Mathematics in Population dynamics: Predator-Prey Model

Predator-Prey Systems

We first consider the following situation.

- One species, the prey, has an ample food supply.
- The second, the predator, feeds on the prey.

Examples of prey and predators include:

- Rabbits and wolves in an isolated forest
- Food fish and sharks
- Aphids and ladybugs
- Bacteria and amoebas

Predator-Prey Model Assumptions

The Lotka-Volterra predator-prey model makes a few important assumptions about the environment and the dynamics of the predator and prey populations:

- The prey population finds food at all times
- In the absence of a predator, the prey grows at a rate proportional to the current population
- The food supply of the predator population depends entirely on the prey populations. The growth can be calculated as **$dx/dt = ax$**
- In the absence of the prey, the predator population would decline at a rate proportional to itself, that is **$dy/dt = -\beta y$, $\beta > 0$**, when **$x=0$**.
- The predator effect is to reduce the prey growth rate, proportional to both the predator and prey populations proportional to the prey and predator populations

Predator-Prey Model Assumptions

The Lotka-Volterra predator-prey model makes a few important assumptions about the environment and the dynamics of the predator and prey populations:

- The number of encounters between predator and prey is proportional to the product of their populations.
- Encounters between predator and prey tends to promote the growth of the predator and inhibit the growth of the prey. Thus, the growth rate of the predator is increased by the term **γxy** and the growth rate of the prey is decreased by the term **$-\delta xy$**.
- During the process, the environment does not change in favor of one species and the genetic adaptation is sufficiently slow.

The General Model

Based on the assumptions for this model we have the following two-equation system of autonomous, first-order, nonlinear differential equations:

x(t): rabbit (prey population) @ time t.

y(t): fox (predator) population @ time t.

(1) $\boldsymbol{dx/dt = \alpha x - \delta xy}$ **,** The term -δxy decreases the natural growth rate of the prey.

(2) $\boldsymbol{dy/dt = -\beta y + \gamma xy}$ **,** *The term γxy increases the natural growth rate of the predators.*

α, β, δ, γ > 0.

Parameters:

α: the growth rate of the prey

β: the death rate of the predator

δ, γ: measure the effect of the interactions of the two species

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The General Model

When (y = 0), equation (1) becomes:

(3) $$dx/dt = \alpha x - \delta xy = \alpha x.$$

$$dx/dt - \alpha x = 0.$$

The general solution to equation

(4) $$x(t) = ce^{\alpha t}$$

So, the rabbit (prey) population will increase exponentially in the absence of a predator.

The General Model

Likewise, in the absence of a prey ($x = 0$), equation (2) becomes:

(5) $$dy/dt = -\beta y + \gamma xy = -\beta y.$$

$$dy/dt + \beta y = 0.$$

The general solution to this equation is:

(6) $$y(t) = ce^{-\beta t}$$

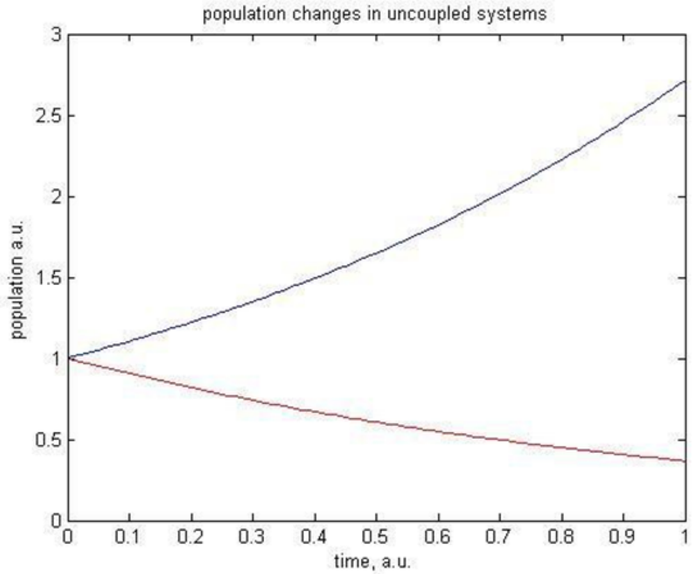
The fox population will experience exponential decay until extinction in the absence of prey.

The Predator-Prey Model

(1) $\frac{dx}{dt} = \alpha x - \delta xy = x(\alpha - \delta y)$,

(2) $\frac{dy}{dt} = -\beta y + \gamma xy = y(-\beta + \gamma y)$;

with $\alpha, \beta, \delta, \gamma > 0$.



