Advanced Calculus

Cardinality

ThinkBS: Basic Sciences in Engineering Education

Kadir Has University, Turkey

The Gaps: How many are there?

As we have seen, there are some gaps between rational numbers, while there are infinitely many rational number between any two arbitrary distinct rational numbers. (Why?)

We filled these gaps with real numbers, and we saw because of supremum property of $\mathbb{R}$ it is a gapless continuum.

One interesting problem here is that 'how many' rational numbers and 'how many' gaps are there. We will look at this problem next.

Equipollent Sets

We say two sets A and B are equipollent (or in a loose sense they have the same amount of elements) and show it by $A \simeq B$, if there is a one-to-one onto function $f : A \to B$ between them. This definition gives us an equivalence relationship, and the sets inside an equivalence class are said to have the same cardinality. For a set A, its cardinality or cardinal number is shown by $|A|$ or $card(A)$.

We call a set A a **finite** set if there exists a natural number n such that $|A| = card(A) = n$.

A set which is not finite (or in other terms, is equipollent with at least one of its proper subsets) is called an **infinite** set.

Countable Sets and Beyond ...

We show the cardinality of natural numbers $\mathbb{N}$ by $\aleph_0$, hence $card(\mathbb{N}) = \aleph_0$ and we call a set A a **countable** or **denumerable** set if it is equipollent with $\mathbb{N}$; if $card(A) = \aleph_0$.

A set which is a finite or a countable set is referred to as **almost countable** set.

The sets or even natural numbers, odd natural numbers, the integers $\mathbb{Z}$ are all examples of countable sets. (Why? Can you construct a one-to-one onto function between them and $\mathbb{N}$?)

For further details see ‘Chap. 2: Finite, countable, and uncountable sets’.

Cardinality of Rational Numbers

Although there are infinitely many rational numbers between any two distinct rational number, but surprisingly there are exactly same 'amount' of rationals as natural numbers; $card(\mathbb{Q}) = \aleph_0$.

This is because one can create a two dimensional table (let's talk only about $\mathbb{Q}^+$ for a moment) and put $\frac{m}{n}$ on the m-th column and n-th row. Then starting from the the corner and moving diagonally, passing numbers that already appeared before, we can construct a one-to-one onto function between positive rationals and natural numbers. A similar argument works for negative rationals and also for all $\mathbb{Q}$.

This shows that the set of rational numbers $\mathbb{Q}$ is a countable set and $card(\mathbb{Q}) = \aleph_0$.

Cardinality of Real Numbers

As of now, we now that $\mathbb{Q}$ is countable and it has gaps. By filling these gaps, we construct the real numbers $\mathbb{R}$. It turns out the number of gaps between rationals is not 'some' but more than even the whole set of rationals!

Cardinality of Real Numbers

As of now, we now that $\mathbb{Q}$ is countable and it has gaps. By filling these gaps, we construct the real numbers $\mathbb{R}$. It turns out the number of gaps between rationals is not 'some' but more than even the whole set of rationals!

Theorem (*Cantor's Diagonal Proof*): The set of real numbers has a cardinality, absolutely greater than the cardinality of rational numbers. No matter how one tries, there is no one-to-one onto correspondence between the set of reals $\mathbb{R}$ and of $\mathbb{N}$ (or $\mathbb{Q}$). We chow the cardinality of real numbers by $\mathfrak{c}$. The set $\mathbb{R}$ is an **uncountable** set.

Infinities, many of them ...

Up to this point, we have encountered by two different kind of infinities. One is the countable infinity $\aleph_0$ and the other one $\mathfrak{c}$ of uncountable infinity. Cantor has also shown that:

Theorem: If A is a set and $P(A)$ is its power set, or the set of all subsets of A, then $card(A) < card(P(A))$.

One can use this to come up with infinitely many infinities!
All sets with cardinality strictly greater than $\aleph_0$ is called an uncountable set.