# Advanced Calculus

## Neighborhoods, Interior and Cluster Points

ThinkBS: Basic Sciences in Engineering Education

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# Neighborhoods

Assume $(X, d)$ is a metric space. The set

$$N_r(x) = B_r(x) = \{y \in X \mid d(x, y) < r\}$$

is called the **open neighborhood** or open ball with radius $r$ and center $x$.
The set

$$N_r[x] = B_r[x] = \{y \in X \mid d(x, y) \leq r\}$$

is called the **close neighborhood** or close ball with radius $r$ and center $x$.

# Neighborhoods: examples

In $\mathbb{R}^2$ with Euclidean metric, $N_1(0,0)$ is the inside of a circle centered at point $(0,0)$ with radius 1.

In $\mathbb{R}^2$ with $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, with $(x = (x_1, x_2), y = (y_1, y_2))$, $N_1(0,0)$ is the inside of a diamond centered at point $(0,0)$ with vertices at points $(1,0)$, $(0,1)$, $(-1,0)$ and $(0,-1)$.

In $\mathbb{R}^2$ with $d(x,y) = max\{|x_1 - y_1|, |x_2 - y_2|\}$, $N_1(0,0)$ is the inside of a square centered at point $(0,0)$ with vertices at points $(1,1)$, $(-1,1)$, $(-1,-1)$ and $(1,-1)$.

# Interior Points

Assume $A \subseteq (X, d)$. A point $a \in A$ is called an interior point if there exist an $r > 0$ such that $N_r(x) \subseteq A$.
The set of interior points of a set $A$ is called the interior and is shown by $int(A)$ or $A^{\mathrm{o}}$.
A set is called open if all of its points are interior points; $A \subseteq A^{\mathrm{o}}$, or equivalently $A = A^{\mathrm{o}}$.

**Example 1**: Assume $A = (0, 1] \subseteq (\mathbb{R}, |.|)$. Then $\frac{1}{5} \in A^{\mathrm{o}}$, $1 \notin A^{\mathrm{o}}$ and $A^{\mathrm{o}} = (0, 1)$ (why?)

**Example 2**: An open neighborhood $N_r(x) \subseteq (X, d)$ is an open set; in other word, all of its points are interior points (why?)

# Cluster Points

Assume $A \subseteq (X, d)$. A point $x \in X$ is called a cluster point if for all $r > 0$, $N_r(x) \cap A \neq \emptyset$.
The set of cluster points of a set $A$ is called cluster points and is shown by $cl(A)$ or $\bar{A}$.
A set is called closed if all of its cluster points belongs to the set itself; $\bar{A} \subseteq A$, or equivalently $A = \bar{A}$, since always $A \subseteq \bar{A}$.

**Example 1**: Assume $A = (0, 1] \subseteq (\mathbb{R}, |.|)$. Then $\frac{1}{5} \in \bar{A}$, $0 \in \bar{A}$ and, $1 \in \bar{A}$, $2 \notin \bar{A}$ and $\bar{A} = [0, 1]$ (why?)

**Example 2**: A closed neighborhood $N_r[x] \subseteq (X, d)$ is a closed set; in other word, all of its points are interior points (why?)

**Example 3**: A set does not need to be either open or close in a metric space. For instance $A = (0, 1]$ is neither open nor close.

**Example 4**: A set $A$ is open if its complement $A^c$ is close (why?)

**Example 5**: In a metric space $(X, d)$, the sets $\emptyset$ and $X$ are always open.

**Example 6**: In a metric space $(X, d)$, the sets $\emptyset$ and $X$ are always close.

**Example 7**: in a discrete metric space, every subset is both open and closed. (why?)