The Predator-Prey Model

Example,

α = 0.02

(the growth rate of the prey, per unit prey)

β = 0.05

(the death rate of the predator, per unit predator)

δ = 0.0005

γ = 0.0004

(measures effect of the interactions of the two species)

$$dx/dt = 0.02x - 0.0005xy,$$
$$dy/dt = -0.05y + 0.0004xy.$$

The Predator-Prey Model

We seek equilibrium points for the model.

So we set:

$$\frac{dx}{dt} = 0.02x - 0.0005xy = 0,$$

$$\frac{dy}{dt} = -0.05y + 0.0004xy = 0.$$

And solve for *x* and *y*…

Two solutions are: $(x, y) = (0, 0)$

& $(x, y) = (125, 40)$.

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If x, y are small (i.e. $0 < x, y < 1$), then The product $xy < x$ and y is even smaller.

So if we consider points close to the origin (0, 0), one of our critical points, then we can drop the terms:

$-0.0005xy$ and $0.0004xy$ …

The original system of equations become a linear system of equations!

$$\boldsymbol{dx/dt = 0.02x,}$$

$$\boldsymbol{dy/dt = -0.05y; \qquad \alpha, \gamma > 0.}$$

The Predator-Prey Model

The original system of equations become a linear system of equations!

dx/dt = 0.02x,
dy/dt = -0.05y;   α, γ > 0.

This linear system can be written as:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.02 & 0 \\ 0 & -0.05 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = A*(x, y)^T.$$

Linear Systems are of the form:
Ax = b (where **x**, **b** are vectors).

The Predator-Prey Model

Remember that for the autonomous first-order linear differential equation $\boldsymbol{dx/dt = ax}$, the solution is $\boldsymbol{x = ce^{at}}$, where c is constant of integration.

$\boldsymbol{x = 0}$ is the only equilibrium solution to this equation if $\boldsymbol{a \neq 0}$.

if $\boldsymbol{a \neq 0}$, then

If $a < 0$, then solution $x(t)$ for $dx/dt = ax$ is an exponential decay over time t.

If $a > 0$, then solution $x(t)$ for $dx/dt = ax$ is an exponential growth over time t.

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Example

Suppose that populations of rabbits (R) and wolves (W) are described by the Lotka-Volterra equations with:

$\alpha = 0.08$, $\delta = 0.001$, $\beta = 0.02$, $\gamma = 0.00002$

The time *t* is measured in months.

a. Find the constant solutions (called the equilibrium solutions) and interpret the answer.

b. Use the system of differential equations to find an expression for dR/dW.

Example 1 a

With the given values, the Lotka-Volterra equations become:

$$\boldsymbol{dR/dt = \alpha R - \delta RW}$$

$$\boldsymbol{dW/dt = -\beta W + \gamma RW}$$

$$dR/dt = 0{,}08R - 0{,}001RW = R(0{,}08 - 0{,}001W)$$

$$dW/dt = -0{,}02W + 0{,}00002RW = W(-0{,}02 + 0{,}00002R)$$

Both R and W will be constant if both derivatives are 0. That is,

$$R' = R(0.08 - 0.001W) = 0$$

$$W' = W(-0.02 + 0.00002R) = 0$$

Example 1 a

One solution is given by: x= 0 and y = 0

- If there are no rabbits or wolves, the populations are certainly not going to increase.

Example 1 a

Two solutions:

One solution is given by: x= 0 and *y* = 0

- If there are no rabbits or wolves, the populations are not going to increase

The other solution is:

$$W = \frac{0.08}{0.001} = 80 \qquad R = \frac{0.02}{0.00002} = 1000$$

- So, the equilibrium populations consist of 80 wolves and 1000 rabbits.

Example 1 a

This means that 1000 rabbits are just enough to support a constant wolf population of 80.

- The wolves aren’t too many—which would result in fewer rabbits.
- They aren’t too few—which would result in more rabbits.

