}{7}$$

$$P(A|A_2) = 1$$

$$P(A|A_3) = \frac{3}{5}$$

If you apply the law of total probability, you get:

$$P(A) = P(A|A_1)P(A_1) + P(A|A_2)P(A_2) + P(A|A_3)P(A_3)$$

In this case, the result is:

$$P(A) = \frac{4}{7}\cdot\frac{1}{3} + 1\cdot\frac{1}{3} + \frac{3}{5}\cdot\frac{1}{3} = \frac{76}{105}$$

b. According to Bayes' theorem, the probability that you drew the ball from the second box is:

$$P(A_2|A) = \frac{P(A|A_2)\,P(A_2)}{P(A)} = \frac{1\cdot\frac{1}{3}}{\frac{76}{105}} = \frac{35}{76}$$

Example 2.12. Three branches of the same bank are offering their clients financial products with a six-month term and a one-year term. The first branch is offering 3 products with a six-month term and 4 with a one-year term; the second, 5 products with a one-year term; and the third, 2 products with a six-month term and 3 with a one-year term. If a client:

a. selects a product from a randomly chosen branch, what is the probability that the selected product has a one-year term?

b. has selected a product with a one-year term from one of the branches, what is the probability that they chose that product from the 2nd branch?

Solution.

An analysis of the problem, however specialised it may be, from a mathematical point of view, shows that there are no differences from example 2.11. You can solve these cases by analogy with those in the examples in the section on basic probabilities. Below, are the analogies (in bold) of the two examples:

Example 2.11	**Example 2.12**
Three **boxes** that look the same	Three **branches** of the same bank
You draw a **ball from a box**	You select a **product from a branch**
You draw a **black ball** from one of the boxes	You select a **product with a one-year term** from one of the branches

In both examples, the characteristics for calculating the total probability and Bayes' formula are the same. Although the statements involve different situations, the mathematical language does not make a distinction. Therefore, the important thing is to identify the characteristics that allow you to calculate the probability.