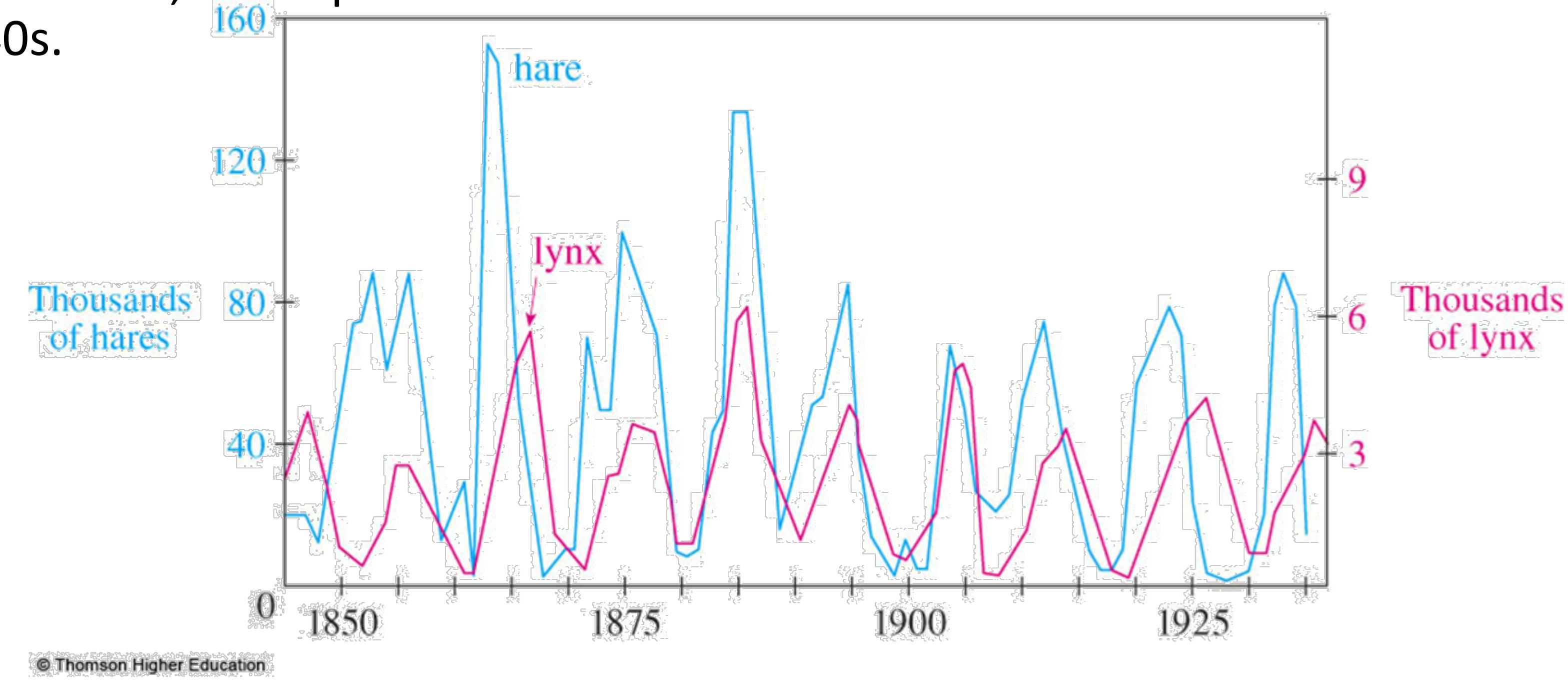
Example 1 b

We use the Chain Rule to eliminate *t* :

$$\frac{dW}{dR} = \frac{\dfrac{dW}{dt}}{\dfrac{dR}{dt}} = \frac{-0.02W + 0.00002RW}{0.08R - 0.001RW}$$

Relationship between real data and the Lotka-Volterra equation

For instance, the Hudson's Bay Company, which started trading in animal furs in Canada in 1670, has kept records that date back to the 1840s.



The graphs show the number of pelts of the snowshoe hare and its predator, the Canada lynx, traded over a 90-year period.

You can see that the coupled oscillations in the hare and lynx populations predicted by the Lotka-Volterra model do actually occur.

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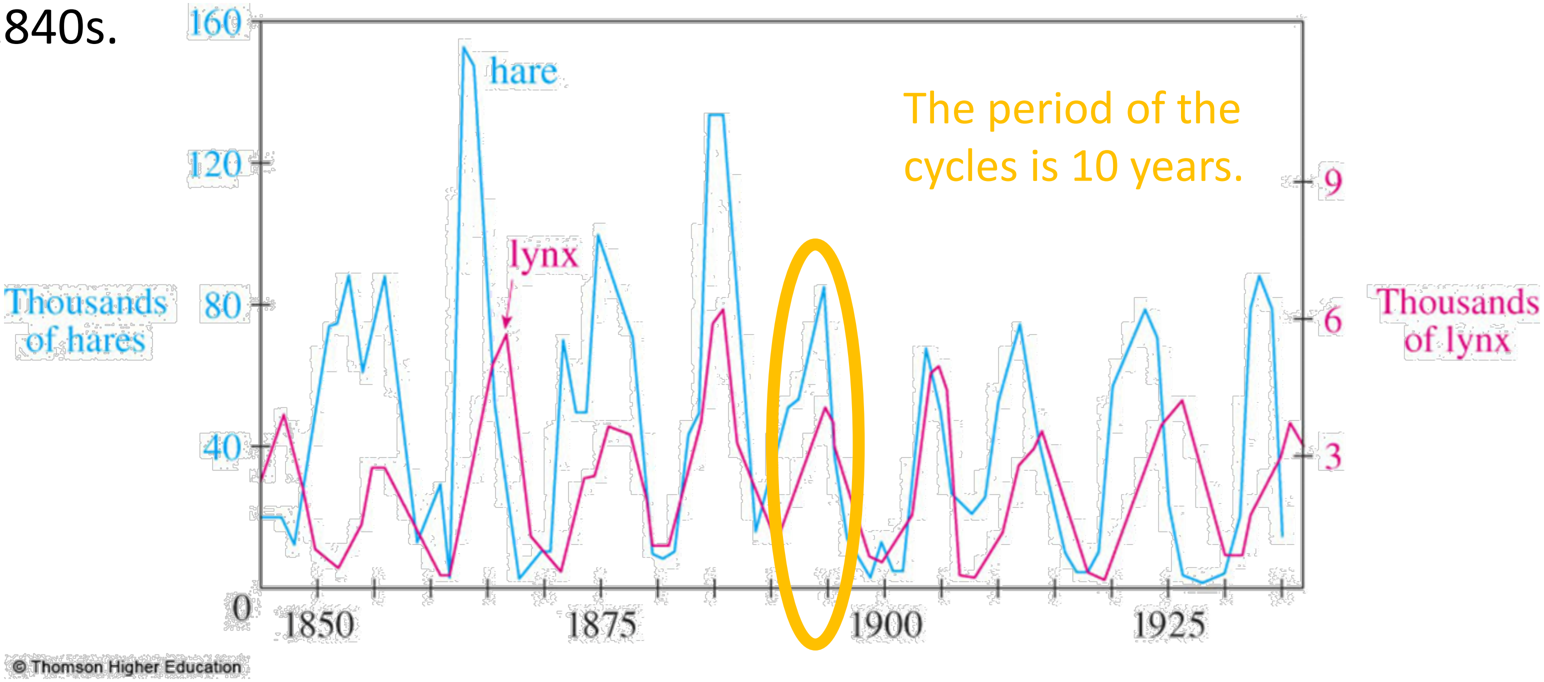

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Simulating populations

```python
# -*- coding: utf-8 -*-

import matplotlib
matplotlib.use('Agg')
import matplotlib.pyplot as plt
import numpy as np
import scipy.integrate
import sys

def derivative(X, t, alpha, beta, delta, gamma):
  x, y = X
  dotx = ((alpha*x) - (x*delta * y))
  doty = (gamma * x *y) - (beta *y  )
  return np.array([dotx, doty])
```

```python
def main():
  values = (sys.argv)
  alpha = (values[1])  #the growth rate of the prey
  if "-" in alpha:
    alpha=alpha.replace("-","")
  if alpha.upper() in ["H","HELP"]:
    print("""dx/dt = αx -δxy, dy/dt = -βy + γxy
    python lotka-volterra.py alpha Beta delta gamma x0 y0
    example data    α = 0.8, δ = 0.01, β= 0.2, γ = 0.002 x0=100 y0=8 tmax=30""")
    quit()
  else:
    alpha=float(alpha)
  beta = float(values[2]) #the death rate of the predator
  delta = float(values[3])  #mortality rate due to predators
  gamma = float(values[4])  #mortality rate due to interaction pray predator
  x0 = float(values[5])  #original prey population
  y0 = float(values[6])  #original predator population
  tmax = float(values[7]) #Time in  years for analisis
  print(values)
  Nt = tmax*100
  t = np.linspace(0.,tmax, Nt)
  X0 = [x0, y0]
  res = scipy.integrate.odeint(derivative, X0, t, args = (alpha, beta, delta, gamma))
  x, y = res.T
  plt.figure()
  plt.grid()
  plt.title("odeint method")
  plt.plot(t, x, 'xb', label = 'Deer')
  plt.plot(t, y, '+r', label = "Wolves")
  plt.xlabel('Time t, [years]')
  plt.ylabel('Population')
  plt.legend()

  plt.savefig('Lotka-volterra.png')
if __name__ == '__main__':
  main()
```

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